A METHOD FOR COMPUTING THE HYDRODYNAMIC BLADE CASCADE FOR A COMPRESSIBLE FLUID BY MEANS OF A STREAM FUNCTION

by

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There are already known some papers dealing with the calculation of hydrodynamic blade cascades. In this paper an additional method for computing should be expounded by a developing of GRUBER's method,* according to which the calculation is elaborated in the case of a steady channel-width and an incompressible fluid. In this paper, by introducing a stream function, we shall present a solution of the case of a compressible fluid and a variable width. Briefly: the method follows the usual way by dividing the running wheel into elementary ones. The main movement equations on a mid-stream surface in the elementary wheel are written down by assuming this surface as being generated by rotation. By this usual assumption we obtain a good approximation. The problem as originally set up has three dimensions and its solution is rather complicated; again, by the above assumption, the problem is reduced to two dimensions. The fluid friction is not taken into consideration, and the solution is based on iteration.

Symbols

r, ϑ variables of the cylindrical co-ordinates:
s meridional line of the mid-stream surface
B width of the elementary wheel
ϱ density of the fluid
$\varkappa$ exponent of the fluid's isentropic change of state
ω angular velocity of the running wheel
u circumferential speed
c absolute velocity
w relative flow velocity
x, y independent variables in the projection plane
N number of blades of the running wheel
t blade pitch in the projection plane
Ψ stream-function
Γ_l blade circulation
γ distribution function of circulation
$\bar{i}, \bar{j}, \bar{k}$ vectorial units in a Cartesian system of co-ordinates
M mass flowing in the time unit referred to an elementary wheel

* Presented at the Dresden Conference, June 1967.

Subscripts

0 state "in the tank"
1,2 entry state, exit state
s, direction s, or, referring to the mid-flow surface
x, y direction x or y, within the projection plane,
p, sc pressure side, suction side

I. The movement equations

According to Fig. 1, the relationship $s=s(r)$, or inversely $r=r(s)$ holds true.

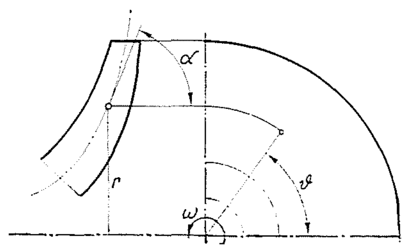

Fig. 1

1/a. The continuity equation on the surface of revolution.

Since no change of the fluid mass takes place in the running wheel during the flow, the continuity equation runs as follows (with the denotations used in Fig. 2):

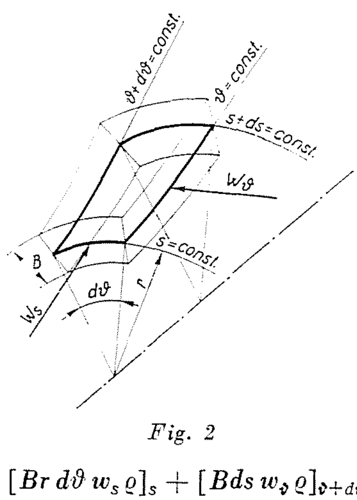

Fig. 2

$$[Br\, d\vartheta\, w_s\, \varrho]_{s+ds} - [Br\, d\vartheta\, w_s\, \varrho]_s + [Bds\, w_\vartheta\, \varrho]_{\vartheta+d\vartheta} - [Bdsw_\vartheta\, \varrho] = 0$$

In order to make calculation easier, width and density should be made dimensionless; this is done by diving the width, by the entry-width B_1, and the density by the so-called tank state density ϱ_0 viz. dividing the equation by the product $B_1\varrho_0$.

After carrying out the division and the necessary operations, we obtain:

$$\frac{\partial}{\partial s}\left(r\frac{B\varrho}{B_1\varrho_0}w_s\right)+\frac{\partial}{\partial\vartheta}\left(\frac{B\varrho}{B_1\varrho_0}w_\vartheta\right)=0$$

1/b. Eddying equation on the surface of revolution

The absolute flow is free from eddies:

$$\text{rot}\ \vec{c}=0$$

Again, the absolute velocity in the vectorial sum of the circumferential speed and the relative velocity:

$$\text{rot}\,\vec{c}=\text{rot}\,(\vec{u}+\vec{w})=0$$

or

$$\text{rot}\,\vec{w}=-2\vec{\omega}$$

since rot $\vec{u}=2\vec{\omega}$, where $\vec{\omega}$ is the angular velocity of the running wheel.

