AN ELEMENTARY INTRODUCTION INTO A CLASS OF DISTRIBUTIONS AND SOME OF THEIR APPLICATIONS

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Some decades ago, physical engineering concepts have developed which could not be described by the classic function concepts, such as concentrated force, concentrated moment, instantaneous electric impulse etc. These phenomena can be given an exact mathematical treatment by means of distributions or of the Mikusiński operators which can be regarded from certain aspects as a generalization of the function concept. Therefore, these concepts are often termed generalized functions. This theory and its applications have a considerable literature, various structures of the theory are known, which should, however, not be explained here. The aim of this paper is to present a correct and secure treatment of the most frequent engineering and physical distributions. This structure should be prone to be readily and rapidly mastered. On this account only a narrow, though important class of the distributions will be discussed here.

As an introduction of the examined set and of the structure to be built up, let us consider the elements of the following type:

1. Sectionally smooth (any times differentiable) and limited complex valued functions with one real variable. Among these the Heaviside Unit Step Function

$$H(x-c)=\begin{cases}0, & \text{for } x<c\\ 1, & \text{for } x\geqq c\end{cases}$$

will have a favoured role.

2. We introduce the delta elements of the form

$$a\delta^{(k)}(x-c),$$

where a is an arbitrary complex, c an arbitrary real number, and $k=0,1,2\ldots$

Let the examined set be the basic set A, and its finite formal linear combinations of elements of types 1 and 2. For the elements of basic set A first algebraic and afterwards infinitesimal operations are defined.

Addition and multiplication by a constant

The definition of addition and multiplication by a constant is analogous with the rule of adding and multiplying by a constant of functions of common sense.

Thus the addition instruction

$$\sum_k a_k \delta^{(k)} + \sum_k b_k \delta^{(k)} = \sum_k (a_k + b_k) \delta^{(k)}$$

is valid also for the delta elements.

E.g. $[2e^x + \delta(x) - 5\delta^{(2)}(x+3)] + [\sin x - 5\delta(x) + 2\delta^{(1)}(x-1) + 3\delta^{(2)}(x+3)] = (2e^x + \sin x) - 4\delta(x) + 2\delta^{(1)}(x-1) - 2\delta^{(2)}(x+3)$.

If $f, g, l, \ldots \in A$, then the addition defined in this way satisfies the following axioms:

1. $f + (g + l) = (f + g) + l$ associativity
2. $f + g = g + f$ commutativity

3. Equation $f + x = g$ has a solution in the case of any (f, g) pair of elements, of which it can be proved that there exists a univocal inverse element. Consequently the addition rules will be valid for subtraction as well, replacing f simply by $-f$.

Further there exists a zero (neutral) element, which is ordered to the pair of elements (f, f).

If a and b are real or complex constants, the following operation rules apply to multiplication by a constant:

4. $a(b f) = (a b) f$
5. $1 f = f$
6. $a(f + g) = a f + a g$
7. $(a + b) f = af + bf$

It follows from the fulfilment of the above axioms that the elements of set A form a linear vector field.

Multiplication

Here the multiplication of the elements type 1 is interpreted as the product of functions of common sense and as the product of elements of basic set A and of elements type 1.

Any element of function class 1 can be produced in the form

$$p(x) = p_c(x) - \sum_{\nu=-1}^{-\infty} \Delta p_\nu H(x_\nu - x) + \sum_{\nu=0}^{\infty} \Delta p_\nu H(x - x_\nu) \tag{1}$$

where $p_c(x)$ is a continuous and sectionally smooth function, $x_\nu < x_{\nu+1}$ and $\Delta p_\nu = p(x_\nu+) - p(x_\nu-)$, and with this there exists a left and right side limit value of the function at every discontinuity. The above series is convergent, since only a finite number of members other than zero belong to every given x value. The product of two functions of the form (1) may also be written in the form (1), although the discontinuity may also disappear, e.g.

