APPLICATION OF THE LAPLACE TRANSFORMATION FOR THE SOLUTION OF DIFFERENTIAL EQUATIONS INVOLVING DISTRIBUTIONS

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In paper [7] the basic set A was introduced, in the present paper the Laplace transformation is defined for the elements of A, and this is employed for the solution of differential equations involving distributions. The basic set A consists of the finite formal linear combinations of ordinary, sectionally smooth (infinitely many times differentiable), bounded, complex-valued functions with one real variable, and of delta elements of the form $a\delta^{(k)}(x-c)$, where a is an arbitrary complex, c an arbitrary real number, and $k = 0, 1, 2, \ldots$ An element from A of this kind is e.g.

$$p(x) + a_0\,\delta(x-c) + a_3\,\delta^{(3)}(x-c)\,,$$

where $p(x)$ is the corresponding ordinary function. For the elements of basic set A algebraic and infinitesimal operations were defined, by means of which the solutions of such ordinary linear differential equations involving distributions and of systems of equations were produced, where the disturbing function and the corresponding column vector consist of the elements of set A.

Let us now consider the linear ordinary differential equation with constant coefficients involving distributions

$$P_n(D)y = f(x) \tag{1}$$

where $f(x) \in A$ and

$$P_n(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0 .$$

The solution of differential equation (1) can be produced in case of $x > 0$ also by the help of the Laplace transformation. This method of solution is in most practical cases even more simple than the classical method. Here a natural requirement would be the existence of the Laplace transforms of the ordinary functions in set A, while the Laplace transforms of the delta elements are to be defined separately, taking into consideration the operation rules valid in structure A.

Accordingly, the Laplace transforms of the delta elements, considering the multiplication rule and the integral definition, will be

$$\mathfrak{L}[\delta(x-c)] \stackrel{\Delta}{=} {}_{0}\!\int^{\infty} \delta(x-c)\, e^{-sx}\, dx = {}_{0}\!\int^{\infty} \delta(x-c)\, e^{-cs}\, dx = e^{-cs}, \quad c > 0$$

Similarly

$$\mathfrak{L}[\delta^{(1)}(x-c)] = s e^{-cs},$$

and in general

$$\mathfrak{L}[\delta^{(k)}(x-c)] = s^k e^{-cs}, \tag{2}$$

where $k = 0, 1, 2, \ldots$ and $c > 0$.

Thus by performing the Laplace transformation both sides of the differential equation involving distributions (1) we obtain for the transforms:

$$(s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0) Y(s) = F(s) + (s^{n-1} +$$
$$+ a_{n-1} s^{n-2} + \ldots + a_1) y(0+) + (s^{n-2} + \ldots + a_2) y^{(1)}(0+) + \ldots$$
$$\ldots + (s + a_{n-1}) y^{(n-2)}(0+) + y^{(n-1)}(0+).$$

Hence:

$$Y(s) = \frac{F(s) + G_{n-1}(s)}{P_n(s)},$$

where $G_{n-1}(s)$ is the polynomial of $(n-1)$th order, and $P_n(s)$ that of nth order of the differential operator s. The Laplace transforms of the delta elements arising in f are included in F. It is easy to see that if the derivative of the highest order of the delta elements is $\delta^{(k)}$ and if $k = n - 1$, then the function $y(x)$ is sectionally continuous, if $k = n - 2$, then $y(x)$ is continuous. In general, if $k = n - l$, then the functions $y, Dy, \ldots D^{l-2}y$ will be continuous. Furthermore, if

1. $k < n$, then $y(x)$ is an ordinary function,
2. $k = n$, then $y(x) \supset \delta(x)$,
3. $k = n + r$, then $y(x) \supset \delta^{(r)}(x)$.

The great practical advantage of the solution by Laplace transformation is that the solution is not to be joined at the section boundaries on account of discontinuities in the disturbing member f and of the delta elements, like in the classical solution, since the Laplace transform already includes the joining conditions.

E.g. 1. $(D + 1)y = H(x) + \delta(x - 1)$,
initial condition: $y(0+) = 0$.

The Laplace transform of function $y(x)$ is

$$Y(s) = \frac{\frac{1}{s} + e^{-s}}{s+1} = \frac{1}{s} - \frac{1}{s+1} + \frac{e^{-s}}{s+1}.$$

Upon performing the retransformation,

$$y(x) = (1 - e^{-x})H(x) + e^{-x+1}H(x-1).$$

Here $k = 0$, $n = 1$, that is $k = n - 1$, accordingly the solution function is discontinuous,

$$y(1+) - y(1-) = 1.$$

2. $(D^2 + 3D + 2)y = 10x + 2\delta(x-3)$,

initial conditions:

$y(0+) = -7.5 + e^{-1} - 2e^{-2}$, $y^{(1)}(0+) = 5 - e^{-1} + 4e^{-2}$.

