EMPLOYMENT OF SINGULARITY CARRIER AUXILIARY CURVE FOR BLADE PROFILE DESIGN

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Introduction

The singularity method as a means for calculating blade profiles and stationary or rotary cascades of blades is generally well known.

The flow induced by a blade profile may be constituted by the singularity distribution located either along the profile contour or over a singularity carrier curve appropriately selected. In practical design usually the latter possibility is made use of. The singularity carrier curve may be represented by a physically feasible curve section, that is, entirely within the profile contour [1] or by one of its approximations such as its chord [2], a logarithmic spiral [3], etc. These latter ones can, of course, represent only a close approximation, if the curve was not contained in the region of the profile.

However, the theorems by Feindt [4] permit the utilization of a calculation-technically advantageous if physically unfeasible singularity carrier curve (not within the profile in every case, of arbitrary profile thickness) in a manner equivalent to the employment of the physically feasible one.

The objective of the present paper is to introduce, on grounds of the Feindt theorem, the existence conditions for the correct employment of physically feasible auxiliary singularity-carrier curves very advantageous from calculation-technical aspects. With the requirements of these conditions satisfied, the flow induced by the singularity distribution over the auxiliary singularity carrier curve would (theoretically) agree with that produced by the blade profile.

The first part of the paper deals with the theorems derived from the geometry of the auxiliary singularity carrier curve, whereas in the second part the conditions of combining the auxiliary curve and the physically feasible singularity carrier curve are discussed.

The statements of the paper are only of theoretical significance. The practical application of these theorems and the presentation of applied calculation methods will be dealt with in a following associated paper.

Principal symbols

ξ, η = Co-ordinate system in the blade lattice plane
$\bar{c}$ = The conjugate of velocity (c)
q = Linear source distribution (normal direction velocity difference)
γ = Linear vortex distribution (tangential direction velocity difference)
γ_ξ = Auxiliary function characteristic of γ
q_ξ = Auxiliary function characteristic of q
Γ = Blade circulation
k = Arch length of curve K
s = Arch length of curve S
c_∞ = Undisturbed flow at the blade lattice
c_{ic} = Part of the induced velocity corresponding to the Cauchy value
c_{ic0} = c_{ic} pertaining to $\gamma_{\xi 0}$ *

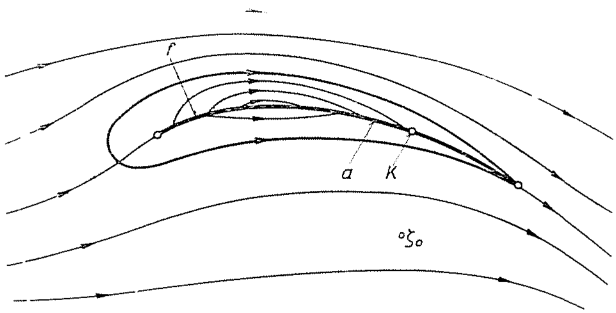

Fig. 1

1. Basic theoretical relations

Let us assume over curve K of *Fig. 1* a conjugate velocity difference of

$$g(\zeta) = \bar{c}_f(\zeta) - \bar{c}_a(\zeta)$$

Thereby, the conjugate of the induced velocity in point ζ_0 is known [4] as

$$\bar{c}_i(\zeta_0) = \frac{1}{2\pi i}\int_K \frac{g(\zeta)}{\zeta - \zeta_0}\, d\zeta = \frac{1}{2\pi}\int_K \frac{q + i\gamma}{\zeta_0 - \zeta}\, |d\zeta| \tag{1}$$

where q and γ represent the normal and tangential velocity component difference, respectively:

$$q = c_{fr} - c_{an}$$

and

$$\gamma = c_{ft} - c_{at}$$

or, as is usually called, the source and vortex distribution, respectively, along the singularity carrier curve.

