# THE USE OF AN INTERESTING PROPERTY OF IDEMPOTENT MATRICES FOR THE INVERSION OF MATRICES

By

K. JÁNKI

Department of Mechanical Engineering Mathematics, Technical University, Budapest

(Received June 29th, 1970)

Presented by Prof. Dr. M. FARKAS

## 1. Introduction

The quadratic matrix $\mathbf{A}$ of order $n$ should be given as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^* \\ \mathbf{a}_2^* \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_n^* \end{bmatrix}$$

where

$$a_i^* = [a_{i1} \; a_{i2} \; . \; . \; . \; a_{in}]$$

Suppose that

$$\det(\mathbf{A}) \neq 0$$

thus $\mathbf{A}$ is not singular or, what is equivalent, the rank of $\mathbf{A}$ is

$$\varrho(\mathbf{A}) = n$$

and so it has an invert $\mathbf{A}^{-1}$, for which

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{E}_n$$

where $\mathbf{E}_n$ is the unit matrix of order $n$.

$$\mathbf{E}_n = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 \end{bmatrix} = [\mathbf{e}_1 \; \mathbf{e}_2 \; . \; . \; . \; \mathbf{e}_n] \, ;$$

where $\mathbf{e}_i$ is the column unit vector with $n$ elements, the $i$-th element being 1, the others 0.

Be $\mathbf{X}$ the reciprocal matrix to be determined where column vectors are vectors $\mathbf{X}_i$ with $n$ elements $(i = 1, 2, \ldots n)$.

The problem can be drawn up as follows: The unique solution of the in-homogeneous linear matrix equation

$$\mathbf{A}\,\mathbf{X} = \mathbf{E}_n \tag{1.1}$$

is to be determined. The existence of a unique solution is assured by the pre-suppositions det $(\mathbf{A}) \neq 0$, and $\varrho\,(\mathbf{A}) = n$.

This problem can be reworded for solving homogeneous linear system of equations as well by introducing certain hypermatrices.

Let us introduce the following hypermatrices:

$$\mathbf{B} = [\mathbf{A} - \mathbf{E}_n]\,,$$

a hypermatrix consisting of $n$ rows and $2n$ columns, with two adjacent blocks being quadratic matrices $\mathbf{A}$ and $\mathbf{E}_n$ of order $n$;

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$$

is a hypermatrix of $2n$ rows and $n$ columns with quadratic matrices $\mathbf{X}$ and $\mathbf{Y}$ of order $n$ as the two superimposed blocks.

By their means (1.1) can be written as:

$$\mathbf{B}\,\mathbf{Z} = 0 \tag{1.2}$$

or more explicitly:

$$[\mathbf{A} - \mathbf{E}_n] \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = 0$$

that is

$$\mathbf{A}\,\mathbf{X} - \mathbf{Y} = 0. \tag{1.3}$$

By comparing (1.1) and (1.3) it is seen that a solution for the homogeneous linear equation system (2.1), with $n$ linearly independent column vectors of $2n$ elements consumed in $Z$, of the following structure:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \\ \mathbf{E}_n \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdot & \cdot & \cdot & \mathbf{X}_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdot & \cdot & \cdot & \mathbf{e}_n \end{bmatrix}$$

is sought for.

The column vectors of $\mathbf{Z}$ that is the column vectors

$$\mathbf{Z}_i = \begin{bmatrix} \mathbf{X}_i \\ e_i \end{bmatrix}$$

and the row vectors

$$\mathbf{b}_i^* = [\mathbf{a}_i^* \;\vdots\; - \mathbf{e}_i^*]$$

of the matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_n^* \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^* & : & -\mathbf{e}_1^* \\ \mathbf{a}_2^* & : & -\mathbf{e}_2^* \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \mathbf{a}_n^* & : & -e_n^* \end{bmatrix}$$

are orthogonal:
Namely, according to (1.2)

$$\mathbf{b}_i^x \mathbf{Z}_i = 0 \qquad i = 1, 2, \ldots n.$$

## 2. An interesting property of idempotent matrices

Idempotent is a quadratic matrix $\mathbf{P}$ of order $n$ satisfying the equation

$$\mathbf{P}^2 = \mathbf{P}.$$

If $\mathbf{P}$ is of rank $r$ that is

$$\varrho(\mathbf{P}) = r$$

and $\mathbf{P}$ is given in the form of a possible minimal sum of diads

$$\mathbf{P} = \sum_{k=1}^{r} \mathbf{u}_k \mathbf{v}_k^* = [\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_r] \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v}_r^* \end{bmatrix} = \mathbf{U}\,\mathbf{V}^*$$

where the columns of $\mathbf{U}$ are linearly independent column vectors of $n$ elements and the rows of $\mathbf{V}^*$ are linearly independent row vectors with $n$ elements, then, according to a well-known theorem (see e.g. page 3 in [1]), the column vectors $\mathbf{u}_k$ and the row vectors $\mathbf{v}_k^*$ form together a biorthogonal system:

$$\mathbf{v}_i^* \mathbf{u}_j = \delta_{ij}; \qquad i, j = 1, 2, \ldots, r$$

where

$$\delta_{ij} = \begin{cases} 0, \text{ for } i \neq j \\ 1, \text{ for } i = j. \end{cases}$$

There exists again a well-known theorem for idempotent matrices of this property (see e.g. pp. 40—42 in [2]).

