ON THE FUNCTIONALITY BETWEEN REYNOLDS NUMBER AND FILM THICKNESS

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1. Introduction

In technical practice, often liquid films are applied to meet various heat transmission problems, owing to the advantageous thermal and fluid mechanical properties. Fields of application are, without aiming at completeness, wet cooling towers, industrial and heat power jet condensers, film evaporation, various chemical industrial installations, etc.

Despite this widespread application, the fluid and mechanical and thermal properties of liquid films are rather seldom known at a sufficient accuracy. Namely, the motion of the viscous fluid is described by the Navier—Stokes differential equation that cannot be solved in its general form. The solution is known for a few simplified basic cases then, however, the model described by the differential equation contains more neglect. If the model approximates the real case, significant mathematical difficulties have to be faced.

Further difficulties appear if the flow is turbulent. Though with the introduction of the conception of local mean velocity the Navier—Stokes differential equation remains valid in form, the so-called turbulent viscosity, different from the common material characteristic "viscosity" may arise. The former can only be exactly determined from the dimensions of velocity distribution as a safe theory, yielding correct quantitative description of the variation of turbulent viscosity.

The velocity distribution of circular pipe flows and flows between two parallel walls is well known from the tests made by NIKURADZE [1], REICHARDT [2], thus in these cases safe results for turbulent viscosity have been obtained. Unfortunately, at the calculation of velocity distribution in liquid films of very small thickness (at most a few mm) serious technical difficulties appear.

The task is further complicated by the liquid film with a free surface, permitting complicated interactions between the liquid film and the surrounding atmosphere. From fluid mechanical aspects one of the most significant among them is the mechanical interaction between the film surface and the gas or vapour atmosphere resulting in the thickening or thinning of the liquid film.

A separate chapter in studying of liquid films is the examination of waves in the free film surface. Without entering into particulars on the wave effect, let us refer to the relevant measurements by BRAUER [3], stating that the wave structure changes several times as a function of the Reynolds number of the liquid film.

Now we do not know functions for the velocity distribution in the liquid film subject to the outlined interactions either in the laminar or the turbulent region, as against flows in circular pipes or between two parallel walls upon duly selecting dimensionless form.

The thermal and fluid mechanical problems of the liquid film have first been dealt with by NUSSELT in 1916 [9], giving the distribution of velocity in liquid films for laminar flow with considerable neglect. In his later works, however, completing his theory, laminar liquid films are only dealt with. His equations describe the phenomena only qualitatively, and even later, when the mechanical interaction on the free film surface is taken into consideration.

Research accelerates from the beginning of the 1930-s and several articles appear on the various kinds of interaction; the turbulent film, the structure of waves on the free surface, etc. The problem is, however, complicated, so even at present no perfect agreement between theory and practice can be spoken of.

In the following experiments are described for the approximation of velocity distribution in liquid films. In this theory, based on test results of turbulence, no strict distinction is made between laminar and turbulent regions. The modern turbulence tests — in the first place the ultramicroscopic tests of FAGE and TOWNEND, as well as the works of DEISSLER showed that even in the region of the so-called laminar flow, no laminar flow can be spoken of in the classical sense of the word. Turbulence occurs in laminar flow, too, only strongly dampened. Thus the turbulent boundary layer, which, on the basis of PRANDTL's work, can be divided into three zones — laminar, buffer and turbulent ones — can be regarded as equally turbulent, only the degree of turbulence differs.

Accordingly the influence of the so-called turbulent viscosity in all three sections must be taken into consideration. It has been stated above that at present turbulent viscosity can only be safely determined from velocity distribution values. In lack of such values turbulent viscosity is approximated by the flow between two parallel walls. This approximation has proved to be acceptable.

2. The velocity distribution equation

The velocity distribution in liquid films, similarly to the flow in circular pipes or between two parallel walls, can be divided into an unformed and a developed section. The velocity distribution in the unformed section strongly depends on the method of development of the liquid film, thus, on several parameters. In this zone the rate of the liquid film may increase or decrease, the film thickness may vary, thus even in the most simple cases the velocity distribution can only be plotted as a function of two variables, involving great mathematical difficulties and owing to the many interactions the result will contain more neglect than that of the developed velocity distribution. In the following only the developed velocity distribution is dealt with.

The suppositions and neglections of the deduction:

1) the influence of the waves on the film surface is negligible;

2) the velocity distribution has developed;

3) the variation of field of gravity is negligible;

4) the mechanical interaction of the surrounding is replaced by interfacial shear stress;

5) a Newtonian liquid is involved.

