EXISTENCE AND USE OF THE SINGULARITY CARRIER AUXILIARY CURVE IN AIRFOIL CASCADES

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In designing airfoil cascades the so-called singularity methods are frequently used. In such cases the flow is the sum of an undisturbed and an induced flow. The effect of the airfoil profiles is usually considered by a source q, and by a γ vortex-distribution (specified boundary conditions on a curve section) placed on a socalled singularity carrier curve section. This paper aims at the generalization of the method.

For singularity carrier is used, as a rule, a section of the curve [1, 2], i.e., an approximation thereof [3, 4], on which, at any point, the components of the velocities, which are on both sides of the carrier and are perpendicular to the carrier, are of the same value but different in direction $\bar{c}_{fn} = -\bar{c}_{an}$. FEINDT has already shown for single airfoil profiles that there may be another singularity carrier as well [5]. FEINDT's idea can be extended for airfoil cascades, too. Thus we come to the theorems that ensure the existence and use of the general singularity carrier, the so-called singularity carrier auxiliary curve. The aim of this paper is to present these theorems.*

Let us consider for simplicity the case $\mathbf{c} = \nabla\Phi$ and $\Delta\Phi = 0$ for a single airfoil, with the signs of Fig. 1 (the conjugate of $\mathbf{c}$ is denoted by $\bar{c}$ and the velocity of undisturbed flow is $\mathbf{c}_\infty$) then

$$\begin{aligned}\bar{c}(z_0) &= \bar{c}_\infty + \frac{1}{2\pi i}\int_{z_1}^{z_2} \frac{\bar{c}_f - \bar{c}_a}{z_s - z_0}\,\mathrm{d}z_s = \bar{c}_\infty + \frac{1}{2\pi \mathrm{i}}\int_{z_1}^{z_2}\frac{g(z_s)}{z_s - z_0}\,\mathrm{d}z_s = \\ &= \bar{c}_\infty + \frac{1}{2\pi}\int_{z_1}^{z_2}\frac{q + i\gamma}{z_0 - z_s}\,|\mathrm{d}z_s| = \bar{c}_\infty + \bar{c}_L .\end{aligned} \tag{1}$$

On this basis the following definitions valid for both single airfoils and cascades can be given:

* The theorems were presented by the author at the Third Conference on Fluid Mechanics and Fluid Machinery [6].

D.1.: The physically feasible singularity carrier is a curve section (S), which lies with its full length in the inside of an airfoil section and the conjugated velocity distribution $\bar{c}(\zeta)$, $\zeta = \xi + i\,\eta$, outside the profile, can be analytically continued through the profile contour thereto; inside the contour the singular points of the distribution $\bar{c}(\zeta)$ are the points of the physically feasible singularity carrier.

D.2.: The complex velocity jump function $g(\zeta)$ is a complex function, whose value at any point of a singularity carrier ($\zeta_s \in S$) is equal to the difference between the velocity conjugates at this point, on both sides of the singularity carrier.

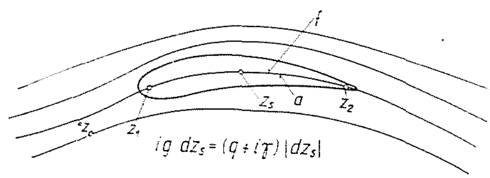

Fig. 1

D.3.: The singularity carrier auxiliary curve (L) is a section of curve whose terminals coincide with those of a physically feasible singularity carrier (S); and the closed curve ($S + L$) makes the boundary, ($\zeta_u \in \bar{U}$) of a simply connected closed region $\bar{U}$, in which the complex velocity jump function $g(\zeta_u)$ is, apart from the common terminals of S and L, holomorphic.

Thus, the singularity carrier characterized by the stipulation $c_{fn} = -c_{an}$ is, according to **D.1.**, always physically feasible; moreover, for infinitely thin profiles with the degradation $c_{fn} = c_{an} = 0$, it is the only physically feasible singularity carrier.

Before discussing the existence theorems allowing the application of the singularity carrier auxiliary curve for cascades, a few more conditions must be drawn up:

1. Let S and L be two curve sections satisfying the assumptions listed below (see Fig. 2):

F.1.: Their terminals coincide in such a way so as to form the boundary of a simply connected region $\bar{U}$ ($\zeta_u \in \bar{U}$).

F.2.: S is the physically feasible singularity carrier belonging to one element of a cascade consisting of cascade elements periodic with distance t in the direction of an imaginary axis.

F.3.: Apart from their terminals ($\nu = 1, 2$) S and L lie in region T ($\zeta_T \in T$), where the complex velocity jump function $g(\zeta_T)$ is holomorphic and, approaching the terminals ζ_ν, ($\zeta_T \to \zeta_\nu$),

$$|g(\zeta_T)| < \frac{M}{|\zeta_T - \zeta_\nu|^\varepsilon}$$

where $\varepsilon < 1$ and $M > 0$,

2. Let T_z ($z \in T_z$), $z = x + iy$, be a region for which the following can be stated:

F.4.: It is simply connected and there exists a function $\zeta = z + i\,\tau(z)$, ($\zeta = \xi + i\,\eta$), single valued, for which **F.6.** is satisfied.

F.5.: It covers the interval J ($0 < x < x_2$) on the real axis: $J \subset T_z$.

F.6.: $\tau(z), q_\xi(z)$ and $\gamma_\xi(z)$ are holomorphic and $\mathrm{d}\tau/\mathrm{d}z \neq i$ ($\mathrm{d}\zeta/\mathrm{d}z \neq 0$), in T_z. The values of $\tau(z), q_\xi(z)$, and $\gamma_\xi(z)$ are real if z is a point of the real axis ($z = x$).