Similarly as in Example 1,

$$Y(s) = \frac{\frac{10}{s^2} + 2e^{-3s} + (-7.5 + e^{-1} - 2e^{-2})s + 3(-7.5 + e^{-1} - 2e^{-2}) + 5 - e^{-1} + 4e^{-2}}{s^2 + 3s + 2} =$$

$$= \frac{5}{s^2} - \frac{7,5}{s} + \frac{e^{-1}}{s+1} - \frac{2e^{-2}}{s+2} + 2e^{-3s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

$$y(x) = (5x - 7.5 + e^{-x-1} - 2e^{-2x-2})\,H(x) + 2(e^{-x+3} - e^{-2x+6})\,H(x-3).$$

Let us consider now differential equation (1) in case where the disturbing term $f(x)$ is of the form

$$r(x) + b_0\delta(x) + \ldots + b_k\delta^{(k)}(x),$$

where $r(x)$ is a sectionally smooth and bounded function, accordingly $f(x) \in A$; the b_i values ($i = 1, \ldots, k$) are real or complex constants.

Let the given constants

$$y(0-),\ y^{(1)}(0-),\ \ldots,\ y^{(n-1)}(0-) \tag{1a}$$

be named the starting values.

At point 0 itself, no initial conditions can be given, since there the function y or some of its derivatives are discontinuous, if $b_i \neq 0$ for at least one i value.

Write the solution of differential equation (1) as the sum of two functions

$$y = y_I + y_{II} \,.$$

Accordingly let us decompose differential equation (1) to the following two differential equations:

$$P_n(D)\, y_I = r(x) \tag{3}$$

$$P_n(D)\, y_{II} = b_0\, \delta(x) + \ldots + b_k\, \delta^{(k)}(x) \tag{4}$$

The starting values pertaining to differential equations (3) and (4) can be given as follows:

$$y_I(0-) = y_I(0+) = y_I(0) = y(0-), \ldots, y_I^{(n-1)}(0-) = y_I^{(n-1)}(0+) = \\ = y_I^{(n-1)}(0) = y^{(n-1)}(0-) \tag{3a}$$

$$y_{II}(0-) = \ldots y_{II}^{(n-1)}(0-) = 0 \tag{4a}$$

Define now function y_{I1} by the equality

$$y_{I1} = y_I\, H(x) \tag{5}$$

where $H(x)$ is the Heaviside unit step function. Hereafter write the expression for $P_n(D)y_{I1}$, using the operation rules of the structure A [7]:

$$Dy_{I1}(x) = H(x)\, Dy_I + y_I(x)\, \delta(x) = H(x)\, Dy_I(x) + y_I(0)\, \delta(x)$$

$$D^2 y_{I1}(x) = H(x) D^2 y_I(x) + y_I^{(1)}(0)\, \delta(x) + y_I(0)\, \delta^{(1)}(x)$$

$$\vdots \qquad\qquad \vdots$$

$$D^n y_{I1}(x) = H(x)\, D^n y_I(x) + y_I^{(n-1)}(0)\, \delta(x) + y_I^{(n-2)}(0)\, \delta^{(1)}(x) + \\ + \ldots y_I(0)\, \delta^{(n-1)}(x)$$

Hence:

$$P_n(D) y_{I1} = P_n(D)\, y_I(x)\, H(x) + (y_I^{(n-1)}(0) + a_{n-1} y_I^{(n-2)}(0) + \ldots + \\ + a_1 y_I(0))\, \delta(x) + (y_I^{(n-2)}(0) + a_{n-1} y_I^{(n-3)}(0) + \ldots + a_2 y_I(0))\, \delta^{(1)}(x) + \ldots + \\ + (y_I^{(1)}(0) + a_{n-1} y_I(0))\, \delta^{(n-2)}(x) + y_I(0)\, \delta^{(n-1)}(x)\,. \tag{6}$$

According to (3) $P_n(D)y_I H(x) = r(x)H(x)$. According to definition (5) $y_{I1}(x) = y_I(x)$, if $x > 0$ and

$$y_{I1}(0-) = y_{I1}^{(1)}(0-) = \ldots = y_{I1}^{(n-1)}(0-) = 0\,.$$

Thus, these values are not identical with the values assumed for $0+$. These latter values are determined by the disturbing term in differential equation (6) involving distributions.