In compliance with Stokes' theorem, we have:

$$[w_s\,ds]_\vartheta+[w_\vartheta\,rd\vartheta]_{s+ds}-[w_s\,ds]_{\vartheta+d\vartheta}-[w_s\,rd\vartheta]_s=-2\omega\frac{dr}{ds}\,d\vartheta\,rds \qquad (1)$$

where $dr/ds=\sin\alpha$ (Fig. 1)

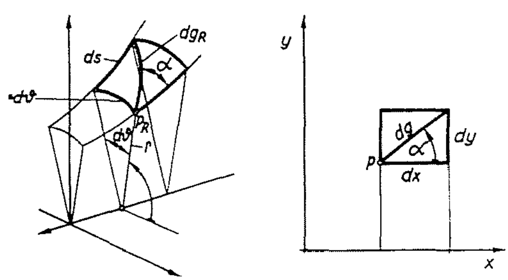

Fig. 3

By carrying out the operation and after due simplification we obtain:

$$\frac{\partial}{\partial s}(w_\vartheta r)-\frac{\partial}{\partial\vartheta}w_s=-2\omega\frac{dr}{ds}r \qquad (2)$$

The next step is to make a transfiguration in which angles and proportions do not change, making the flow problem planimetric. This conform transfiguration is characterized by the relationship:

$$x=x(s)$$

$$y=y(\vartheta)$$

in other words, x and y depend only on s, and ϑ, respectively. Such a transfiguration was carried out already in some former papers (e. g. [1], [2]).

With reference to Fig. 3, this transfiguration is given by the equations:

$$dx = \frac{Nt}{2\pi}\,\frac{ds}{r(s)}$$
$$dy = \frac{Nt}{2\pi}\,d\vartheta \tag{3a}$$

from which:

$$x = \frac{Nt}{2\pi}\int_0^s \frac{ds}{r(s)}$$
$$y = \frac{Nt}{2\pi}\,[\vartheta]_0^{\vartheta} \tag{3b}$$

Now we introduce the stream function as follows:

$$\frac{\partial\psi}{\partial\vartheta} = r\,\frac{B\varrho}{B_1\varrho_0}\,w_s$$
$$-\frac{\partial\psi}{\partial s} = \frac{B\varrho}{B_1\varrho_0}\,w_\vartheta \tag{3c}$$

As can be seen, equations (3c) are in full conformity with equation (1) and equation (2) can be transformed by the corresponding use of equations (3a), (3b), (3c).

I/c. Equation of turbulence in the projection plane

According to the rule of indirect differentiation, we obtain from equation [2].

$$\frac{\partial}{\partial x}\left(-\frac{B_1\varrho_0}{B\varrho}\,r\,\frac{\partial\psi}{\partial x}\,\frac{dx}{ds}\right)\frac{dx}{ds} - \frac{\partial}{\partial y}\left(\frac{B_1\varrho_0}{B\varrho}\,\frac{1}{r}\,\frac{\partial\psi}{\partial y}\,\frac{dy}{d\vartheta}\right)\frac{dy}{d\vartheta} = - 2r\omega\,\frac{dr}{ds}$$

After having carried out the operations of differentiation, due substitutions and modifications, we obtain

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \Delta\psi = -\,\mathrm{grad}\left(\ln\frac{B_1\varrho_0}{B\varrho}\right)\mathrm{grad}\,\psi + \frac{B\varrho}{B_1\varrho_0}\Big)\left(\frac{2\pi r}{Nt}\right)^2 2\omega\sin\alpha \tag{4}$$

II. Calculation of $\Delta\psi$

We have to bear in mind that equation (4) represents the exact mathematical definition of the problem, since no kind of neglection has been made as yet. In the following, this strictness should be abandoned, because the above partial differential equation cannot be solved in a closed form, since boundary conditions are too much complicated. The iterative method shall be applied, assuming that $B_1\varrho_0/B\varrho$ does not change but in the direction of x:

$$\operatorname{grad}\left(\frac{B_1\varrho_0}{B\varrho}\right)=\frac{\partial}{\partial x}\left(\frac{B_1\varrho_0}{B\varrho}\right)\bar{i}$$

and that the function $\psi(x, y)$ represents a third degree parabola in the y direction.

The parabola is expressed as follows:

$$\psi_1=\psi_1\left\{0,5+\frac{y-f(x)}{t}+a(x)\left[\left(\frac{y-f(x)}{t}\right)^2-0,25\right]+\right.$$
$$\left.+b(x)\left(\frac{y-f(x)}{t}\right)\left[\left(\frac{y-f(x)}{t}\right)^2-0,25\right]\right.$$

where $a(x)$, and $b(x)$ are yet unknown functions of the independent variable x; $f(x)$ represents the equation of the camber line of the channel between two adjacent blades, and ψ is a constant that will be determined later on.