Having fixed the definitions **D.1.** $\sim$ **D.3.**, and the conditions **F.1.** $\sim$ **F.6.**, we can formulate the existence theorems that prove the existence of the singularity carrier auxiliary curve, and make its use possible.

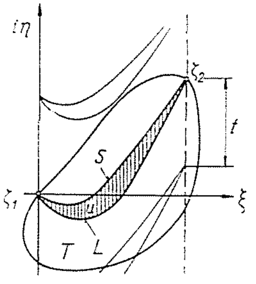

Fig. 2

Existence theorems

The use of the singularity carrier auxiliary curve is possible by the following existence theorems:

T.1.: If the conditions **F.1.** $\sim$ **F.3.** are valid, then for every integer μ, with the restriction $\zeta \neq \zeta_u + i\,\mu\,t$ the induced velocity $\bar{c}(\zeta)$ can be calculated with the help of curve L ($\zeta_L \in L$) and the pertaining distributions $q(\zeta_L)$, $\gamma(\zeta_L)$, instead of curve S ($\zeta_s \in S$) and the pertaining distributions $q_s(\zeta_L)$, $\gamma_s(\zeta_s)$. In other words, with a given S and $q_s(\zeta_s)$, $\gamma_s(\zeta_s)$ to a given L there always exists a $q(\zeta_L)$ and $\gamma(\zeta_L)$ so that

$$\bar{c}_L(\zeta) = \frac{1}{2\pi}\int_{\zeta_1}^{\zeta_2}[q_s(\zeta_s)+i\gamma_s(\zeta_s)]\sum_{\mu=-\infty}^{+\infty}\frac{1}{\zeta-\zeta_S-i\mu t}\,|\mathrm{d}\zeta_s| =$$

$$= \frac{1}{2\pi}\int_{\zeta_1}^{\zeta_2}[q(\zeta_L)+i\gamma(\zeta_L)]\sum_{\mu=-\infty}^{+\infty}\frac{1}{\zeta-\zeta_L-i\mu t}\,|\mathrm{d}\zeta_L|\,.$$

(The extension of Feindt's theorem [5] to a straight airfoil cascade.)

Proof: Let us denote the class of holomorphic functions in T by $F_T(\zeta_T)$ and the holomporhic ones in the closed U by $F_u(\zeta_u)$. If, apart from terminals ζ_1 and ζ_2, $U \subset T$, then ($f_T \in F_T$, $f_u \in F_u$)

$$\oint_{L+S} f_T(\zeta_u) f_u(\zeta_u)\, d\zeta = 0$$

on condition that $f_T(\zeta_u)$, proceeding to points ζ_ν ($\nu = 1, 2$), satisfies the conditions $|f_T(\zeta_u)| < M/|\zeta_u - \zeta_\nu|^\varepsilon$, $\varepsilon < 1$ and $M > 0$, resp. According to conditions **F.1.** $\sim$ **F.3.**, and for every integer μ, with the restriction $\zeta \neq \zeta_u + i\mu t$, the $g(\zeta_T) \in F_T(\zeta_T)$ (for any fixed ζ) and

$$\sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta - \zeta_u - i\mu t} \in F_u(\zeta_u)$$

is also valid, so for a fixed ζ also:

$$\bar{c}_L(\zeta) = \frac{1}{2\pi i} \int_{\xi_1}^{\xi_2} g(\zeta_s) \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta_s + i\mu t - \zeta}\, d\zeta_s = \frac{1}{2\pi i} \int_{\xi_1}^{\zeta_2} g(\zeta_L) \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta_L + i\mu t - \zeta}\, d\zeta_L$$

which is identical to the conclusion of **T.1.**, since on the basis of Equation (1)

$$ig(\zeta_s)\, d\zeta_s = [q_s(\zeta_s) + i\gamma_s(\zeta_s)]\, |d\zeta_s|$$

and

$$ig(\zeta_L)\, d\zeta_L = [q(\zeta_L) + i\gamma(\zeta_L)]\, |d\zeta_L|\,.$$

Theorem **T.1.** clears the circumstances of the existence of the singularity carrier auxiliary curve. It is the basis and starting point for all further conclusions. This theorem makes it possible in designing profiles, also for airfoil cascades, to use an L singularity carrier auxiliary curve, which is taken on in advance and has simple shape from the point of view of computation technique, in place of the physically feasible S singularity carrier; thus the quantity of the necessary computation work is considerably reduced.

T.2.: Let $\mathbf{c} = \nabla\Phi$ and $\Delta\Phi = 0$ be valid in the cascade flow, and assume the conditions **F.1.** $\sim$ **F.3.** If these are satisfied, no matter whether curve S or L is chosen as singularity carrier, the conjugated velocity distribution $\bar{c}(\zeta)$ can be analytically continued, through the other, to the actual singularity carrier; and in both cases $\bar{c}(\zeta)$ is holomorphic in U.

Proof: The flow velocity distributions in question, when singularity carriers S or L are used, can be calculated from the relationships respectively

$$\bar{c}(\zeta) = \frac{1}{2\pi i} \int_{\zeta_1}^{\zeta_2} g(\zeta_S) \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta_S + i\mu t - \zeta} \, \mathrm{d}\zeta_S + \bar{c}_\infty$$

i.e.

$$\bar{c}(\zeta) = \frac{1}{2\pi i} \int_{\zeta_1}^{\zeta_2} g(\zeta_L) \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta_L + i\mu t - \zeta} \, \mathrm{d}\zeta_L + \bar{c}_\infty .$$