$$H(x)[H(x) - 1] = H(x) - H(x) = 0.$$

When multiplying elements of the basic set A and the elements of function class 1, both the products of the elements of function class 1, and the products of these and of elements of type 2 occur. The latter ones are defined as

$$p(x)\,\delta(x - c)\,\Delta\,p(c)\,\delta(x - c)\,,$$

if the function $p(x)$ is continuous at $x = c$. Further,

$$p(x)\,\delta^{(1)}(x - c)\;\Delta\;p(c)\,\delta^{(1)}(x - c) - p^{(1)}(c)\,\delta(x - c)\,,$$

if $p^{(1)}(x)$ is continuous at $x = c$.

In general, if k is a positive whole number,

$$\begin{aligned} p(x)\,\delta^{(k)}(x - c) = p(c)\delta^{(k)}(x - c) - \tbinom{k}{1}\,p^{(1)}(c)\,\delta^{(k-1)}(x - c) + \\ + \ldots + (-1)^k p^{(k)}(c)\,\delta(x - c)\ldots \end{aligned} \tag{2}$$

if $p^{(k)}(x)$ is continuous at $x = c$. At the right side the use of the symbol $(\delta - p)^k$ is also customary.

Multiplication (2) is seen to be reduced by this rule to the multiplication of elements of type 2 by a constant.

E.g. $$\begin{aligned} \sin x \cdot \delta^{(3)}(x - c) = \sin c \cdot \delta^{(3)}(x - c) - 3\cos c \cdot \delta^{(2)}(x - c) - \\ - 3\sin c \cdot \delta^{(1)}(x - c) + \cos c \cdot \delta(x - c)\,. \end{aligned}$$

For the multiplication of functions type 1 and of the elements of basic set A the following definitions are still introduced. Let p and q be functions type 1 and $f \in A$, then

$$pqf\;\Delta\;(pq)f \qquad \text{and} \qquad pf + qf\;\Delta\;(p+q)\,f$$

E.g. 1. $H(x-c)H(x)\delta(x) = H(x-c)\delta(x) = 0$, if $c > 0$

2. $H(x)\delta(x) - H(x)\delta(x) = (H(x) - H(x))\delta(x) = 0$.

Hereafter the operation axioms of the multiplication performed in the above sense can be written. Let function $p(x)$ be of the type 1, and $f, g \in A$, then

$$pf = fp \text{ and } p\,(f+g) = pf + pg\,.$$

Before introducing the differentiation and integration operations, let us consider a physical example.

Examine an electric circuit with a single voltage source and a switch. Voltage source E_0 is assumed to be constant in time and the switch is closed at the moment $t = 0$. Then the voltage of the network will be $E(t) = E_0 H(t)$.

Let us now consider a circuit containing a resistance, an induction coil, and a capacitor in series. In this case the current intensity function $I(t)$ satisfies the following integro-differential equation.

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(t)\,dt = E(t), \tag{3}$$

where L is the inductance, R the ohmic resistance, and C the capacitance. By formally differentiating both sides of the equation we obtain the differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{c}I = \frac{dE}{dt}, \tag{4}$$

where

$$\frac{dE}{dt} = E_0\frac{dH}{dt}\,.$$

$dH/dt = f$ taken formally as a function exists everywhere but at $t=0$, and is equal to 0. Thus it would be $\int\limits_{-\infty}^{t} f(x)dx = 0$, or $\int\limits_{-\infty}^{t} p(x)f(x)dx = 0$. This would physically mean that both the inhomogeneous and the homogeneous differential equations describe the same process, since the two solutions are identical. This is, however, a contradiction. Trying now to reintegrate differential equation (4), other than the original integro-differential equation (3) would result, since the right side will be 0. Thus it looks like as if (3) ought not to be differentiated.