From what is exposed above, we obtain

$$\Delta\psi=\psi_1\left\{\frac{2a}{t^2}+\frac{6b}{t^2}\left(\frac{y-f}{t}\right)-\frac{f''}{t}+a''\left[\left(\frac{y-f}{t}\right)^2-0.25\right]-\right.$$
$$-4a'\left(\frac{y-f}{t}\right)\frac{f'}{t}+2a\left[\left(\frac{f'}{t}\right)^2-\left(\frac{y-f}{t}\right)\frac{f''}{t}\right]+b''\left(\frac{y-f}{t}\right)\left[\left(\frac{y-f}{t}\right)^2-\right.$$
$$\left.-0.25\right]--2b'\left[3\left(\frac{y-f}{t}\right)^2-0.25\right]\frac{f'}{t}+b'\left\{6\left(\frac{y-f}{t}\right)\left(\frac{f'}{t}\right)^2-\right.$$
$$\left.\left.-\frac{f''}{t}\left[3\left(\frac{y-f}{t}\right)^2-0.25\right]\right\}\right\} \tag{5}$$

By introducing the denotation

$$\frac{y-f}{t}=\bar{\eta}$$

we shall find the function $a(x)$ with the restriction, that the equation (4) should be satisfied only at the points

$$_{+}\bar{\eta}_1=+\sqrt{\frac{0.25}{3}}=0.298 \quad \text{and} \quad _{-}\bar{\eta}_1=-\sqrt{\frac{0.25}{3}}=-0.298$$

In this way we have

$$\Delta\psi_{+\bar{\eta}_1} = \psi_1 \left\{ \frac{2a}{t^2} + \frac{6b}{t^2}\,\bar{\eta}_1 - \frac{f''}{t} + a''\,[\bar{\eta}_1^2 - 0.25] - \right.$$

$$- 4a'\,\bar{\eta}_1 \left(\frac{f'}{t}\right) + 2a\left[\left(\frac{f'}{t}\right)^2 - \bar{\eta}_1\left(\frac{f''}{t}\right)\right] +$$

$$\left. + b''\,\bar{\eta}_1\,[\bar{\eta}_1^2 - 0.25] + 6b\bar{\eta}_1\left(\frac{f'}{t}\right)^2 \right\} \tag{5a}$$

and

$$\Delta\psi_{-\bar{\eta}_1} = \psi_1 \left\{ \frac{2a}{t^2} - \frac{6b}{t^2}\,\bar{\eta}_1 - \frac{f''}{t} + a''\,[\bar{\eta}_1^2 - 0.25] + 4a'\,\bar{\eta}_1\left(\frac{f'}{t}\right) + \right.$$

$$\left. + 2a\left[\left(\frac{f'}{t}\right)^2 + \bar{\eta}_1\left(\frac{f''}{t}\right)\right] - b''\,\bar{\eta}_1\,[\bar{\eta}_1^2 - 0.25] - 6b\bar{\eta}_1\left(\frac{f'}{t}\right)^2 \right\} \tag{5b}$$

For the final calculation of the function $a\,(x)$ we shall develop the expression $(\Delta\psi_{+\bar{\eta}_1} + \Delta\psi_{-\bar{\eta}})/2\psi_1$ as follows:

$$\frac{\Delta\psi_{+\bar{\eta}_1} + \Delta\psi_{-\bar{\eta}_1}}{2\psi_1} = \frac{1}{t^2}\,[2a(1+f'^2) - f''\,t - 0.166\,a''\,t]$$

from where the differential equation is obtained:

$$2(1+f'^2)\,a - 0.166\,t^2\,a'' = \frac{\Delta\psi_{+\bar{\eta}_1} + \Delta\psi_{-\bar{\eta}_1}}{2\psi_{\bar{1}}} + f''\,t \tag{6}$$

With equal divisions h, the derivative can be substituted by a difference quotient as an approximation:

$$a'' \simeq \frac{1}{h^2}\,(a_{i+1} + a_{i-1} - 2a_i)$$

By substituting this in (6), we obtain:

$$a_i = \frac{1}{2\left[(1+f'^2) + 0.166\left(\frac{t}{h}\right)^2\right]} \left\{ \frac{\Delta\psi_{+\bar{\eta}_1} + \Delta\psi_{-\bar{\eta}_1}}{2\psi_1}\,t^2 + f''\,t + \right.$$

$$\left. + 0.166\left(\frac{t}{h}\right)^2 (a_{i-1} + a_{i+1}) \right\} \tag{7}$$

The partial differential equation (4) can be solved only when the troubling function on the right side is calculated first. This can be done as follows:

The relationship is known as being true:

$$w = |\vec{w}| = \frac{1}{\dfrac{B\varrho}{B_1 \varrho_0}} |\operatorname{grad} \psi| \frac{Nt}{2\pi r} \tag{8}$$

in which

$$\operatorname{grad} \psi = \bar{i} \frac{\partial \psi}{\partial x} + \bar{j} \frac{\partial \psi}{\partial y}$$

on the basis of equation (3a).