Let $m \leqq n$ and $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be a sequence of quadratic matrices of order $n$ such that

$$\sum_{i=1}^{m} \mathbf{P}_i = \mathbf{E}_n$$

and

$$\mathbf{P}_i \mathbf{P}_j = 0 \quad \text{for } i \neq j$$

then

1. $\mathbf{P}_i^2 = \mathbf{P}_i$ for each i, i.e., each element of the sequence is an idempotent matrix;

2. for $\varrho(\mathbf{P}_i) = \varrho_i$, i.e.

$$\mathbf{P}_i = \sum_{k=1}^{\varrho_i} \mathbf{u}_{ik} \mathbf{v}_{ik}^* = \mathbf{U}_i \mathbf{V}_i^*$$

$$\sum_{i=1}^{m} \varrho_i = n$$

the sum of the element ranks in the sequence equals $n$;

3. the set of the column and row vectors in the diads determined for all $i$ forms a complete biorthogonal system of dimension $n$, i.e.

$$\mathbf{v}_{jk}^* \mathbf{u}_{il} = \delta_{kl}\, \delta_{ij} = \begin{cases} 1 \text{ for } i = j \text{ and } k = l \text{ simultaneously} \\ 0 \text{ for either } i \neq j \text{ or } k \neq 1 \end{cases}$$

where

$$l = 1, 2, \ldots \varrho_i; \qquad i, j = 1, 2, \ldots, m.$$
$$k = 1, 2, \ldots \varrho_j.$$

## 3. A new algorithm for matrix inversion

Based on the theorems drawn up in the preceding item, the following algorithm is proposed for solving the problem (1.2) in item 1. (The advantage of the algorithm suggested by the author consists in at least halving the number of necessary multiplications as compared to other known algorithms of basis factorization.)

First of all, let us replace $\mathbf{B}$ by the following hypermatrix of order 2n

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & -\mathbf{E}_n \\ \cdots & \cdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \cdots \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{c}_i^*$ being the $i$-th row vector.
For $$1 \leqq i \leqq n$$

these row vectors are linearly independent by definition. All the other row vectors are identically 0, causing the rank of $\mathbf{C}$ to equal $n$. An arbitrary column vector of the unknown matrix $\mathbf{Z}$ is the vector $\mathbf{z}$ with $2n$ elements.

The equations in the equation system to be solved are

$$\mathbf{c}_i^* \mathbf{Z} = 0 \qquad i = 1, 2, \ldots n .$$

To solve this equation system means to find all the linearly independent column vectors $\mathbf{z}$ orthogonal to the row vectors $\mathbf{c}_i^*$. The algorithm to this aim consists of the following steps:

*Step 1:* Be the $i_1$-th element $(i_1 \leqq n)$ of the row vector $\mathbf{c}_1^*$ non-zero (such an $i_1$ exists by definition), hence

$$\mathbf{c}_1^* \mathbf{e}_{i_1} \neq 0 .$$

Then

$$\mathbf{P}_1 = \frac{\mathbf{e}_{i_1} \mathbf{c}_1^*}{\mathbf{c}_1^* \mathbf{e}_{i_1}}$$

(here and further on the column unit vectors $\mathbf{e}_k$ have 2n elements) is an idempotent matrix each row vector of which is a product of $\mathbf{c}_1^*$ by a scalar:

$$\mathbf{P}_1^2 = \frac{\mathbf{e}_{i_1} (\mathbf{c}_1^* \mathbf{e}_{i_1}) \mathbf{c}_1^*}{(\mathbf{c}_1^* \mathbf{e}_{i_1})^2} = \frac{\mathbf{e}_{i_1} \mathbf{c}_1^*}{\mathbf{c}_1^* \mathbf{e}_{i_1}} = \mathbf{P}_1 .$$

Thus

$$\mathbf{P}_2 = \mathbf{E}_{2n} - \mathbf{P}_1$$

is an idempotent matrix

$$\mathbf{P}_2^2 = \mathbf{E}_{2n} - 2\mathbf{P}_1 + \mathbf{P}_1^2 = \mathbf{E}_{2n} - \mathbf{P}_1 = \mathbf{P}_2$$

with linearly independent (i.e. non-zero) columns orthogonal to $\mathbf{c}_1^*$:

$$\mathbf{c}_1^* \mathbf{P}_2 = \mathbf{c}_1^* - \mathbf{c}_1^* \frac{\mathbf{e}_{i_1} \mathbf{c}_1^*}{\mathbf{c}_1^* \mathbf{e}_1} = \mathbf{c}_1^* - \mathbf{c}_1^* = 0 .$$