In fact, only absolutely continuous functions are featured by the function itself in place of the indefinite integral of the derivative, that is differentiation and integration are inverse operations only in the class of absolutely continuous functions. E.g. invertibility is not valid for a stepwise function.

Now it stands to reason to try to define two mutually inverse operations on set A and in all sections where a function is absolutely continuous, the new operations should be identical with the classical differentiation and integration operations. With this the above characteristics of absolutely continuous functions would be extended to set A. Accordingly, let us try to generalize the operations of differentiation and integration what will imply the generalization of the function concept.

Differentiation

Differentiation is defined only for the elements of set A. If such an element is a differentiable function of common sense then differentiation corresponds to the classic differentiation. When differentiating a product, if multiplication is performed first, only elements from A are to be differentiated. By definition,

$$H^{(1)}(x-c)\ \Delta\ \delta(x-c),\ H^{(2)}(x-c)\ \Delta\ \delta^{(1)}(x-c)\,,\ \ldots \tag{5}$$

and

$$\delta^{(k)}(x-c)=0, \qquad \text{if} \qquad x \neq c, k=0,1,2,\ \ldots\ .$$

For sake of illustration, $\delta(x-c)$ can be regarded as the common limit case of the derivatives of the smooth function series approximating the function $H(x-c)$. An analogous approach is valid for the $\delta^{(k)}(x-c)$ elements.

The derivative of functions of the form (1) is found to be accordingly

$$p^{(1)}(x)=p_c^{(1)}(x)+\sum_{\nu=-\infty}^{\infty} \Delta p_\nu\, \delta(x-x_\nu)$$

Of course, it is assumed that left and right side derivatives exist at the discontinuities.

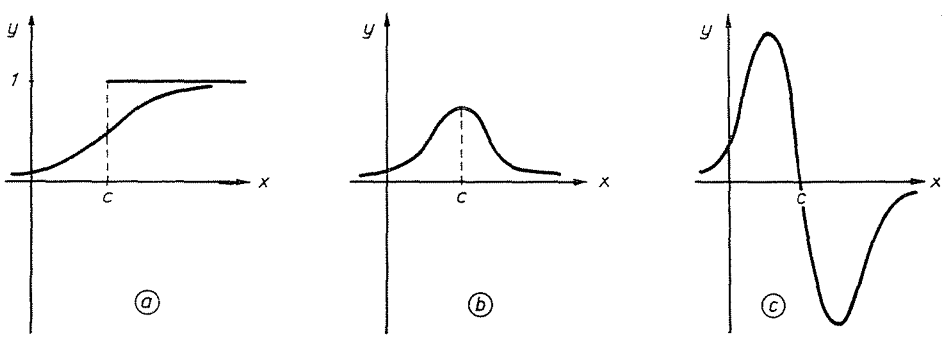

Fig. 1. Approximation of function $H(x-c)$ and of its derivatives by smooth functions

Consequently, the generalized derivation orders to the sectionally continuous and smooth function, at the places of discontinuity, the delta element multiplied by the algebraic measure of discontinuity.

If $f, g \in A$ and a is constant, then $(f+g)^{(1)} = f^{(1)} + g^{(1)}$ and $(af)^{(1)} = = a f^{(1)}$.

The multiplication rule as introduced by definition (2) for the product of a smooth function $p(x)$ and an element $\delta^{(k)}(x-c)$ can be proved, if the rule for the differentiation of a product function is extended by definition to the products of the form $p(x)\,H(x-c)$, $p(x)\delta(x-c), \ldots$