The first step of iteration is based on the assumption that the blade cascade is infinitely dense, blades are infinitely thin, and consequently all the stream lines coincide. The basic idea of infinitely thin blades is in common use for an approximative calculation of turbomachines; it is applied by the authors [1], [4]. Now, the relative velocity is expressed by means of the blade form and by ψ_1 as a characteristic of the flowing mass according to Fig. 4. As can be seen:

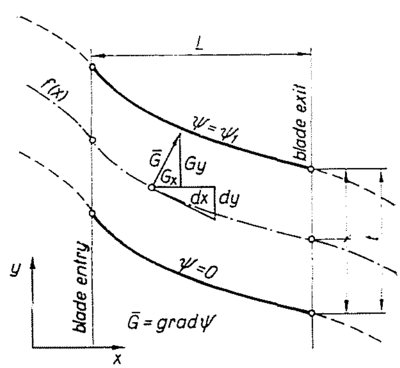

Fig. 4

$$f'_\infty = \frac{dy}{dx} = -\frac{G_x}{G_y}$$

and thus

$$G_x = -G_y f'_\infty$$

where the subscript ∞ refers to the idea of a stream line belonging to a system having an infinite number of blades. Since for such a system the formulae

$$G_y = \operatorname{grad} \psi)_\infty]_y = \frac{\psi_1}{t}$$

and

$$G_x = \operatorname{grad} \psi)_\infty \big]_x = -\frac{\psi_1}{t} f'_\infty$$

hold true, we obtain for the relative velocity

$$\frac{1}{\dfrac{B\varrho}{B_1\varrho_0}} \frac{\psi_1}{t} \sqrt{1+f'^2_\infty} \cdot \frac{Nt}{2\pi r} \tag{9}$$

As is known, the quotient ϱ/ϱ_0 is a function of velocity [1], if the inflow velocity comprises a component C_{1u}, this quotient is expressed as follows:

$$\frac{\varrho}{\varrho_0} = \left\{1 + \frac{\varkappa-1}{2}\left[\left(\frac{u}{a_0}\right)^2 - \left(\frac{w}{a_0}\right)^2 - \frac{2c_{1u}u_1}{a_0^2}\right]\right\}^{\frac{1}{\varkappa-1}} \tag{10a}$$

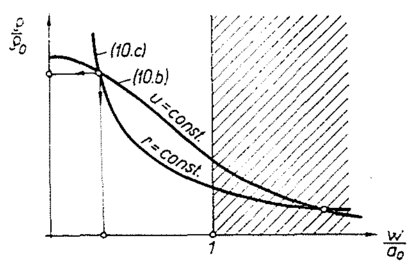

Fig. 5

Without the said component, the following formula should be used:

$$\frac{\varrho}{\varrho_0} = \left\{1 + \frac{\varkappa-1}{2}\left[\left(\frac{u}{a_0}\right)^2 - \left(\frac{w}{a_0}\right)^2\right]\right\}^{\frac{1}{\varkappa-1}} \tag{10b}$$

where a_0 denotes the so-called "tank state" sonic speed.

Now, the ratio ϱ/ϱ_0 shall be expressed on the basis of formula (9)

$$\frac{\varrho}{\varrho_0} = \frac{1}{a_0} \frac{1}{\dfrac{w}{a_0}} \frac{1}{\dfrac{B}{B_1}} \frac{\psi_1}{t} \sqrt{1+f'^2_\infty} \cdot \frac{Nt}{2\pi r} \tag{10c}$$

in which the relative velocity w is made dimensionless (being divided by a_0).

Obviously, ϱ/ϱ_0 should comply with both the formula (10c) and either (10a) or (10b). Taking it as an independent variable, while u/a_0 and r are considered as parameters, the value ϱ/ϱ_0 can be determined as the point of intersection of two corresponding curves (Fig. 5).

To this end, it is necessary to describe the quantities B/B_1 and f'_∞ as functions of r. For the function f'_∞, the distribution of circulation in the case of an infinite number of blades should be determined. By applying Stokes theorem on the surface of revolution for the area as shown in Fig. 6, the formula is obtained:

$$d\Gamma_l - 2\omega \frac{dr}{ds} \frac{2\pi r}{N} ds = \frac{\partial}{\partial s}\left[\frac{2\pi r}{N} w_{\vartheta,\infty}\right] ds \tag{11a}$$

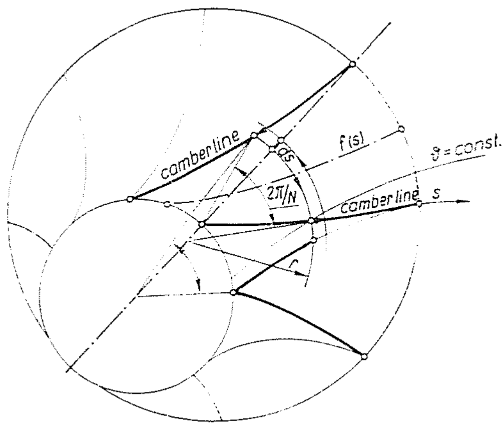

Fig. 6

By transforming it as a function of the variable x in the projection plane, we have:

$$\gamma^* (x)\, dx - 2\omega \frac{dr}{ds} \left(\frac{2\pi r}{Nt}\right)^2 t dx = \frac{\partial}{\partial x}\left[\psi_1 \frac{B_1 \varrho_0}{B\varrho} f'_\infty\right] dx \tag{11b}$$

for which the relationship

$$w_{\vartheta,\infty} = w_{s,\infty} f'_\infty$$

and

$$w_{s,\infty} = \frac{\psi_1}{\frac{2\pi r}{N}} \frac{B_1 \varrho_0}{B\varrho}$$

has been used.