Examine hereafter the differential equation involving distributions

$$P_n(D)\,y_{II} = b_0\,\delta(x) + \ldots + b_k\,\delta^{(k)}(x) \tag{7}$$

which relates to y_{II}. By considering conditions (1a)

$$y_{II}(0-) = y_{II}^{(1)}(0-) = \ldots = y_{II}^{(n-1)}(0-) = 0\,. \tag{7a}$$

A function y_{IIu} can be defined here which is identical with y_{II} for $x > 0$ and satisfies the homogeneous differential equation

$$P_n(D)\,y_{IIu} = 0 \tag{8}$$

where the initial conditions are given by differential equation (7) involving distributions and y_{IIu} is continuous for all values of x.
Upon considering the above differential equation (7) involving distributions can be written in the form

$$\begin{aligned} P_n(D)\,y_{II} = {} & \big(y_{IIu}^{(n-1)}(0) + a_{n-1}y_{IIu}^{(n-2)}(0) + \ldots + a_1 y_{IIu}(0)\big)\,\delta(x) + \\ & + \big(y_{IIu}^{(n-2)}(0) + a_{n-1}y_{IIu}^{(n-3)}(0) + \ldots + a_2 y_{IIu}(0)\big)\,\delta^{(1)}(x) + \\ & + \ldots + \big(y_{IIu}^{(1)}(0) + a_{n-1}y_{IIu}(0)\big)\,\delta^{(n-2)}(x) + y_{IIu}(0)\,\delta^{(n-1)}(x) \end{aligned} \tag{9}$$

If in differential equation (6) involving distributions $k = n-1$, that is $b_{n-1} \neq 0$, and according to (9) $y_{IIu}(0) = b_{n-1}$, then y_{II} is discontinuous at $x = 0$. If $k = n-2$, then y_{II} is continuous, but Dy_{II} is already discontinuous at $x = 0$. In general, if $k = n-r$, then y_{II}, $Dy_{II}, \ldots, D^{(r-2)}y_{II}$ are continuous, but $D^{r-1}y_{II}$ is discontinuous. By the help of the given coefficients b_i ($i = 0,1,\ldots,k$)

the initial values

$$y_{IIu}(0)\,,\quad y_{IIu}^{(1)}(0)\quad \ldots\,,\quad y_{IIu}^{(n-1)}(0)$$

can be determined, since the distribution

$$b_0\,\delta(x) + \ldots + b_k\,\delta^{(k)}(x)$$

can be expressed in a single way (see [4]).

The solution of differential equations (8) and (9) with the given conditions is identical for $x > 0$.

Let us now return to the original differential equation (1) and try to determine the solution for $x > 0$, if the starting values

$$y(0-), \quad y^{(1)}(0-), \quad \ldots, \quad y^{(n-1)}(0-) \tag{1a}$$

are given. Define the function

$$y_1(x) = y_{I1}(x) + y_{II}(x)$$

which is equal to the function $y(x)$ in case of $x > 0$ and for which the differential equation involving distributions

$$\begin{aligned} P_n(D)\,y_1 &= r_1(x) + \big(b_0 + y^{(n-1)}(0-) + a_{n-1}y^{(n-2)}(0-) + \ldots + \\ &+ a_1 y(0-)\big)\,\delta(x) + \big(b_1 + y^{(n-2)}(0-) + a_{n-1}y^{(n-3)}(0-) + \ldots + \\ &+ a_2 y(0-)\big)\,\delta^{(1)}(x) + \ldots + \big(b_{n-2} + y^{(1)}(0-) + a_{n-1}y(0-)\big)\,\delta^{(n-2)}(x) + \\ &+ \big(b_{n-1} + y(0-)\big)\,\delta^{(n-1)}(x) \end{aligned} \tag{10}$$

can be written, where $r_1(x) = r(x)H(x)$.

For (10) the two-sided Laplace transformation can be easily employed, namely

$$\mathfrak{L}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x)\, e^{-sx}\, dx = 1$$

$$\mathfrak{L}[\delta^{(k)}(x)] = \int_{-\infty}^{\infty} \delta^{(k)}(x)\, e^{-sx}\, dx = (-1)^k\, (e^{-sx})^{(k)}\Big|_{x=0} = s^k$$

and if

$$\mathfrak{L}[y_1(x)] = Y_1(s),$$

then

$$\mathfrak{L}[y_1^{(1)}(x)] = s\mathfrak{L}[y_1(x)] = sY_1(s),$$

since $y_1(x) = y(x)H(x)$, further

$$y_1^{(1)}(x) = y^{(1)}(x)\,H(x) + y(0)\,\delta(x)$$

and

$$\mathfrak{L}[y_1^{(1)}(x)] = \mathfrak{L}[y^{(1)}(x)\,H(x)] + y(0) = s\mathfrak{L}[y(x)\,H(x)] - y(0) + y(0) = s\mathfrak{L}[y_1(x)].$$