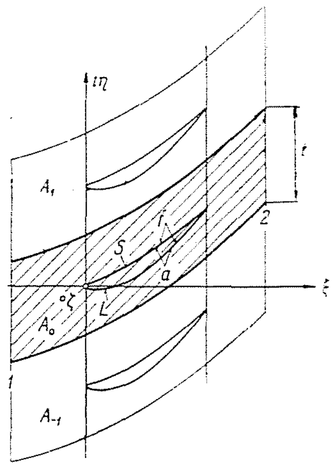

Fig. 3

At the right-hand side of these equations, there are Cauchy-integrals and $\bar{c}_\infty$ is constant. Therefore any of the two relationships produce holomorphic $\bar{c}(\zeta)$ distributions in U, except at its boundary. If the conditions of Theorem **T.1.** hold good, then, at any $\zeta \neq \zeta_u + i\,\mu\,t$ the two expressions give identical $\bar{c}(\zeta)$ values. From this follows, because of the unicity theorem that the two formulae give identical $\bar{c}(\zeta)$ distribution, meaning that, if singularity carriers S or L are alternately used, two different analytical continuations of the same function will apear in U.

Theorem **T.2.** is significant because, proving the existence of the analytical continuations of the F(ζ) distributions, it makes possible to produce the distribution by convergent geometrical series that can be integrated for each member.

T.3.: Let $\mathbf{c} = \nabla \Phi$ and $\Delta\Phi = -c_\xi \dfrac{\mathrm{d}}{\mathrm{d}\xi} \ln b$ in the cascade flow [1] be bounded and integrable, and assume the conditions **F.1.** $\sim$ **F.3.** to be valid. In this case the division $\bar{c}(\zeta) = \bar{c}_B(\zeta) + \bar{c}_H(\zeta)$ where (see Fig. 3)

$$\bar{c}_B(\zeta) = \bar{c}_{\infty B} + \frac{1}{2\pi}\left[\int_{A_0} \Delta\Phi(\zeta') \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta-\zeta'-i\mu t}\,\mathrm{d}A(\zeta') + \right.$$

$$\left. + \int_{\zeta_1}^{\zeta_2} [q_B(\zeta_s) + i\gamma_B(\zeta_s)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta-\zeta_s-i\mu t}\,|\mathrm{d}\zeta_s|\right]$$

and

$$\bar{c}_H(\zeta) = \bar{c}_{\infty H} + \frac{1}{2\pi}\int_{\zeta_1}^{\zeta_2} [q_H(\zeta_s)+i\gamma_H(\gamma_s)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta-\zeta_s-i\mu t}\,|\mathrm{d}\zeta_s|$$

are always possible in such a way that, using the signs of Fig. 3,

$$\bar{c}_B(\zeta) = \frac{1}{2\pi}\int_{A_0} \Delta\Phi(\zeta') \sum_{\eta=-\infty}^{+\infty} \frac{1}{\zeta-\zeta'-i\mu t}\,\mathrm{d}A(\zeta')$$

and

$$\bar{c}_H(\zeta) = \frac{1}{2\pi}\int_{\zeta_1}^{\zeta_2} [q_s(\zeta_s)+i\gamma_s(\zeta_s)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta-\zeta_s-i\mu t}\,|\mathrm{d}\zeta_s| + \bar{c}_\infty =$$

$$= \frac{1}{2\pi}\int_{\zeta_1}^{\zeta_2} [q(\zeta_L)+i\gamma(\zeta_L)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta-\zeta_L-i\mu t}\,|\mathrm{d}\zeta_L| + \bar{c}_\infty ,$$

where, if the two sides of the singularity carrier are denoted by subscripts „f" and „a", and subscript n refers to the normal, while t to the tangential direction,

$$q_s(\zeta_s) = c_{nf}(\zeta_s) - c_{na}(\zeta_s)\,;\ \gamma_s(\zeta_s) = c_{tf}(\zeta_s) - c_{ta}(\zeta_s)$$

and

$$q(\zeta_L) = c_{nf}(\zeta_L) - c_{na}(\zeta_L)\,;\ \gamma(\zeta_L) = c_{tf}(\zeta_L) - c_{ta}(\zeta_L)$$

finally with the conditions

$$\partial\bar{c}/\partial\eta|_{\xi=-\infty} = \partial\bar{c}/\partial\eta|_{\xi=+\infty} = c_{B\eta}|_{\xi=-\infty} = c_{B\eta}|_{\xi=+\infty} = 0$$

satisfies, and with the notations

$$\bar{c}|_{\xi=-\infty} = \bar{c}_1 ,\ \text{ and }\ \bar{c}|_{\xi=+\infty} = \bar{c}_2 ,\ \ c_\infty = (\bar{c}_1+\bar{c}_2)/2 .$$

For the distribution $\bar{c}_H(\zeta)$d the conclusions of theorems **T.1.** and **T.2.** are valid.

Proof: If, on curve S, the conjugated velocity jump of the distribution $\bar{c}_B(\zeta)$ is denoted by $g_B(\zeta)$, and that of $\bar{c}_H(\zeta)$ by $g_H(\zeta)$, then, to show the validity of **T.3.** it will be sufficient to prove that $g_B(\zeta) = 0$; further, that $\bar{c}_\infty = \bar{c}_{\infty H}$ ($\bar{c}_{B\infty} = 0$). As in the periodically repeating region A_0, which contains curve S, $\Delta\Phi$ is bounded and integrable, thus, the Newtonian potential being continuous, the distribution $\bar{c}_B(\zeta)$ will be continuous throughout the whole A_0, so it will have no jump on S: $g_B(\zeta) = 0$. With the condition $c_{1B\eta} = c_{2B\eta} = 0$,

$$\bar{c}_\infty = [c_{1B\xi}+c_{2B\xi}+c_{1H\xi}+c_{2H\xi}-\mathrm{i}(c_{1\eta}+c_{2\eta})]/2\,.$$