Now, integrate the latter equation between the limites $x(0)$ and $x(s)$

$$\Gamma_l\big(x(s)\big) - \frac{2\pi\omega}{N}\left[r^2(x) - r^2\big(x(0)\big)\right] = \psi_1\left[\frac{B_1 \varrho_0}{B(x)\,\varrho(x)} f'_\infty(x) - \right.$$

$$\left. - \frac{B\varrho}{B\big(x(0)\big)\,\varrho\big(x(0)\big)} f'_\infty\big(x(0)\big)\right] \tag{12a}$$

where

$$\Gamma_l\big(x(s)\big) = \int_0^x \gamma^*(x)\, ds$$

and

$$f'_\infty\big(x(0)\big) = \frac{w_\vartheta(0)}{w_s(0)}$$

For the present problem the appropriate form of the function $\gamma^*(x)$ will be chosen and thus the function $\Gamma_l(x)$ can be determined. Again, the values $w_\vartheta(0)$ and $w_s(0)$ are known as characteristics of the inflow. Now, when $\gamma^*(x)$ is defined, the value $\varrho/\varrho_0 f'_\infty$ viz. the values ϱ/ϱ_0 and f'_∞ can be separately calculated by using equation (10c), because the second term at the left side of equation (12a) and the expression $B_1\varrho_0/B(x)\varrho(x)$ are known quantities. Thus, from (12a) we obtain:

$$f'_\infty(x) = \frac{\varrho}{\varrho_0}\frac{1}{\psi_1}\frac{B}{B_1}\Big\{\Gamma_l(x) - \frac{2\pi\omega}{N}\big[r^2(x) - r^2\big(x(0)\big)\big] + \\ + \psi_1 \frac{B_1\varrho_0}{B\big(x(0)\,\varrho\big(x(0)\big)} f'_\infty\big(x(0)\big)\Big\} \tag{12b}$$

By putting this into $(10-c)$ we can write

$$\frac{\varrho}{\varrho_0} = \frac{1}{a_0}\frac{1}{\dfrac{w}{a_0}}\frac{1}{\dfrac{B}{B_1}}\frac{\psi_1}{t}\frac{Nt}{2\pi r} \times \tag{13}$$
$$\times \sqrt{1 + \left\{\frac{\varrho}{\varrho_0}\frac{1}{\psi_1}\frac{B}{B_1}\left[\Gamma_l(x) - \frac{2\pi\omega}{N}\big(r^2 - r^2(0)\big)\right] + \psi_1\frac{B_1\varrho_0}{B\big(x(0)\big)\,\varrho\big(x(0)\big)} f'_\infty\big(x(0)\big)\right\}}.$$

We shall express ψ_1 for the case of an infinite number of blades:

$$w_s \frac{B\varrho}{B_1\varrho_0} r = \frac{\partial\psi}{\partial\vartheta}$$

and

$$\left(\frac{\partial\psi}{\partial\vartheta}\right)_\infty = \left(\frac{\partial\psi}{\partial y}\right)_\infty \frac{dy}{d\vartheta} = \frac{\psi_1}{t}\frac{Nt}{2\pi r}$$

we can write:

$$\psi_1 = \frac{B\varrho}{B_1\varrho_0} w_{s\infty}\frac{2\pi r}{N} = \frac{M}{B_1\varrho_0 N} \tag{14}$$

The expression of ψ_1 does not change for the case of a finite number of blades.

Now, the relationship (13) can be transformed as the explicit expression of w/a_0; thus this can be plotted in Fig. 5.

With the known value of ϱ/ϱ_0 the expression

$$\operatorname{grad}\left(\ln \frac{B_1 \varrho_0}{B\varrho}\right) = \frac{\partial}{\partial x}\left[\ln\left(\frac{B_1 \varrho_0}{B\varrho}\right)\right]\bar{i}$$

can be calculated, because the function (B/B_1) is known in any case, either the blade cascades are projected or existing ones are to be checked.

Thus, by accepting the above assumption, the troubling function on the right side of equation (4) can be determined as a function being dependent only on the unique independent variable x:

$$z(x) = -\frac{\partial}{\partial x}\left[\ln\left(\frac{B_1 \varrho_0}{B\varrho}\right)\right]\frac{\partial\psi}{\partial x} + \frac{B\varrho}{B_1 \varrho_0}\left(\frac{2\pi r}{Nt}\right)^2 2\omega \sin\alpha$$

where

$$\frac{\partial\psi}{\partial x} = -\frac{\psi_1}{t} f'_\infty$$

as already stated.