Be the surface supporting the liquid film a vertical, smooth, flat plate. The datum line x of the co-ordinate system is in the plane of the plate, vertical, pointing downward. The datum line y is one of the normals to the plane surface. The origin is in a point of the flat plate from where the velocity distribution can be regarded as developed.

The Newtonian equation of the dynamic viscosity:

$$\tau = \mu \frac{dw}{dy} . \tag{1}$$

This is made valid for the turbulent flow:

$$\tau = (\mu + \mu_t) \frac{d\overline{w}}{dy} \tag{2}$$

where μ_t turbulent dynamic viscosity;

$\overline{w}$ local mean velocity.

As the liquid does not accelerate (developed velocity distribution) and the field of gravity can be taken as constant, the distribution of shear strength in the liquid film is linear. Denote the value of shear stress along the wall by τ_0, along the free surface by $\pm\tau_\delta$ (signs indicating two opposite directions of the vapour-to-liquid flow).

Shear stress distribution can be written as:

$$\tau = \frac{\pm\tau_\delta - \tau_0}{\delta} y + \tau_0 \tag{3}$$

where δ is the film thickness.

After substituting $\mu = \varrho\nu$ and $\mu_t = \varrho\varepsilon$, eliminate τ_0, from the left side of Eq. (4) and from the right side ϱ. After arrangement:

$$\left(\pm\frac{\tau_\delta}{\tau_0} - 1\right)\frac{y}{\delta} + 1 = (\nu + \varepsilon)\frac{d\overline{w}}{dy} \cdot \frac{\varrho}{\tau_0} . \tag{5}$$

Put this equation in dimensionless form known from the theory of turbulent flow. Introducing friction velocity $w^* = \sqrt{\frac{\tau_0}{\varrho}}$. The dimensionless velocity is $u = \frac{\overline{w}}{w^*}$, the dimensionless leng this. $\eta = \frac{yw^*}{\nu}$, Thus, Eq. (5) will be:

$$\left(\pm\frac{\tau_\delta}{\tau_0} - 1\right)\frac{\eta}{s} + 1 = \left(1 + \frac{\varepsilon}{\nu}\right)\frac{du}{d\eta} . \tag{6}$$

Introducing $\beta = 1 \pm \frac{\tau_\delta}{\tau_\delta}$:

$$1 - \beta\frac{\eta}{s} = \left(1 + \frac{\varepsilon}{\nu}\right)\frac{du}{d\eta} \tag{7}$$

where s dimensionless film thickness $\left(\frac{\delta w^*}{\nu}\right)$;

ε turbulent kinematic viscosity.

The mechanical interaction between the liquid film and the surrounding gas or vapour atmosphere let be denoted by β eq. For $\beta = 1$ the interaction can be neglected, for $\beta < 1$ the interaction makes the liquid film thinner (d.c.), for $\beta > 1$ the liquid film is made thicker (a.c.).

On the basis of the work by LIN, MOULTON, PUTNAM [6] the ε/ν value with the following three functions can be given to the flow between two parallel walls:

$$\text{for} \quad 0 \leqq \eta \leqq 5 \ \text{then} \ \varepsilon/\nu = \left(\frac{\eta}{14.5}\right)^3 \tag{8}$$

$$5 \leqq \eta \leqq 20 \ \text{then} \ \varepsilon/\nu = \eta/5 - 0.959 \tag{9}$$

$$\eta \geqq 20 \ \text{then} \ \frac{\varepsilon}{\nu} = \varkappa^2 \frac{\left|\frac{du}{d\eta}\right|^3}{\left|\frac{d^2u}{d\eta^2}\right|^2} . \tag{10}$$

The valid relationship in the first two intervals originate from the authors mentioned, whilst the third is the result of the similarity hypothesis by Todor KÁRMÁN.

According to the tests by the authors mentioned the velocity distribution in the flow in the turbulent core between two parallel walls, calculated on the basis of the Kármán similarity hypothesis agrees well with the measured values.

The $\varkappa$ is the constant determined from the measurements. Its value can be assumed as between 0.36 and 0.4. In further dealings it will be taken as 0.4. Eq. (7) can be solved taking up the value of the β/s quotient, — which is the parameter in the problem — as well as of (8), (9) and (10) separately after its substitution in (7).