It can be seen that if, in a cascade flow of varying layer thickness and with spacing t, the volumetric flow Q_t passes between two profiles, then, with layer thickness marked b, and introducing $b|_{\xi=-\infty} = b_1$ and $b|_{\xi=+\infty} = b_2$, in plane flow

$$c_{1B\xi} = \frac{Q_t}{2t}\left[\frac{1}{b_1} - \frac{1}{b_2}\right] + \frac{1}{2t}\int_{\zeta_1}^{\zeta_2} q_s(\zeta_s)\,|\mathrm{d}\zeta_s|$$

and

$$c_{2B\xi} = \frac{-Q_t}{2t}\left[\frac{1}{b_1} + \frac{1}{b_2}\right] - \frac{1}{2t}\int_{\zeta_1}^{\zeta_2} q_s(\zeta_s)\,|\mathrm{d}\zeta_s| = -c_{1B\xi}$$

that is, after substitution

$$\bar{c}_\infty = [c_{1H\xi}+c_{2H\xi}-i(c_{1H\eta}+c_{2H\eta})]/2\,.$$

Theorem **T.3.** extends the use of the singularity carrier auxiliary curve over the case of the source-type flows occuring in airfoil cascades. The calculation of the $\bar{c}_L(\zeta)$ blade induction is the same for both the source-free ($\Delta\Phi = 0$) and the source-type ($\Delta\Phi = -c_\xi \dfrac{\mathrm{d}}{\mathrm{d}\xi} \ln b$) flow.

T.4.: If conditions **F.4.** $\sim$ **F.6.** are satisfied, then the complex function

$$g(z) = [\gamma_\xi(z) - iq_\xi(z)]\bigg/\left[1+i\,\frac{\mathrm{d}\tau}{\mathrm{d}z}(z)\right]$$

will be holomorphic in the T_ζ ($\zeta \in T_\zeta$) which is the mapping of T_z, by the function $\zeta = z + i\,\tau(z)$, $(z = x + iy)$, see Fig. 4, and it can be considered as such complex velocity jump function of J $(0 < x < x_2)$ belonging to its mapping $L(\zeta_L)$ to which on L the velocity component differences

$$q(\zeta_L) = q_\xi(_L)/\sqrt{1+\eta_L'(\xi_L)^2} \quad \text{and} \quad \gamma(\zeta_L) = \gamma_\xi(\xi_L)/\sqrt{1+\eta_L'(\xi_L)^2}$$

as source-, and vortex distributions, respectively, belong. If conditions **F.1.** $\sim$ $\sim$ **F.3.** are satisfied, then L can be used as singularity carrier auxiliary curve, for which the source-and vortex distributions have been produced with the usual [2] auxiliary functions $q_\xi(\xi_L)$ and $\gamma_\xi(\xi_L)$.

Proof: The conditions **F.4.** $\sim$ **F.6.** give assurance that the $z = f_i(\zeta)$ inverse of the mapping function $\zeta = f(z)$ is holomorphic in the T_ζ which is the mapping of T_z and thus

$$g(z(\zeta)) = \frac{\gamma_\xi(z(\zeta)) - iq\,(z(\zeta))}{1+i\,\dfrac{d\tau}{dz}\,(z(\zeta))}$$

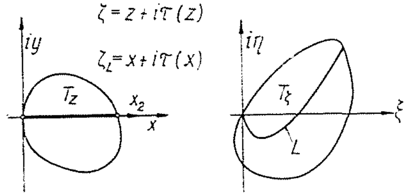

Fig. 4

as the function of ζ is also holomorphic in T_ζ. If $\zeta_L \in L$ then, on the basis of $\zeta_L = x + i\,\tau(x)$

$$g(\zeta_L) = g(x) = \frac{\gamma_\xi(x) - iq_\xi(x)}{1+i\,\dfrac{d\tau}{dx}\,(x)} = [\gamma(x) - iq(x)]\,\frac{\sqrt{1+\left|\dfrac{d\tau}{dx}\right|^2}}{1+i\,\dfrac{d\tau}{dx}} =$$

$$= [\gamma(x) - iq(x)]\,\frac{|d\zeta_L|}{d\zeta_L}\,.$$

From this, according to the known relationship

$$ig(\zeta_L)\,d\zeta_L = [q(x) + i\gamma(x)]\,|d\zeta_L|\,,$$

(see Equation (1)), the difference in the velocity components on L is really $q(\zeta_L)$ and $\gamma(\zeta_L)$.

T.5.: If we are contented in conditions **F.4.** $\sim$ **F.6.** that T_z is the inside of a circle with diameter J, then, for the practice, condition **F.6.** for **T.4.** can be modified to:

F.6.1. $\tau(z)$ will be holomorphic in T_z and $q_\xi(z)$, $\gamma_\xi(z)$ in J $(0 < x < x_2)$ will be continuous; further $d\tau/dz \neq i$.

The $\tau(z)$, $q_\xi(z)$ and $\gamma_\xi(z)$ are real-valued functions on the real axis $(z = x)$.