Since the troubling term in equation (4) is considered as a function that depends only on the variable x, the values of $\Delta\psi$ are assumed as not depending on y; and so we can write:

$$\frac{\Delta\psi_{+\bar{\eta}_h} + \Delta\psi_{-\bar{\eta}_h}}{2} = z(x)$$

Now, the values of $z(x)$ calculated on the basis of an infinite number of blades should be substituted into the formula (7):

$$a_i = \left[2\left(\frac{1+f'^2}{0{,}166\left(\frac{t}{h}\right)^2} + 1\right)\right]^{-1} \cdot \left\{\frac{z(x_i)\frac{t^2}{\psi_1} + f'' t}{0.166\left(\frac{t}{h}\right)^2} + a_{i-1} + a_{i+1}\right\} \qquad (15)$$

For an approximative calculation it is important to assess the probable error, to precise the formula of a possible correction. So we should find the relationship between the functions $a(x)$ and $\gamma^*(x)$. In compliance with the blade circulation we have:

$$d\Gamma_l = (w_{sc} - w_p)\, dl = (w_{sc,s} - w_{p,s})\frac{1}{\cos^2\beta}\, ds =$$

$$= \left[\frac{1}{r}\left(\frac{\partial\psi}{\partial\vartheta}\right)_{sc} - \frac{1}{r}\left(\frac{\partial\psi}{\partial\vartheta}\right)_p\right](1+f'^2)\frac{B_1\varrho_0}{B\varrho}\, ds =$$

$$= \left[\left(\frac{\partial\psi}{\partial y}\right)_{sc} - \left(\frac{\partial\psi}{\partial y}\right)_p\right]\left(\frac{B_1\varrho_0}{B\varrho}\right)(1+f'^2)\, dx = \gamma(x)\, dx\,,$$

where

$$\gamma(x) = \frac{B_1 \varrho_0}{B \varrho} \left[\left(\frac{\partial \psi}{\partial y} \right)_{sc} - \left(\frac{\partial \psi}{\partial y} \right)_{p} \right] (1 + f'^2) \tag{16}$$

is valid on the basis of

$$\left(\frac{\partial \psi}{\partial y} \right)_{sc} = \psi_1 \left\{ \frac{1}{t} + a(x) + b(x) \frac{0.5}{t} \right\}$$

and

$$\left(\frac{\partial \psi}{\partial y} \right)_{p} = \psi_1 \left\{ \frac{1}{t} - a(x) + b(x) \frac{0.5}{t} \right\}$$

Consequently:

$$\gamma(x) = \frac{B_1 \varrho_0}{B \varrho} 2\psi_1 \frac{a(x)}{t} (1 + f'^2) \tag{17}$$

Starting by assuming an infinite number of blades (i.e. $f' = f'_\infty$) the function $\gamma(x)$ will differ from the starting function $\gamma^*(x)$. Thus, the value of f' should differ from the value of f'_∞. In order to find the measure of modification, the relationship between the blade shape and $\gamma^*(x)$ for an infinite number of blades should be examined. By using the equation (11b) we have:

$$\gamma^* (x) = 2\omega \left(\frac{2\pi r}{Nt} \right)^2 t \sin \alpha + \psi_1 \left[f'_\infty \frac{\partial}{\partial x} \frac{B_1 \varrho_0}{B \varrho} + \frac{B_1 \varrho_0}{B \varrho} f''_\infty \right] \tag{11c}$$

As can be seen, the variations of $\gamma^*(x)$ are related to the variation of f'_∞, ϱ/ϱ_0 and f''_∞. Taking all these variations into consideration would make the correction of the blade camber line too complicated. So a possible neglection will be applied by assuming that a small variation of $\gamma^*(x)$ does not bear on the values of f'_∞ and of $\partial/\partial x\, B_1\varrho_0/B\varrho$. In this way, the approximation

$$\delta\gamma^* (x) \simeq \psi_1 \delta f''_\infty \frac{B_1 \varrho_0}{B \varrho} \tag{18}$$

should be accepted as the relationship between a small variation of the distribution of calculation and the variation of f''_∞ in the case of an infinite number of blades.

For obvious reasons we can assume, in the case of a finite number of blades, a similar relationship between the variation of $\gamma(x)$ and the variation of f''; namely, in the case of a finite number of blades, the formula

$$\delta f'' = \frac{1}{\psi_1} \frac{B \varrho}{B_1 \varrho_0} \delta\gamma(x) \tag{19}$$

serves to express the necessary correction of the value f''.
Again, for $\delta\gamma(x)$ we can write:

$$\delta\gamma(x) = \gamma^*(x) - \frac{B_1 \delta_0}{B\varrho} 2\psi_1 \frac{a(x)}{t} (1 + f'^2) \tag{20}$$

By substituting the result obtained from (20) in (19), the value $\delta f''$ can be calculated, and the channel profile line can be corrected. The approximation is repeated as many times as is necessary to make $\delta f''$ disappear.