The boundary condition of the problem: the liquid is a Newtonian liquid (it adheres to the wall) thus on the wall the value of the velocity is zero ($y = 0$). In dimensionless form:

$$\text{for} \quad \eta = 0 \qquad \text{then} \quad u = 0 .$$

The velocity distribution is built up of 3 functions according to (8), (9) and (10). It appears reasonably from this that on the boundaries of the validity regions the velocity distribution is continuous and unbroken.

This requirement at $\eta = 20$ can be fulfilled forthwith as the differential equation to be solved in the third section is of second order. However, it cannot be fulfilled without a residual at $\eta = 5$. As in the first and second sections the differential equations to be plotted are first order the continuity condition can be fulfilled, but the unbrokenness is only in that case where the derivates of (8) and (9) are equal at $\eta = 5$. Unfortunately this condition does not materialise. As the integration of the differential equation for the first and second section can be carried out in closed form, the break at $\eta = 5$ in (8) and (9) does not mean a further mathematical problem.

3. The solution of the differential equation of the velocity distribution

The velocity distribution in the interval is given by the solution of the differential equation

$$1 - \left(\frac{\beta}{s}\right)\eta = \left[1 + \left(\frac{\eta}{14.5}\right)^3\right]\frac{du}{d\eta}\,. \tag{11}$$

The equation is of the divisable type.

After the introduction of the new variable $x = \dfrac{\eta}{14.5}$ dividing and integrating:

$$u = 14{,}5\left[\frac{1 + 14.5\,\dfrac{\beta}{s}}{6}\ln\frac{(1+x)^2}{1-x+x^2} + \frac{1 - 14.5\,\dfrac{\beta}{s}}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}\right] + c. \tag{12}$$

Making use of the boundary condition of the problem — the liquid adheres to the wall — the equation of the velocity distribution in the region $0 \leqq \eta \leqq 5$, after a few alterations and replacements is

$$u = 14{,}5\left[\frac{1 + 14.5\left(\dfrac{\beta}{s}\right)}{6}\ln\frac{\left(1 + \dfrac{\eta}{14.5}\right)^2}{1 - \dfrac{\eta}{14.5} + \left(\dfrac{\eta}{14.5}\right)} + \frac{1 - 14{,}5\left(\dfrac{\beta}{s}\right)}{\sqrt{3}}\operatorname{arctg}\frac{\sqrt{3}\left(\dfrac{\eta}{14.5}\right)}{2 - \dfrac{\eta}{14.5}}\right]. \tag{13}$$

The valid differential equation of velocity distribution in the interval $5 \leqq \eta \leqq 20$ is obtained after the substitution of (9) into (7):

$$1 - \left(\frac{\beta}{s}\right)\eta = \left(\frac{\eta}{5} + 0.041\right)\frac{du}{d\eta} \tag{14}$$

(14) is also divisable. Dividing and integrating:

$$u = 5\left[1 + 0.205\left(\frac{\beta}{s}\right)\right]\ln(\eta + 0.205) - 5\left(\frac{\beta}{s}\right)\eta + c\,. \tag{15}$$

The condition is that at $\eta = 5$ the velocity distribution gives a continuous curve. The substituting value of (13) at $\eta = 5$ is

$$u = 4.9474 - 12.2786 \left(\frac{\beta}{s}\right). \tag{16}$$

The integrating constant can be determined by equalising (16) and the substituting the value of (15) at $\eta = 5$. After arranging in the interval $5 \leqq \eta \leqq 20$, the valid velocity distribution is as follows:

$$u = 5\left[1 + 0.205\left(\frac{\beta}{s}\right)\right]\ln\,(\eta + 0.205) + \left(\frac{\beta}{s}\right)(11.0306 - 5\eta) - 3.3006. \tag{17}$$

In the case of $\eta > 20$ the velocity distribution is obtained from (7) and (11) which will take the following form:

$$1 + \left(\frac{\beta}{s}\right)\eta = \left[1 + \varkappa^2 \frac{\left|\dfrac{du}{d\eta}\right|^3}{\left|\dfrac{d^2u}{d\eta^2}\right|^2}\right]\frac{du}{d\eta}. \tag{18}$$

In the starting point ($\eta = 20$) of the curve described by (18) both the substitution value and the gradient of the function was given. At $\eta = 20$ the function (17) and its derivatives are identical with their substitutional values. The value of the derivative of (17) at $\eta = 20$ is

$$\left.\frac{du}{d\eta}\right|_{\eta=20} = \frac{5 + 1.025\left(\dfrac{\beta}{s}\right)}{20.205} - 5\left(\frac{\beta}{s}\right). \tag{19}$$

The differential equation (18) cannot be solved in closed form, thus the velocity distribution in the region $\eta > 20$ can only be given by a numerical process. The Runge—Kutta process has been chosen for this purpose.