Proof: On the basis of Theorem I. of WEIERSTRASS, to functions $q_\xi(x)$ and $\gamma_\xi(x)$, which are continuous on the section J $(0 < x < x_2)$, for any arbitrarily small $\varepsilon > 0$ one can find such $q'_\xi(x)$ and $\gamma'_\xi(x)$, analytical in J, that $| q'_\xi - q_\xi | < \varepsilon$ and $| \gamma'_\xi - \gamma_\xi | < \varepsilon$. With these functions, substituting z for x, and with the extensions $q'_\xi(z)$ and $\gamma'_\xi(z)$, the function

$$g(z) = [\gamma'_\xi(z) - iq'_\xi(z)] \Big/ \left[1 + i \frac{\mathrm{d}\tau}{\mathrm{d}z}(z)\right]$$

will be holomorphic in the T_ζ mapping of the inside of the circle drawn on J as on a diameter; thus, the conditions of Theorem **T.4.** are satisfied. However, if the continuous distributions are accounted with, then by force of the relation for the absolute value of the velocity

$$\left| \int_{\zeta_1}^{\zeta_2} [q(\zeta_L) + i\gamma(\zeta_L)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta - \zeta_L - i\mu t} |\mathrm{d}\zeta_L| - \right.$$

$$\left. - \int_{\zeta_1}^{\zeta_2} [q'(\zeta_L) + i\gamma'(\zeta_L)] \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta - \zeta_L - i\mu t} |\mathrm{d}\zeta_L| \right| <$$

$$< \varepsilon \left| \int_{\zeta_1}^{\zeta_2} (1+i) \sum_{\mu=-\infty}^{+\infty} \frac{1}{\zeta - \zeta_L - i\mu t} |\mathrm{d}\zeta_L| \right|$$

the deviation could be maintained as to be arbitrarily small.

Theorems **T.4.** and **T.5.** prove the conditions of **T.1.** thereby the practicability of the singularity carrier auxiliary curve. The case set forth in **T.5.** — which appears in the overwhelming majority of the cases — is of particular importance: the source and vortex distributions employed heretofore can be used on the singularity carrier auxiliary curve in the future too.

By these theorems, the singularity carrier auxiliary curves could be used for the computation of airfoil cascades. The difficulty of holding the singularity carrier curves inside the airfoil sections, inconvenient especially in case of thin profiles when calculated by using physically feasible ones, is avoided by this method.

Further on, let the singularity carrier auxiliary curve L be a section on the assumption $0 \leq x \leq x_2$ of the curve $\zeta_L = x + i\,\tau(x)$, $(\zeta_L = \xi_L + i\,\eta_L)$. Further, let us be contented with the conditions given by **T.5.**; that is, T_ζ $(\zeta \in T_\zeta)$ is only the mapping made by $\zeta = z + i\,\tau(z)$ of the inside of the circle T_z $(z \in T_z)$, $(z = x + iy)$, drawn on the interval J $(0 \leq x \leq x_2)$ as on a diameter. The source- and vortex-distribution to be applied on L will be made in the well-known way [2]:

$$\gamma = \gamma_\xi(\xi_L)\,|\mathrm{d}\xi_L|/|\mathrm{d}\zeta_L| = \gamma_\xi(\xi_L)/\sqrt{1+\eta_L'(\xi_L)^2}$$

i.e.

$$q = q_\xi(\xi_L)\,|\mathrm{d}\xi_L|/|\mathrm{d}\zeta_L| = q_\xi(\xi_L)/\sqrt{1+\eta_L'(\xi_L)^2}$$

and the distributions $\gamma_\xi(\xi_L)$, i.e. $q_\xi(\xi_L)$ in view of $\xi_L = x$ will be continuous in J $(0 < x < x_2)$. The terminals of the physically feasible singularity carrier S and L belonging thereto must coincide according to **D.3.** and **T.1.** In the case $\mathbf{c} = \nabla\Phi$ and $\Delta\Phi = 0$, the coincidence can be realized in several ways:

1) by choosing adequate undisturbed flow (c_∞);
2) by choosing adequate blade circulation (Γ);
3) by choosing adequate vortex distribution (γ_ξ).

The limit value of velocity $\bar{c}_L$, if $\zeta \to \zeta_L$ is known as:

$$\begin{aligned}\bar{c}_L(\zeta_L) &= \pm\frac{1}{2}[\gamma(\zeta_L)-iq(\zeta_L)]\frac{|\mathrm{d}\zeta_L|}{\mathrm{d}\zeta_L}+\\ &+\frac{1}{2\pi}\int\limits_{\zeta_1}^{\zeta_2}[q(\zeta_{L'})+i\gamma(\zeta_{L'})]\sum_{\mu=-\infty}^{+\infty}\frac{1}{\zeta_L-\zeta_{L'}-i\mu t}\,|\mathrm{d}\zeta_{L'}|+\bar{c}_\infty=\\ &=\pm\frac{1}{2}[\gamma(\zeta_L)-iq(\zeta_L)]\frac{|\mathrm{d}\zeta_L|}{\mathrm{d}\zeta_L}+\bar{c}_c+\bar{c}_\infty=\\ &=\pm\frac{1}{2}[\gamma(\zeta_L)-iq(\zeta_L)]\frac{|\mathrm{d}\zeta_L|}{\mathrm{d}\zeta_L}+\bar{c}_k\end{aligned} \qquad (2)$$

where in calculating $\bar{c}_c = c_{c\xi} - ic_{c\eta}$ the Cauchy-main value comes in. The terminals will coincide if (see Fig. 5), with the notation

$$c_{fn} = c_{kn}+q/2,\ \int\limits_{\zeta_1}^{\zeta_2} c_{fn}\,|\mathrm{d}\zeta_L| = \int\limits_{\zeta_1}^{\zeta_2} c_{kn}\,|\mathrm{d}\zeta_L| = 0\,,$$

since the condition of a losed profile contour is

$$\int\limits_{\zeta_1}^{\zeta_2} q\,|\mathrm{d}\zeta_L| = 0\,.$$

(If $\Delta\Phi \neq 0$, then the condition is given by the relationship

$$\int\limits_{\zeta_1}^{\zeta_2} c_{fn}\,|\mathrm{d}\zeta_L| = 0\,.$$

Of the three (but not exclusive) possibilities of terminal coincidence the first on means that, with regard to

$$|d\xi_L| = d\xi_L \sqrt{1+\eta_L'(\xi_L)^2}$$

the condition

$$c_{\infty\xi} = \left\{ \int_0^{\xi_2} [c_{\infty\eta} + c_{c\eta}(\xi_L) - \eta_L'(\xi_L)\, c_{c\xi}(\xi_L)]\, d\xi_L \right\} \Big/ \int_0^{\xi_2} \eta_L'(\xi_L)\, d\xi_L \tag{3}$$