Proof:

$$\int_c^x p^{(1)}(t)\,H(t-c)\,dt = [p(x)-p(c)]\,H(x-c).$$

Upon differentiating both sides:

$$p^{(1)}(x)\,H(x-c) = p^{(1)}(x)\,H(x-c) + [p(x)-p(c)]\,\delta(x-c),$$

that is

$$p(x)\delta(x-c) = p(c)\delta(x-c).$$

Let us hereafter determine element $p(x)\delta^{(1)}(x-c)$. On the one hand

$$[p(x)\,\delta(x-c)]^{(1)} = [p(c)\,\delta(x-c)]^{(1)} = p(c)\,\delta^{(1)}(x-c),$$

on the other

$$[p(x)\,\delta(x-c)]^{(1)}\ \Delta\ p^{(1)}(x)\,\delta(x-c) + p(x)\,\delta^{(1)}(x-c) = \\ = p^{(1)}(c)\,\delta(x-c) + p(x)\,\delta^{(1)}(x-c),$$

and thus

$$p(x)\,\delta^{(1)}(x-c) = p(c)\,\delta^{(1)}(x-c) - p^{(1)}(c)\,\delta(x-c).$$

Continuing this process we obtain the formula defined by (2).

Integration

The interpretation of the integral of function class 1 is identical with the classic (Riemann or Lebesgue) integral concept.

By definition:

$$\begin{aligned} &\int \delta(x-c)\,dx\ \Delta\ H(x-c) + c, \\ &\int \delta^{(k)}(x-c)\,dx\ \Delta\ \delta^{(k-1)}(x-c) + c_1. \end{aligned} \tag{6}$$

A sum can be integrated by terms, and the integral of an element from set A multiplied by a constant is equal to the integral of the element multiplied by the constant.

The definite integral is defined directly by

$$\begin{aligned}&\int_a^b \delta(x-c)\,dx = H(c-a) - H(c-b)\\&\int_a^b \delta^{(k)}(x-c)\,dx = 0\,;\; k = 1, 2, \ldots\end{aligned} \qquad a, b \neq c \qquad (7)$$

If $p(x)$ is a function of class 1, it is continuous at $x = c$, and $a < c < b$, then according to (2) and (7)

$$\int_a^b p(x)\,\delta(x-c)\,dx = p(c)\,,$$

and in general

$$\int_a^b p(x)\,\delta^{(k)}(x-c)\,dx = (-1)^k p^{(k)}(c)\,;\; k = 1, 2, \ldots$$

provided $p^{(k)}(x)$ is continuous at $x = c$.
For $c < a < b$ or $c > b > a$, these last definite integrals will be zero. Namely $\delta^{(k)}(x-c) = 0$ $(k = 0, 1, 2, \ldots)$ in all intervals not containing c.
E.g.

$$\int_{-1}^{1} e^{-x}\left[2H(x) - \delta(x) + i\delta^{(1)}(x)\right] dx = 2\int_{-1}^{1} e^{-x} H(x)\,dx -$$

$$- \int_{-1}^{1} e^{-x}\,\delta(x)\,dx + i\int_{-1}^{1} e^{-x}\,\delta^{(1)}(x)\,dx = 2\left(1 - \frac{1}{e}\right) - 1 + i = 1 - \frac{2}{e} + i\,.$$

The limits of integration may be infinity, e.g.

$$\int_{-\infty}^{\infty} \cosh 2x \cdot \delta^{(1)}(x-1)\,dx = -2\sinh 2\,.$$

Application to ordinary linear differential equations

Consider the ordinary linear differential equation with constant coefficients

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0)\, y = f(x)\,,$$

where $D = d/dx$ is the operator of differentiation.

Be function $f(x)$ sectionally continuous in the examined interval, having both right and left side limit values at all the discontinuities. In this case a solution $y(x)$ can be defined, which can be differentiated continuously $(n-1)$ times in the examined interval and satisfies the differential equation but at the discontinuities.