The requirement for the applicability of the Runge—Kutta process (but for almost every numerical process), is that the solving function in the tested interval fulfills the so-called Lipschitz condition. Therefore, although physically such a solution can be expected except for the horizontal point of tangency arising in the case of $\beta \geqq 1$, a few more important function tests have been performed.

The tests showed that the function has a local maximum at $\eta = \dfrac{s}{\beta}$, and this point is at the same time the singular point of the differential equation.

The singular point is the junction of every summation curve. The sumation curves looked at from the $\eta = \frac{s}{\beta}$ line are specular (as the result of absolute value) and cross at the singular point with identical tangents. In the tested interval the function is concave throughout from underneath and its value is greater than zero.

As the singular point cannot be crossed with the above-mentioned process, the following method was applied. The singular point could be approached by the controlling of the step of the Runge—Kutta process, then the part beyond the singular point was produced by reflection (using the symmetrical properties of the function).

The otherwise rather tedious calculations were made in an ICT computer in ICT-ALGOL at the Central Institute of Physical Research. The programming was done by Ferenc Kolonits (Erőterv), who besides doing the programming was also of great assistance in the function testing.

4. The determination of the Reynolds number of liquid film

The value of the Reynolds number of the film was also necessary for the evaluation of the results. The definition of the Reynolds number for liquid films is: $Re = w_m \cdot \delta/\nu$ where w_m is the average velocity of the film and δ the thickness of the film. As w_m is the velocity distribution of the integrate centre according to y, the integrating formula of the average making was substituted in the relationship of the Reynolds number:

$$Re = \int_0^{\delta} \frac{w\,dy}{\nu}\,. \tag{20}$$

Introducing the velocity distribution already known and the new variables generally applied in the turbulent flows.

$$Re = \int_0^{s} u d\eta\,. \tag{21}$$

Thus after the determination of the velocity distribution the value of the Reynolds number can be obtained by a simple integration. This can be easily carried out in a function given in grating points, by applying the Simpson formula.

5. Results of the calculation

On the basis of the calculations carried out on a few β/s values in Fig. 1, the velocity distribution is formulated in the dimensionless steps used in the calculations $[u = u(s)]$. The velocity steps are linear, the longitudinal ones

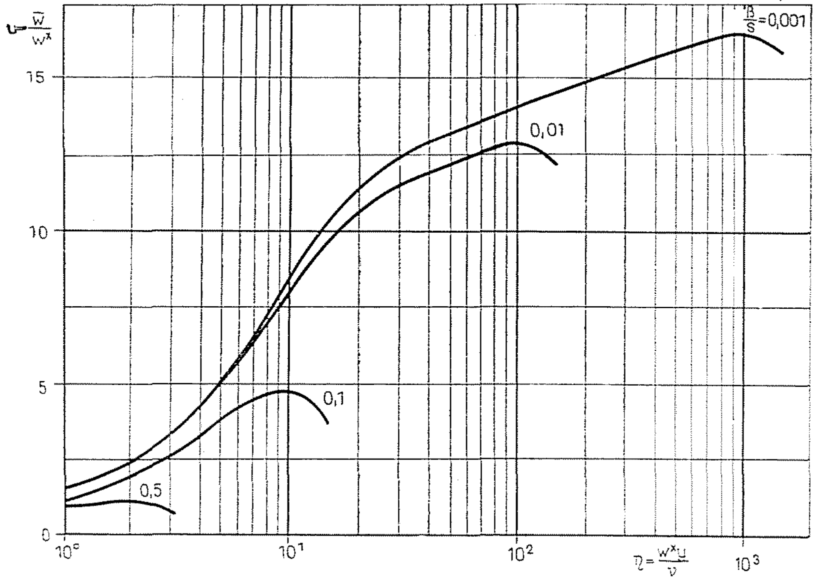

Fig. 1. Dimensionless velocity distribution in liquid film for a few $\frac{\beta}{s}$ parameter values

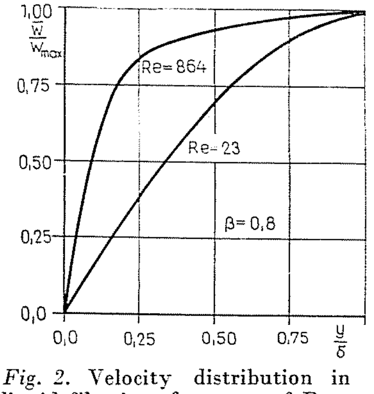

Fig. 2. Velocity distribution in liquid film in a few cases of Reynolds number for $\beta = 0.8$ $\left(\frac{\beta}{s} = 0.01\right)$