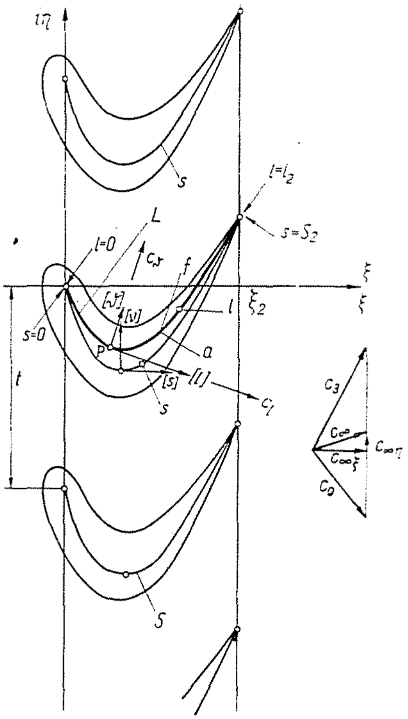

Fig. 5

must be satisfied, while with the second $\Gamma = t(c_{2\eta} - c_{1\eta})$ must be taken into account and with the notation $c_c^* = tc_c/\Gamma$ the condition

$$\Gamma = \left\{ \int_0^{\xi_2} [c_{\infty\eta} - \eta_L'(\xi_L)\, c_{\infty\xi}]\, d\xi_L \right\} \Big/ \frac{1}{t} \int_0^{\xi_2} [\eta_L'(\xi_L)\, c_{c\xi}^*(\xi_L) - c_{c\eta}^*(\zeta_L)]\, d\xi_L \tag{4}$$

must be met. In the third case the vortex distribution will be made in the form $\gamma_\xi + \varkappa\gamma_{\xi 0}$, where

$$\int_0^{\xi_2} \gamma_{\xi 0}\, d\xi = 0 \,,$$

and with the notation $c_c|_{\gamma_\xi=\gamma_{\xi 0}} = c_{c0}$ the corresponding condition is

$$\varkappa = \frac{\int\limits_0^{\xi_2} \left[c_{\infty\eta} + c_{c\eta}(\xi_L) - \eta_L'(\xi_L)\left(c_{\infty\xi} + c_{c\xi}(\xi_L)\right)\right] d\xi_L}{\int\limits_0^{\xi_2} \left[\eta_L'(\xi_L)\, c_{c0\xi}(\xi_L) - c_{c0\eta}(\xi_L)\right] d\xi_L} . \tag{5}$$

Should any of the three possibilities be used, it is desirable to have these conditions not to influence the basic data considerably. A good onset for the point ζ_{L2} is obtained if the formula

$$\eta_{L2} = \int\limits_0^{\xi_2} \frac{c_{\infty\eta} + \frac{1}{t}\int\limits_0^{\xi_L} \gamma_\xi(\xi')\, d\xi'}{c_{\infty\xi} + \frac{1}{t}\int\limits_0^{\xi_L} q_\xi(\xi')\, d\xi'}\, d\xi_L \tag{6}$$

is used.

In the following, the theorems based on the foregoing theorems which can help the use of the singularity carrier auxiliary curve will be dealt with. These theorems give the relationships between the velocity distributions on the L singularity carrier auxiliary curve and those on the physically feasible S singularity carrier with identical terminals.

According to **T.2.** the $\bar{c}(\zeta)$ conjugate velocity distribution; $\mathbf{c} = \nabla\Phi$ and $\Delta\Phi = 0$, whichever of the S and L curves is taken as singularity carrier (see Fig. 6); can be analytically continued through the other curve to the actual singularity carrier. Thus, within the curves denoted I and II in the Figure $\nabla \mathbf{c} = \nabla \times \mathbf{c} = 0$ hold true; it means that applying the Stokes and Gauss theorem for the curves, four integral relationships can be written. (If the flow in the blade cascade is source-type, i.e. $\mathbf{c} = \nabla\Phi$ and $\Delta\Phi = -c_\xi \frac{d}{d\xi} \ln b \neq 0$ then the further considerations are valid only for the distribution $\bar{c}_H(\zeta)$ defined in theorem **T.3.** Denoting the arc length of curve L by l, and that of curve S by s, and using to an arbitrary point of the curves the system of coordinates according to Fig. 6 and with regard to

$$c_{fl} = c_{kl} - \gamma/2\,, \quad c_{al} = c_{kl} + \gamma/2 \quad \text{i.e.} \quad c_{f\vartheta} = c_{k\vartheta} + q/2\,, \quad c_{a\vartheta} = c_{k\vartheta} - q/2$$

the four relationships will be as follows:

$$\int_0^l \left[-c_{k\vartheta}(l') - \frac{q(l')}{2}\right] dl' + \int_0^{s(l)} \left[c_{kn}(s') + \frac{q_s(s')}{2}\right] ds' = \int_{\vartheta_s(l)}^0 c_{fl}(l,\vartheta)\, d\vartheta =$$

$$= -\left[c_{kl}(l) - \frac{\gamma(l)}{2}\right]\vartheta_s(l) - \frac{\partial}{\partial l}\left[c_{k\vartheta}(l) + \frac{q(l)}{2}\right]\frac{\vartheta_s^2(l)}{2!} + \frac{\partial^2}{\partial l^2}\left[c_{kl}(l) - \frac{\gamma(l)}{2}\right]\frac{\vartheta_s^3(l)}{3!} + \dots \tag{7}$$