*Step 2:* On the other hand, there exists an $i_2$ (obviously not greater than $n$) such that

$$\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2} \neq 0 .$$

Thus

$$\mathbf{P}_3 = \mathbf{P}_2 - \frac{\mathbf{P}_2 \mathbf{e}_{i_2} \mathbf{c}_2^* \mathbf{P}_2}{\mathbf{c}_2^* \mathbf{P}_2 e_{i_2}}$$

is an idempotent matrix

$$\mathbf{P}_3^2 = \mathbf{P}_2^2 - 2\,\mathbf{P}_2 \frac{\mathbf{P}_2 \mathbf{e}_{i_2} \mathbf{c}_2^* \mathbf{P}_2}{\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2}} + \frac{\mathbf{P}_2 \mathbf{e}_{i_2} (\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2}) \mathbf{c}_2^* \mathbf{P}_2}{(\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2})^2} = P_3$$

with linearly independent (i.e. non-zero) column vectors orthogonal to both $\mathbf{c}_1^*$ and $\mathbf{c}_2^*$:

$$\mathbf{b}_1^* \mathbf{P}_3 = \mathbf{c}_1^* \mathbf{P}_2 - \mathbf{c}_1^* \frac{\mathbf{P}_2 \mathbf{e}_{i_2} \mathbf{c}_2^* \mathbf{P}_2}{\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2}} = 0$$

namely

$$\mathbf{c}_1^* \mathbf{P}_2 = 0$$

$$\mathbf{c}_2^* \mathbf{P}_3 = \mathbf{c}_2^* \mathbf{P}_2^* - \frac{(\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2}) \mathbf{c}_2^* \mathbf{P}_2}{\mathbf{c}_2^* \mathbf{P}_2 \mathbf{e}_{i_2}} = \mathbf{c}_2^* \mathbf{P}_2 - \mathbf{c}_2^* \mathbf{P}_2 = 0.$$

*Step k:* Be $\mathbf{P}_k$ determined in the previous step such that

$$\mathbf{P}_k^2 = \mathbf{P}_k$$

and

$$\mathbf{c}_1^* \mathbf{P}_k = \mathbf{c}_2^* \mathbf{P}_k = \ldots = \mathbf{c}_{k-1}^* \mathbf{P}_k = 0, \quad \text{but}$$

$$\mathbf{c}_k^* \mathbf{P}_k \neq 0$$

so there exists an $i_k (\leq n)$ for which

$$\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{i_k} \neq 0.$$

Then

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \frac{\mathbf{P}_k \mathbf{e}_{i_k} \mathbf{c}_k^* \mathbf{P}_k}{\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{i_k}}$$

is an idempotent matrix:

$$\mathbf{P}_{k+1}^2 = \mathbf{P}_k^2 - 2\,\mathbf{P}_k \frac{\mathbf{P}_k \mathbf{e}_{i_k} \mathbf{c}_k^* \mathbf{P}_k}{\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{i_k}} +$$

$$+ \frac{\mathbf{P}_k \mathbf{e}_{i_k} (\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{i_k}) \mathbf{c}_k^* \mathbf{P}_k}{(\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{i_k})^2} = \mathbf{P}_k + {}_1$$

with linearly independent (i.e. non-zero) column vectors equally orthogonal to $\mathbf{c}_1^*, \ldots, \mathbf{c}_{k-1}^*$ and $\mathbf{c}_k^*$:

$$\mathbf{c}_k^* \mathbf{P}_{k+1} = \mathbf{c}_k^* \mathbf{P}_k - \frac{(\mathbf{c}_k^* \mathbf{P}_k \mathbf{e}_{ik}) \mathbf{c}_k^* \mathbf{P}_k}{\mathbf{c}_k^* P_k \mathbf{e}_{ik}} = \mathbf{c}_k^* \mathbf{P}_k - \mathbf{c}_k^* \mathbf{P}_k = 0.$$

Finally, for $k = n$ the linearly independent (i.e. non-zero) column vectors of the idempotent matrix $\mathbf{P}_{n+1}$ got in the $n$-th step are just the solution vectors $\mathbf{Z}$ sought for.

The construction of the algorithm is such that the last $n$ columns of matrix $\mathbf{P}_{n+1}$ form the hypermatrix:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{E}_n \end{bmatrix}$$

where $\mathbf{X} = \mathbf{A}^{-1}$
that was to be determined.