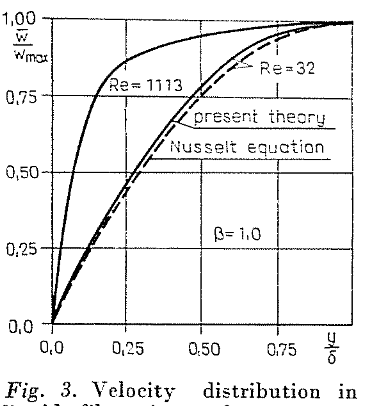

Fig. 3. Velocity distribution in liquid films in a few cases of Reynolds number for $\beta = 1$ $\left(\frac{\beta}{s} = 0.01\right)$

are logarithmic. Fig. 1 can be used to good advantage for determining the suitable velocity distribution. However, owing to the scales applied, the velocity distribution is shown distortedly. Therefore, in Figs 2, 3 and 4, in the linear scales some velocity distribution has been formulated in the more illustrative $\frac{\bar{w}}{\bar{w}_{\max}} = f\left(\frac{y}{\delta}\right)$ dimensionless form on a few *Re* and β values on the basis of Fig. 1. In Fig. 5 the relationship between Reynolds number and

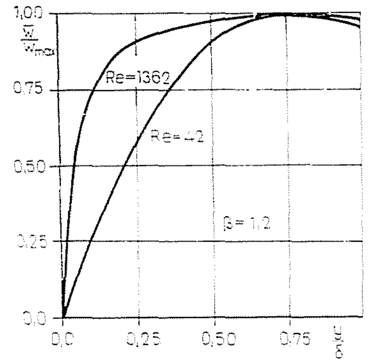

Fig. 4. Velocity distribution in liquid films in a few cases of Reynolds number for $\beta = 1.2$ $\left(\frac{\beta}{s} = 0.01\right)$

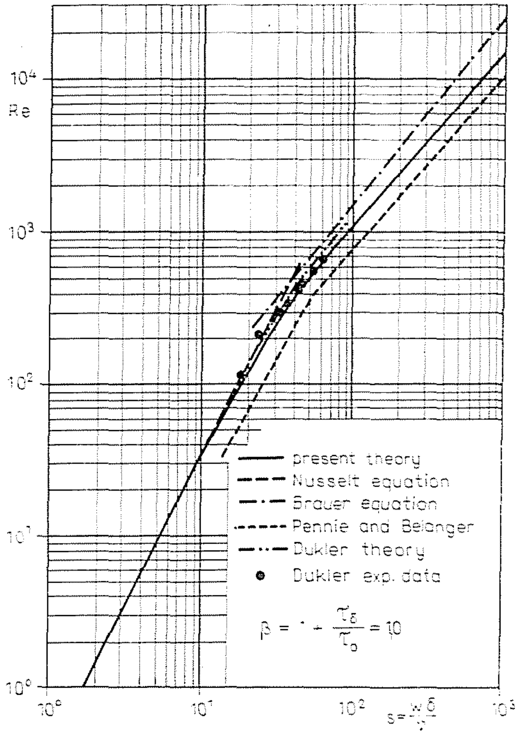

Fig. 5. Relationship between Reynolds number and dimensionless film thickness in case of $\beta = 1$

dimensionless film thickness has been plotted in the case of $\beta = 1$, thus when the mechanical interaction between the film surface and the flowing gas or vapour can be neglected. Both axis are logarithmic! It can well be seen from Fig. 5 that the classical laminar and turbulent films are two clearly distinct regions. However, these do not meet in one break point, but one curve continuously bends over into another.

In the literature numerous semi-empirical and clearly empirical connections can be found on the relationship between film thickness and Reynolds number, which show a large scatter. In the following the results of the calculation are compared with some more reliable results.

On the basis of the Nusselt theory, if $Re < 400$ then the following relationship is valid

$$\delta = \left(\frac{3\nu^2}{g}\right)^{1/3} \cdot Re^{1/3}. \tag{21}$$

Transforming Eq. (21) used in the calculations into dimensionless paces, then plotting Fig. 5 of it is found that at the value $Re < 100$ the agreement is perfect. At $Re = 100$ the two results begin to deviate. The result of the Nusselt theory gives a smaller film thickness than the values calculated by the author. Alongside the growth of the Reynolds number the deviation increases, and at $Re = 400$, which is the critical Reynolds number of the classical theory, the deviation is already about 30%.