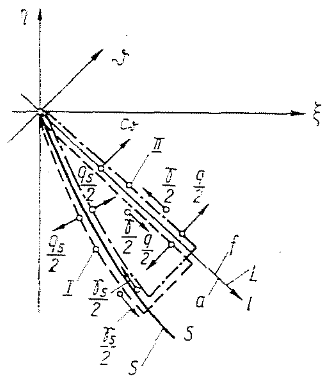

Fig. 6

and

$$\int_0^l \left[-c_{k\vartheta}(l') + \frac{q(l')}{2}\right] dl' + \int_0^{s(l)} \left[c_{kn}(s') - \frac{q_s(s')}{2}\right] ds' = \int_{\vartheta_s(l)}^0 c_{al}(l,\vartheta)\, d\vartheta =$$

$$= -\left[c_{kl}(l) + \frac{\gamma(l)}{2}\right]\vartheta_s(l) - \frac{\partial}{\partial l}\left[c_{k\vartheta}(l) - \frac{q(l)}{2}\right]\frac{\vartheta_s^2(l)}{2!} + \frac{\partial^2}{\partial l^2}\left[c_{kl}(l) + \frac{\gamma(l)}{2}\right]\frac{\vartheta_s^3(l)}{3!} + \dots \tag{8}$$

i.e.

$$\int_0^l \left]- c_{kl}(l') + \frac{\gamma(l')}{2}\right] dl' + \int_0^{s(l)} \left[c_{ks}(s') - \frac{\gamma_s(s')}{2}\right] ds' = -\int_{\vartheta_s(l)}^0 c_{f\vartheta}(l,\vartheta)\, d\vartheta =$$

$$= \left[c_{k\vartheta}(l) + \frac{q(l)}{2}\right]\vartheta_s(l) - \frac{\partial}{\partial l}\left[c_{kl}(l) - \frac{\gamma(l)}{2}\right]\frac{\vartheta_s^2(l)}{2!} - \frac{\partial^2}{\partial l^2}\left[c_{k\vartheta}(l) + \frac{q(l)}{2}\right]\frac{\vartheta_s^3(l)}{3!} + \dots \tag{9}$$

and

$$\int_0^l \left[-c_{kl}(l') - \frac{\gamma(l')}{2}\right] \mathrm{d}l' + \int_0^{s(l)} \left[c_{ks}(s') + \frac{\gamma_s(s')}{2}\right] \mathrm{d}s' = -\int_{\vartheta_s(l)}^0 c_{a\vartheta}(l,\vartheta)\, \mathrm{d}\vartheta =$$

$$= \left[c_{k\vartheta}(l) - \frac{q(l)}{2}\right] \vartheta_s(l) - \frac{\partial}{\partial l}\left[c_{kl}(l) + \frac{\gamma(l)}{2}\right] \frac{\vartheta_s^2(l)}{2!} - \tag{10}$$

$$- \frac{\partial^2}{\partial l^2}\left[c_{k\vartheta}(l) - \frac{q(l)}{2}\right] \frac{\vartheta_s^3(l)}{3!} + \dots$$

For equations (7) $\sim$ (10) the following conditions must be realized:

F.7.: $\mathbf{c} = \nabla \Phi$ and $\Delta\Phi = 0$. Denoting the distance perpendicular to L of the physically feasible singularity carrier S and of the singularity carrier auxiliary curve L by $\vartheta_s(\zeta_L)$ and the boundary points of T_ζ ($\zeta_T \in T_\zeta$) in which the complex velocity jump function is holomorphic, by ζ_{TP}, let $|\vartheta_s(\zeta_L)| < |\zeta_L - \zeta_{TP}|$ be valid.

Of the combinations of Equations (7) $\sim$ (10), relationships of basic significance can be obtained.

T.6. If condition **F.7.** is realized, the following relationship will exist between the source distribution $q_s(s)$ on the physically singularity carrier S and the source- and vortex distribution, $q(l)$ and $\gamma(l)$, respectively, on the singularity carrier auxiliary curve L, denoting the distance of the two curves measured perpendicularly to L by ϑ_s:

$$\int_0^{s(l)} q_s(s')\, \mathrm{d}s' = \int_0^l q(l')\, \mathrm{d}l' + \gamma(l)\, \vartheta_s(l) - \frac{\mathrm{d}q}{\mathrm{d}l}(l) \frac{\vartheta_s^2(l)}{2!} - \frac{\mathrm{d}^2\gamma}{\mathrm{d}l^2}(l) \frac{\vartheta_s^3(l)}{3!} + \dots \tag{11}$$

Proof: Subtracting equation (7) from equation (8) the theorem is proved at once. The convergence of the series on the right-hand side is realized by condition **F.7.** and existence theorem **T.2.**

Thus, according to Theorem **T.6.**, if any of the $q_s(s)$ or $q(l)$ is fixed, the other will be given from Equation (11). So if we want to produce the profile thickness with the usual $q_s(s)$ distributions, a $q(l)$ must be chosen which satisfies Eq. (11). The deviation between $q_s(s)$ and $q(l)$ will be all the greater according to the growth of $\vartheta_s(l)$. Accordingly, it will be practicable to use a singularity carrier auxiliary curve which is near to a physically feasible singularity carrier with the same terminals.