## 4. The use of the algorithm in a concrete numerical example

Be the matrix to be inverted:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & -2 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 \end{bmatrix}$$

Transform it into a matrix of order $2n = 8$;

$$\mathbf{C} = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -2 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{c}_1^* \\ \mathbf{c}_2^* \\ \mathbf{c}_3^* \\ \mathbf{c}_4^* \\ \hline \mathbf{0}^* \\ \mathbf{0}^* \\ \mathbf{0}^* \\ \mathbf{0}^* \end{bmatrix}$$

Furthermore, let the unit matrix of order 8 be

$$\mathbf{E} = [\mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3\, \mathbf{e}_4 \mid \mathbf{e}_5\, \mathbf{e}_6\, \mathbf{e}_7\, \mathbf{e}_8]$$

Since

$$\mathbf{c}_1^* \mathbf{e}_1 = 1$$

therefore

$$\mathbf{P}_1 = \frac{\mathbf{e}_1\,\mathbf{c}_1^*}{\mathbf{c}_1^*\,\mathbf{e}_1} \quad \text{and}$$

$$\mathbf{P}_2 = \mathbf{E} - \frac{\mathbf{e}_1\,\mathbf{c}_1^*}{\mathbf{c}_1^*\,\mathbf{e}_1} = \mathbf{E} - \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since $\mathbf{c}_2^*\ \mathbf{P}_2 = [0 -1 -2 -2 \mid 2 -1\ 0\ 0]$;

$$\mathbf{c}_2^*\,\mathbf{P}_2\,\mathbf{e}_2 = -1$$

therefore

$$\mathbf{P}_3 = \mathbf{P}_2 = \frac{\mathbf{P}_2\,\mathbf{e}_2\,\mathbf{c}_2^*\,\mathbf{P}_2}{\mathbf{c}_2^*\,\mathbf{P}_2\,\mathbf{e}_2} = \mathbf{P}_2 - \left[\begin{array}{c} 0 \\ -1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \left[\begin{array}{cccc|cccc} 0 & -1 & -2 & -2 & 2 & -1 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since

$$\mathbf{c}_3^* \mathbf{P}_3 = [\,0\;\;0\;\;1\;\;1 \mid -1\;\;0\;\;-1\;\;0\,]; \qquad \mathbf{c}_3^* \mathbf{P}_3 \mathbf{e}_3 = 1$$

therefore

$$\mathbf{P}_4 = \mathbf{P}_3 - \frac{\mathbf{P}_3 \mathbf{e}_3 \mathbf{c}_3^* \mathbf{P}_3}{\mathbf{c}_3^* \mathbf{P}_3 \mathbf{e}_3} = \mathbf{P}_3 - \left[\begin{array}{c} 0 \\ -2 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] [\,0\;\;0\;\;1\;\;0 \mid -1\;\;0\;\;-1\;\;0\,] =$$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since

$$\mathbf{c}_4^* \mathbf{P}_4 = [\,0\;\;0\;\;0\;\;1 \mid -2\;\;-1\;\;-2\;\;-1\,]; \qquad \mathbf{e}_4 \mathbf{P}_4 \mathbf{e}_4 = 1$$

therefore

$$\mathbf{P}_5 = \mathbf{P}_4 - \frac{\mathbf{P}_4 \mathbf{e}_4 \mathbf{c}_4 \mathbf{P}_4}{\mathbf{c}_4^* \mathbf{P}_4 \mathbf{e}_4} =$$

$$= \mathbf{P}_4 - \left[\begin{array}{c} -1 \\ 0 \\ -1 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] [\,0\;\;0\;\;0\;\;1 \mid -2\;\;-1\;\;-2\;\;-1\,] =$$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse matrix sought for is the upper right block of order 4 of $\mathbf{P}_5$:

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & -1 & -2 & -1 \\ 0 & -1 & -2 & 0 \\ -1 & 1 & -1 & -1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

## Summary

A method is presented to lead in a finite number of iterations for any given non-singular $n$-square matrix to the unknown system of linearly independent column vectors, which, together with the linearly independent row vectors of the matrix, form a biorthogonal system. The above algorithm applies a well-known property of suitably constructed projector matrices the sum of which is the unit matrix.

## References

1. Egerváry, E.: On a property of the projector matrices and its application to the canonical representation of matrix functions. Acta Scientiarum Mathematicarum Tomus XV. Fasc. 1. Szeged. 1953. p. 1—6.
2. Egerváry, E.: Über eine Konstruktive Methode zur Reduktion einer Matrix auf die Jordansche Normalform. Acta Mathematica Academica Scientiarum Hungarica. Tomus X. Fasc. 1—2. Budapest, 1959. p. 31—54.

Dr. Kálmán Jánki, Budapest XI., Sztoczek u. 2—4. Hungary