$y, Dy, \ldots, D^{n-1}y$ may assume any initial value at $x = x_0$. For instance, be $x_0 = 0$ and C_k, C_{k+1} the first and second discontinuities of $f(x)$ to the right from 0. Accordingly, the function is continuous in the interval $(0, C_k\text{-})$, the solution does exist also in the closed interval, the derivatives exist at 0 from the right, and at C_k from the left. The left side limit values of $y, Dy, \ldots, D^{n-1}y$ at C_k will be the initial values in the interval $C_k \leqq x \leqq C_{k+1}$. Accordingly, the solution is being continued in the last named interval and it remains continuous at $(0, C_{k+1})$ up to the $(n-1)$th derivative inclusively. The n-th derivative is discontinuous at C_k. It is evident from the foregoing that the solution can be extended both to the right and to the left.

The solution will be consequently valid also, if x_0 is a discontinuity of function $f(x)$, further if coefficients $a_n, \ldots, a_0$ are continuous or sectionally continuous functions of x. In this latter case the solution is obtained by sections and in all such intervals the coefficients $a_n(x), \ldots, a_0(x)$ should be continuous. It has still to be supposed that between the discontinuities, $a_n(x) \neq 0$, including the right and left side limit values.

E.g.
$$(D^2 + 3D + 2)y = H(x)$$
$$y(0) = 0\,, \quad Dy|_{x=0} = 0$$
$$y = k_1 e^{-x} + k_2 e^{-2x} + \frac{1}{2}(1 + e^{-2x} - 2e^{-x})\, H(x)\,.$$

Let us now consider a linear differential equation with constant coefficients, where $f(x) \in A$.

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0)\, y = f(x)\,. \tag{8}$$

Let m designate the highest order of the derivatives of the delta elements arising in $f(x)$. Thus $f(x)$ can be written also in the form $D^{m+1}F(x)$, where $F(x)$ is already a sectionally continuous function. The pertaining differential equation is

$$D^{m+1}\, u = f(x)\,.$$

If $u = F(x)$ is one of its solutions, then $\delta(x)$ and its derivatives are not involved in $F(x)$, since the order of the derivative is reduced by one by each integration. Accordingly, differential equation (8) can be written in the form

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0)\, y = D^{n+1} F(x)$$

yielding two equations:

$$D^{m+1} v = y\,, \quad (a_n D^n + \ldots + a_0)\, v = F(x)\,.$$

If $v(x)$ is the solution of the second differential equation, then deriving this equation $(m + 1)$ times

$$D^{m+1}[(a_n D^n + \ldots + a_0)\, v] = D^{m+1} F(x)$$

and from this

$$(a_n D^n + \ldots + a_0)\, D^{m+1} v = D^{m+1} F(x)$$

$y = D^{m+1}v$ satisfies the differential equation for $y(x)$, yielding one of the solutions of differential equation (8). The general solution can be found by adding to this solution the general solution of the pertaining homogeneous differential equation.

E.g.

$$(D^2 + 3D + 2)\, y = 10x + 2\,\delta(x-3) - \delta^{(1)}(x+1)\,, \quad \text{or}$$

$$(D^2 + 3D + 2)\; y = D^2\left[\frac{5\,x^3}{3} + (2\,x - 3)\; H(x-3) - H(x+1)\right].$$

Of this two differential equations result:

$$D^2 v = y$$

$$(D^2 + 3D + 2)\, v = \frac{5x^3}{3} + (2x - 3)\, H(x-3) - H(x+1).$$

A particular solution of the latter one, which is continuous together with its first derivative is

$$v(x) = \frac{5}{6}\,x^3 - \frac{15}{4}\,x^2 + \frac{35}{4}\,x - \frac{75}{8} + \left(e^{-x-1} - \frac{1}{2}\,e^{-2x-2} - \frac{1}{2}\right) H(x+1) +$$

$$+ \left(x - \frac{9}{2} + 2\,e^{-x+3} - \frac{1}{2}\,e^{-2x+6}\right) H(x-3)\,.$$

This solution can be obtained by joining the sections. A particular solution of the original differential equation can be obtained by differentiating $v(x)$ twice. Thus

$$y_p = 5\,x - \frac{15}{2} + (e^{-x-1} - 2\,e^{-2x-2})\, H(x+1) + 2\,(e^{-x+3} - e^{-2x+6})\, H(x-3)\,.$$

This function is discontinuous already at $x = -1$.