T.7.: If condition **F.7.** is realized, the following relationship will exist between vortex distribution $\gamma_s(s)$ on the physically feasible singularity carrier S and the source- and vortex-distribution $q(l)$ and $\gamma(l)$, respectively, on the singularity carrier auxiliary curve L:

$$\int_0^{s(l)} \gamma_s(s')\,\mathrm{d}s' = \int_0^{l} \gamma(l')\,\mathrm{d}l' - q(l)\,\vartheta_s(l) - \frac{\mathrm{d}\gamma}{\mathrm{d}l}(l)\,\frac{\vartheta_s^2(l)}{2!} + \frac{\mathrm{d}^2 q}{\mathrm{d}l^2}\,\frac{\vartheta_s^3(l)}{3!} + \dots \tag{12}$$

Proof: Subtracting equation (10) from equation (9), the validity of the theorem is seen immediately. The convergence of the series on the right hand side is realized again by condition **F.7.** and the existence theorem **T.2.**

Theorem **T.7.** gives the relationship between the velocity distributions $\gamma_s(s)$ and $\gamma(l)$. Experience shows that it is less significant than relationship (11) of the source-distributions. Its importance is enhanced if a fixed $\gamma_s(s)$ distribution is attempted.

T.8.: If condition **F.7.** is realized, the following relationships exist between the values of the components of the mean velocity $\bar{c}_k$ defined in equation (2), taken on the physically feasible singularity carrier auxiliary curve L,

$$\int_0^{s(l)} c_{kn}(s')\,\mathrm{d}s' = \int_0^{l} c_{k\vartheta}(l')\,\mathrm{d}l' - c_{kl}(l)\,\vartheta(l) - \frac{\mathrm{d}c_{k\vartheta}(l)}{\mathrm{d}l}\,\frac{\vartheta_s^2(l)}{2!} + \frac{\mathrm{d}^2 c_{kl}(l)}{\mathrm{d}l^2}\,\frac{\vartheta_s^3(l)}{3!} + \dots \tag{13}$$

$$\int_0^{s(l)} c_{ks}(s')\,\mathrm{d}s' = \int_0^{l} c_{kl}(l')\,\mathrm{d}l' + c_{k\vartheta}(l)\,\vartheta_s(l) - \frac{\mathrm{d}c_{kl}(l)}{\mathrm{d}l}\,\frac{\vartheta_s^2(l)}{2!} - \frac{\mathrm{d}^2 c_{k\vartheta}(l)}{\mathrm{d}l^2}\,\frac{\vartheta_s^3(l)}{3!} + \dots \tag{14}$$

Proof: The proposition is proved by summing equations (7) and (8), i.e. (9) and (10), respectively. The convergence of the right-hand side series is realized by condition **F.7.** and existence theorem **T.2.** On the right-hand side of the relationships in theorems **T.6.** $\sim$ **T.8.** it is generally sufficient to go up to second or third degree approximation.

The significance of the first relationship of **T.8.** is emphasized by the fact that the value of the integral

$$\int_0^{s(l)} c_{kn}(s')\,\mathrm{d}s'$$

on the left-hand side can usually be prescribed, and thus a relationship for use in the distribution $\vartheta_s(l)$ is obtained. In a stationary cascade, when the members of higher than third degree are neglected, for $\vartheta_s(l)$ the approximating relationship

$$\frac{1}{6}\,\frac{\mathrm{d}^2 c_{kl}(l)}{\mathrm{d}l^2}\,\vartheta_s^3 - \frac{1}{2}\,\frac{\mathrm{d}c_{kl}(l)}{\mathrm{d}l}\,\vartheta_s^2 - c_{kl}(l)\,\vartheta_s + \int_0^{l} c_{k\vartheta}(l')\,\mathrm{d}l' = 0 \tag{15}$$

can be obtained, if S is taken to be a singularity carrier, on which $c_{kn} = 0$. Knowing L, this makes possible the determination of the points of S.

In the case of a rotating airfoil cascade, the straight blade cascade sketched in Fig. 5 should be considered as the mapping of the rotating blade cascade obtained by the usual conform mapping method. Here the conjugate of the mapping of the peripheral velocities shall be $\bar{c}_\zeta = -iu_\eta$. For either of curves I or II in Fig. 6 since $\nabla u_\zeta = 0$, the relationship will be

$$-\int_0^l u_{\eta\vartheta}(l')\,dl' + \int_0^{s(l)} u_{\eta n}(s')\,ds' \cong -u_{\eta l}(l)\,\vartheta_s(l) - \left(\frac{\partial u_{\eta l}(l)}{\partial l} - \Omega\right)\frac{\vartheta_s^2(l)}{2} + \left(\frac{\partial^2 u_{\eta l}(l)}{\partial l^2} - \frac{\partial \Omega}{\partial \vartheta}\right)\frac{\vartheta_s^3(l)}{6} \tag{16}$$

where $\Omega = \partial u_{\eta\vartheta}/\partial l - \partial u_{\eta l}/\partial\vartheta$. Now on the singularity carrier S let us have. $w_{kn} = c_{kn} - u_{\eta n} = 0$. Substracting equation (15) from the first relationship of **T.8.** (introducing the notation $w_{\varepsilon,\lambda} = c_{\varepsilon\lambda} - u_{\eta\lambda}$ for every ε, λ) the approximating equation, for the mapping of the relative flow:

$$\frac{1}{6}\left[\frac{d^2 w_{kl}(l)}{dl^2} - \frac{\partial\Omega}{\partial\vartheta}\right]\vartheta_s^3 - \frac{1}{2}\left[\frac{dw_{kl}(l)}{dl} + \Omega(l)\right]\vartheta_s^2(l) - w_{kl}(l)\,\vartheta_s(l) + \int_0^l w_{k\vartheta}(l')\,dl' = 0 \tag{17}$$

will be given at any fixed l and it is easy to solve for ϑ_s.

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Summary

The paper presents the generalization of the singularity method used for design in airfoil cascades. It proves that it is not necessary for the singularity carrier to lie in full length in the inside of the airfoil. There are outlined the existence theorems securing the existence and use of the generalized singularity carrier curve section, the so called singularity carrier auxiliary curve, and their demonstration as well.

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