In order to determine the cascade of blade, first the values according to (12b) are applied; after $a(x)$ is determined, the distribution $\gamma(x)$ is calculated on the basis (17), and with these the values of $\delta f''$ can be determined.

The subsequent steps of calculation are:

1 — Calculation of frequency distribution ϱ/ϱ_0 by using equations (10a), (10b), and (13), resp., on the basis of a function of distribution suitably chosen.

2 — Determination of the value of f'_∞ on the basis of the frequency distribution as given by formula (12b).

3 — Calculation of the values $z(x)t^2/\psi_1$, and $f''t$ on the basis of known values of ϱ/ϱ_0 and f''_∞; calculation of $a(x)$ by using formula (15).

4 — Determination of the circulation distribution by means of the relationship (17).

5 — Determination of $\partial\gamma(x)$ by means of formula (20).

6 — Determination of $\partial f''$ by the formula (19); calculation of $\partial f'$ from $\partial f''$.

7 — Correction of the values $f''(x)$ and $f'(x)$.

By these 7 steps the first iteration is completed. The second iteration starts with the 3rd step, by keeping the value $[z(x)t^2/\psi_1]$ unchanged. When the iteration is carried out, $b(x)$ is calculated from $a(x)$ by using the formula

$$\frac{\Delta\psi_{+\bar{\eta}_1} - \Delta\psi_{-\bar{\eta}_1}}{2\psi_1} = 0$$

and consequently:

$$b_i = \frac{1}{2\left[18\left(\frac{h}{t}\right)^2 (1 + f'^2) + 1\right]} \left\{ \frac{24a'f't + 12af''t}{\left(\frac{t}{h}\right)^2} + b_{i-1} + b_{i+1} \right\} \tag{21}$$

In the case of but a small number of blades, the deviation caused by the assumption $\partial/\partial y \; \varrho/\varrho_0 = 0$ is larger, and the correction according to (19) is no more effective. So another type of a correcting formula should be applied.

As is known, the velocity distribution and the statical pressure distribution along the direction y can be considered as a rather straight line [3]. On this basis the following assumption seems to be justified:

$$\left.\frac{w}{a_0}\right]_{\text{actual,mean}} \cdot \left.\frac{\varrho}{\varrho_0}\right]_{\text{actual,mean}} \cdot \cos\beta_{\text{actual}} = \left.\frac{w}{a_o}\right]_n \cdot \left.\frac{\varrho}{\varrho_o}\right]_n \cdot \cos\beta_n, \tag{22}$$

where the subscript n denotes the values obtained after the n^{th} iteration of the correcting formula (19), and we have

$$\left.\frac{w}{a_0}\right] = \frac{1}{2}\left\{\left.\frac{w}{a_0}\right]_p + \left.\frac{w}{a_0}\right]_{sc}\right\}$$

Again,

$$\left.\frac{w}{a_0}\right]_{\text{actual,mean}} = \left.\frac{w}{a_0}\right]_\infty \tag{23}$$

The assumption

$$\left.\frac{\varrho}{\varrho_0}\right]_{\text{actual,mean}} \simeq \left.\frac{\varrho}{\varrho_0}\right]_n$$

is justified.
Therefore,

$$\left.\frac{w}{a_0}\right]_{\text{actual,mean}} \cdot \cos\beta_{\text{actual}} = \left.\frac{w}{a_0}\right]_n \cos\beta_n \tag{24}$$

Only, the quantity $\cos\beta_{\text{actual}}$ is not known as yet. Since

$$\cos\beta = \frac{1}{\sqrt{1+\mathrm{tg}^2\beta}} = \frac{1}{\sqrt{1+f'^2}}$$

we obtain, from formulae (23) and (24):

$$\left.\frac{w}{a_0}\right]_\infty \sqrt{1+f_n'^2} = \left.\frac{w}{a_0}\right]_n \sqrt{1+f'^2_{\text{actual}}}$$

and also

$$f'_{\text{actual}} = \sqrt{\left\{\frac{\left.\frac{w}{a_0}\right]_\infty}{\left.\frac{w}{a_0}\right]_n}\right\}^2 (1+f_n'^2) - 1} \tag{25a}$$

By taking this value of f'_{actual} and using formula (17), $a(x)$ can be determined by keeping $\gamma^*(x)$ unchanged. Then the new values of $b(x)$ are calculated from formula (21). Using the obtained values of $a(x)$ and $b(x)$ we can use the two equations which precede formula (17), and thus we obtain the new values of

$$\left(\frac{\partial\psi}{\partial y}\right)_{sc} \quad \text{and} \quad \left(\frac{\partial\psi}{\partial y}\right)_p, \qquad \text{and also}$$

the value of

$$\frac{w}{a_0} = \frac{1}{2}\left[\left(\frac{w}{a_0}\right)_{sc} + \left(\frac{w}{a_0}\right)_{p}\right]$$

This is now the value $w/a_0]_{n+1}$.