In general

$$\mathfrak{L}[y_1^{(k)}(x)] = s^k\mathfrak{L}[y_1(x)] = s^kY_1(s)\,.$$

Thus by performing the Laplace transformation of the differential equation (10) with distributions

$$Y_1(s) = \frac{1}{P_n(s)}\left(F_1(s) + (b_0 + y^{(n-1)}(0-) + \ldots + a_1 y(0-)) + (b_1 + \right.$$

$$\left. + y^{(n-2)}(0-) + \ldots + a_2 y(0-))\, s + \ldots + (b_{n-1} + y(0-))\, s^{n-1}\right).$$

From this, by inverse transformation

$$y_1(x) = \mathfrak{L}^{-1}[Y_1(s)]$$

Let us now consider the following example:

$$(D^2 + 3D + 2)y = 2\delta(x) + 3\delta^{(1)}(x)$$

$$y(0-) = 1,\ y^{(1)}(0-) = 2.$$

Upon rewriting the equation we obtain for the function $y_1(x)$ the differential equation involving distributions

$$(D^2 + 3D + 2)y_1 = (2 + 2 + 3)\delta(x) + (3 + 1)\delta^{(1)}(x).$$

By Laplace transformation

$$Y_1(s) = \frac{7 + 4s}{s^2 + 3s + 2} = \frac{3}{s+1} + \frac{1}{s+2}.$$

By retransforming

$$y_1(x) = 3e^{-x} + e^{-2x}.$$

The solution corresponding to the homogeneous differential equation

$$(D^2 + 3D + 2)y_h = 0$$

and calculated by the starting value

$$y_h(0-) = 1,\ \ y_h^{(1)}(0-) = 2$$

will be

$$y_{1h} = 4e^{-x} - 3e^{-2x}.$$

The step caused by the disturbing term at point $0+$ is

$$y_1(0+) - y_{1h}(0+) = 3 \quad \text{and} \quad y_1^{(1)}(0+) - y_{1h}^{(1)}(0+) = -7.$$

Consider now this same example without Laplace transformation, by the direct determination of the distribution solution. The previously given starting values $y(0-) = 1$ and $y^{(1)}(0-) = 2$ can be regarded here also as the final state of the solution considered in the section $-\infty < x < 0$. Let us now write the equation in the form

$$(D^2 + 3D + 2)y = D^2(2xH(x) + 3H(x))$$

from which

$$(D^2 + 3D + 2)v = (2x + 3)H(x) \quad \text{and} \quad y = D^2 v.$$

1. Regard first the case $x < 0$. For this the above differential equation has the form

$$(D^2 + 3D + 2)v = 0$$

and from this

$$v = c_1 e^{-x} + c_2 e^{-2x}.$$

Taking the final state into consideration we find that

$$v(0-) = c_1 + c_2 = 1 \quad \text{and} \quad v^{(1)}(0-) = -c_1 - 2c_2 = 2.$$

Namely for the case $x < 0$ we have $v = y$. By calculating the two constants from the above equations, the solution for $x < 0$ is

$$y = 4e^{-x} - 3e^{-2x}.$$

2. Regard hereafter the case $x > 0$. For this the differential equation is found to be

$$(D^2 + 3D + 2)v = 2x + 3,$$

from which

$$v = k_1 e^{-x} + k_2 e^{-2x} + x.$$

Constants k_1 and k_2 are now determined in such a way that functions v and $v^{(1)}$ should be continuous at point 0, hence

$$k_1 + k_2 = 1 \quad \text{and} \quad -k_1 - 2k_2 = 1,$$

thus for $x > 0$

$$v = 3e^{-x} - 2e^{-2x} + x.$$

3. Now consider the complete range $-\infty < x < \infty$, for which

$$v = 4e^{-x} - 3e^{-2x} + (-e^{-x} + e^{-2x} + x)\,H(x)$$

and from this

$$y = 4e^{-x} - 3e^{-2x} + (-e^{-x} + 4e^{-2x})H(x)\,.$$

Summary

In the paper Laplace transformation is defined for the elements of basic set A, and this is employed for the solution of ordinary constant coefficient linear differential equations where the elements of A are figuring in the disturbing term. The solution process will thus be more simple than the classical one. The solution is determined for $x > 0$ also in that case where the finite formal linear combination of $\delta^{(k)}(x)$, $(k = 0, 1, \ldots, (n-1))$, is similarly figuring in the disturbing term and the starting values are given at $x = (0-)$.

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