In case of $Re > 400$ one of the most reliable relationships originates from Brauer [3]. Its relationship is empirical:

$$\delta = 0{,}302 \left(\frac{3\nu^2}{g}\right)^{1/3} \cdot Re^{8/1}. \tag{22}$$

Also after the suitable transformation in Fig. 5, it can be said that the coincidence between the two relationships in the region $600 < Re < 3000$ is suitable in the first approximation. The average deviation is about 20%, and the two relationships proceed approximately parallel.

The two results begin to markedly deviate at the value $Re > 3000$, but this can be expected, as it is the empirical result of the Brauer theory formula up to $Re = 2000$.

The coincidence of the results is better than the theoretical results of Dukler and Bergelin [8]. The average deviation is about 15%. However, Dukler and Bergelin determined the relationship $Re = f(\delta)$, too. The results gave a greater film thickness in the case of $Re > 400$ than its theoretical results. In the case of $Re < 400$ the dimensions coincided with the theory. In the region $100 < Re < 800$ its dimensions coincide perfectly with the values calculated by the author. The author did not obtain results from measurements made in the $Re > 800$ region.

For $Re < 100$ the test and theoretical results of DUKLER and BERGELIN also coincide with the values calculable from the Nusselt theory.

PENNIE and BELANGER also tested the $Re = f(\delta)$ function in the region $400 < Re < 3000$. Their results gave an essentially larger film thickness than did the calculation results of the author.

Fig. 5 can also be easily applied for practical calculations if the following are taken into consideration. In the counter of the Reynolds number the product $w_m \cdot \delta$ is the same as in aliquid film 1 m wide, the volume of the falling liquid during unit time (a special volume rate), easy to be determined.

The dimensionless film, thickness was plotted on the horizontal axis $s = \frac{w^* \delta}{\nu}$ by definition. In the friction velocity relationship $w^* = \sqrt{\frac{\tau_0}{\varrho}}$ the $\tau_0 = \varrho g \delta$ substitution is permissible on the basis of the balance of forces, in the case of $\beta = 1$. After simplifications

$$s = \frac{\sqrt{g}}{\nu} \delta^{3/2} \tag{23}$$

(23) is a relationship between (δ) and the dimensionless film thickness (s) in meters.

The calculations made show that Fig. 5 can be used not only in the case $\beta = 1$, but in the whole interval $0.7 \leq \beta \leq 1.3$. The practical error is within the reading-off accuracy.

Summary

Owing to the interactions on the free film surface, the velocity distribution in liquid films, cannot be described by one dimensionless function — as can be done for circular pipe flows — but with a set of curves, with the interaction as parameter. Neglecting the wave effect on the film surface, using the turbulent viscosity relationship calculable from the velocity distribution of flow between two parallel walls, the approximate velocity distribution of the liquid film can be determined. The arising relationship between Reynolds number and film thickness coincides well with the results of tests and is suitable for practical calculations. The results of the turbulence tests suitably meet the "classical" laminar and turbulent regions not as break points but as continuous curves. This is also supported by the measurements.

References

1. NIKURADZE, I.: VDI Forschungsheft 356. VDI Verlag (1932).
2. REICHARDT, H.: Z. angew. Mat. Mech. **20**, 297—328 (1940).
3. BRAUER, H.: VDI Forschungsheft 457. VDI Verlag (1956).
4. FEIND, K.: VDI Forschungsheft 481. VDI Verlag (1960).
5. REICHARDT, H.: Allg. Wärmetechnik **2**, 129 (1951).

6. LIN, C. S., MOULTON, R. W., PUTNAM, G. L.: Ind. Eng. Chem. **45,** 636—640 (1953).
7. BÁLINT, E.: Közelítő matematikai módszerek, Műszaki Könyvkiadó Budapest, 1966.
8. DUKLER, A. E.—BERGELIN, O. P.: Chem. Engng. Progress **48,** 557—563 (1952).
9. NUSSELT, W.: Zeitschrift VDI **60,** 541—546 and 569—575 (1916).
10. NUSSELT, W.: Zeitschrift VDI **67,** 206—220 (1923).
11. JAKOB, M.: Heat transfer, John Willey and Sons, New York 1957.
12. ROHSENOW, W. M.: Trans. ASME **78,** 1637—1643 (1956).

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