The general solution is

$$y = y_p + c_1\, e^{-x} + c_2\, e^{-2x}\,.$$

Special initial conditions

If the initial conditions are to be given at $x = x_0$ so that the disturbing function $f(x)$ includes also terms of the form $\delta^{(k)}(x - x_0)$, then for $k < n$ the initial conditions may be given by the right or left side limit values. Let us consider an example.

$$(D^2 - 1)y = \delta(x) + 2\delta^{(1)}(x); \qquad y(0+) = a, \qquad Dy(0+) = b$$

$$y = c_1 e^{-x} + c_2 e^{x} + \left(\frac{1}{2} e^{-x} + \frac{3}{2} e^{x}\right) H(x);$$

from this at $x > 0$: $y = c_1 e^{-x} + c_2 e^{x} + \frac{1}{2} e^{-x} + \frac{3}{2} e^{x}$

$$Dy = - c_1 e^{-x} + c_2 e^{x} - \frac{1}{2} e^{-x} + \frac{3}{2} e^{x}.$$

The equations following from the initial conditions are

$$c_1 + c_2 = - 2 + a \qquad c_1 = \frac{1}{2}(a - b - 1)$$

$$- c_1 + c_2 = - 1 + b \qquad c_2 = \frac{1}{2}(a + b - 3)$$

The step of the function and of its derivative at $x = 0$ is

$$y(0+) - y(0) = 2, \qquad Dy(0+) - Dy(0-) = 1.$$

Fig. 2 illustrates the case $c_1 = c_2 = 0$.

A similar treatment may be employed if the left-side limit value is given. Thus for $x < 0$

$$y = c_1 e^{-x} + c_2 e^{x}; \qquad Dy = - c_1 e^{-x} + c_2 e^{x}$$
$$y(0-) = c_1 + c_2 \qquad Dy(0-) = -c_1 + c_2 .$$

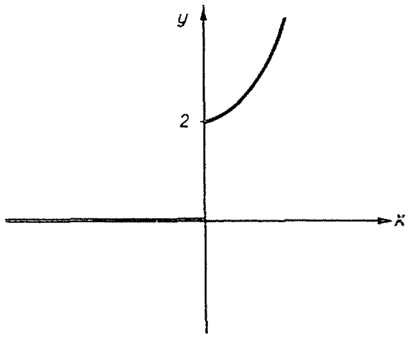

Fig. 2. Discussion of the solution of differential equation $(D^2 - 1)y = \delta(x) + 2\delta^{(1)}(x)$ with given initial conditions at $0+$

Simultaneous differential equations

Consider the system of differential equations

$$D\bar{y} = \mathbf{A}\bar{y} + \bar{f},$$

where the elements of **A** are constants and the elements of column vector $\bar{f}$ contain delta elements as well. This system of differential equations can be solved by using our results obtained so far, e.g. by matrix calculus.

Let us consider a simple problem.

$$(D + 1)y_1 + y_2 = H(x)$$
$$y_1 + (D - 2)y_2 = H(x - 2).$$

By elimination this can be reduced immediately to the following second order differential equation:

$$(D^2 - 3D - 3)y_2 = - H(x) + H(x - 2) + \delta(x - 2).$$

Though the original system contains no delta elements, they obviously may enter during the elimination process.

Summary

The set A formed as the finite formal linear combination of sectionally smooth functions the Dirac delta and its derivatives is examined as a structure. Algebraic and infinitesimal operations are defined which are identical to the corresponding classical operations in the common case. Differentiation and integration are defined so that they should be mutually inverse operations in every case. In consequence, each sectionally smooth function can be differentiated any times. Afterwards, the foregoing results are applied to the solution of ordinary differential equations and systems of differential equations with disturbing functions taken from the elements of the set A.

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