As a second step of approximation, we calculate:

$$f'_{n+2} = \sqrt{\left\{\frac{\left.\frac{w}{a_0}\right]_{\infty}}{\left.\frac{w}{a_0}\right]_{n}}\right\}^2 (1 + f'^2_{n+1}) - 1} \tag{25b}$$

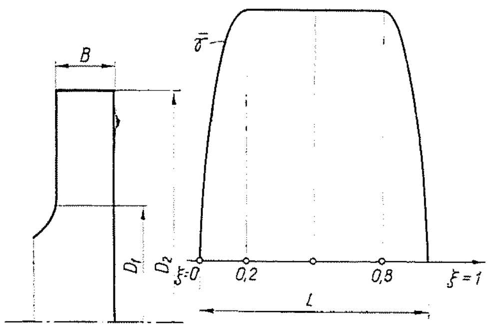

Fig.7

Starting from this with a new series of iterations as described above, we continue iterating as long as the quotient

$$\left\{\left.\frac{w}{a_0}\right]_n / \left.\frac{w}{a_0}\right]_{n+k}\right\}$$

becomes equal to 1 (with $k = 1, 2, 3 \ldots$).

As seen from what has been expounded above, the first step of this second series of iterations starts with values of $w/a_0]_n$ and f'_n obtained from formula (19).

As our last argumentation, let us compare the numerical results of our method, dealing with a running wheel of radial flow, with the results as obtained by the method of singularities [4]. (Fig. 7)
The numerical data are:

$$\frac{D_2}{D_1} = 1.6; \quad (D = 2r)$$

$$N = 16$$

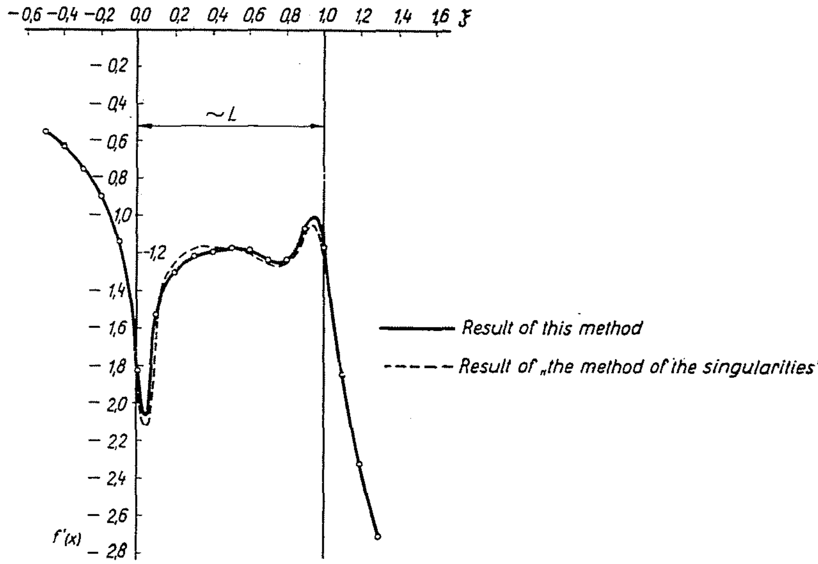

Fig. 8

$$\psi_{id} = \frac{2c_{2u}}{u_2} = 1.2$$

$$\varphi = \frac{C_{2r}}{u_2} = 0.3$$

$$M^* = \frac{u_2}{a_0} = 0.7$$

$$\gamma^*(x) = \frac{\psi_{id}\, r_2 u_2}{NL}\, \bar{\gamma}(\xi)$$

where

$$\bar{\gamma}(\xi) = \left[\frac{2220}{16}(\xi - 0.2)^3 - 1.111\right] h(\xi) - \frac{2220}{16}\left[(\xi - 0.2)^3 h(\xi - 0.2) + \right.$$

$$\left. + (\xi - 0.8)^3 h(\xi - 0.8)\right] + \left[\frac{2220}{16}(\xi - 0.8)^3 - 1.111\right] h(\xi - 1)$$

and

$$L = \frac{Nt}{2\pi} \ln \frac{r_2}{r_1}$$

Notably, $\xi = 0$ corresponds to the value

$$x_1 = \frac{Nt}{2\pi} \ln r_1$$

and

$\xi = 1$ corresponds to the value $x_2 = Nt/2\pi \ln r_2$ and the functions h are step-wise functions of the unit-step.

The diagram in Fig. 8, is a comparison of the results of both methods.

Summary

This method for calculation of a hydrodynamical cascade of blades is suitable to project the blade-system of turbomachines having a great number of relatively thin blades. The task is to determine the camber-line of a running wheel in the meridional section, when pressure, flowing quantity and the recommended number of blades are given. The problem, having originally three dimensions, is reduced — by dividing the wheel and by means of a conform transfiguration — to a planimetric question. The approximative solution is carried out by the introduction of a stream-function, through iterations. In the case of a relatively great number of very thin blades, this method proves to be quicker than the known method of calculating the singularities.

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