If point ζ_{ok} approximates a certain point of curve K beyond any limitation then, as shown by CZIBERE [1],

$$\bar{c}_i(\zeta_{0k}) = \pm \frac{g(\zeta_{0k})}{2} + \frac{1}{2\pi} \int_K \frac{q + i\gamma}{\zeta_{0k} - \zeta} |d\zeta| \tag{2}$$

where the positive sign refers to the side of curve K marked "f" and the negative sign to that marked "a", and where a conditional limit value, that is, the Cauchy's principal value should be calculated for the right-hand side integral.

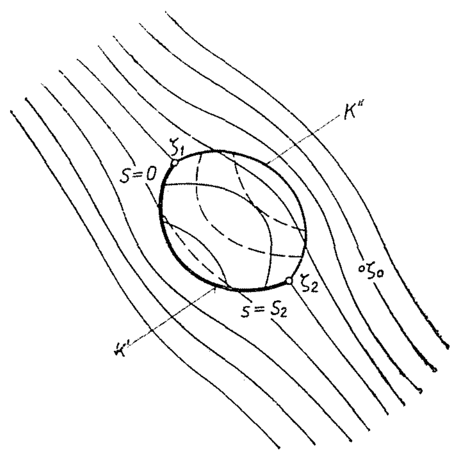

Fig. 2

According to (1),

$$ig\, d\zeta = (q + i\gamma)\, |d\zeta|$$

that is

$$g(\zeta) = [\gamma(\zeta) - iq(\zeta)] \frac{|d\zeta|}{d\zeta} \,. \tag{3}$$

Assuming that the carrier curve K' of *Fig. 2* and the pertaining source/vortex-distribution $q' + i\gamma'$ is given, and the zone function $g(\zeta)$ is holomorphic which means $g(s)$ is given at each point of K' determined by the chord (s), and in the neighbourhood of the point there is an analytic continuation. In this case each K'' curve having its ζ_1 and ζ_2 terminals coinciding with those of curve K' and being contained in the region where $g(\zeta)$ is holomorphic, may carry a $q'' + i\,\gamma''$ distribution so as to induce, in the closed external region enclosed by the two curves, a velocity distribution identical to that induced by the distribution imposed on curve K'. In terminals ζ_ν

$$|g(\zeta)| < \frac{M}{|\zeta - \zeta_\nu|^\gamma} \tag{4}$$

where $\gamma < 1$ and $M > 0$ are permissible (4). With the intention of making use of this theorem, the extent of the region should be determined whereon the function $g(\zeta)$ defined by means of the given curve K and the associated distribution $q + i\gamma$ might be continuable.

By means of function $\Phi(x)$ where $\Phi(x)$ is real-analytical in the open interval J $(0 < x < x_2)$ and continuous at the terminals, let us assume curve K

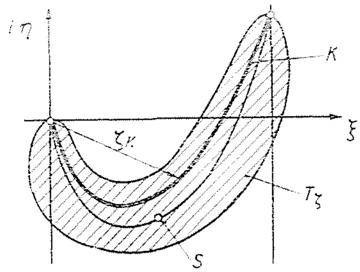

Fig. 3

of *Fig. 3* as given by the function

$$\zeta_K = x + i\Phi(x)\,. \tag{5}$$

At this curve, the function

$$g(x) = [\gamma(x) - iq(x)]\frac{|d\zeta|}{d\zeta} =$$

$$= [\gamma(x) - iq(x)]\frac{\sqrt{1 + \left[\frac{d\Phi}{dx}(x)\right]^2}}{1 + i\frac{d\Phi}{dx}(x)}$$

i.e., considering the roots with a positive sign,

$$\gamma_\xi(x) = \gamma(x)\sqrt{1 + \left[\frac{d\Phi}{dx}(x)\right]^2}$$

$$q_\xi(x) = q(x)\sqrt{1 + \left[\frac{d\Phi}{dx}(x)\right]^2}$$

and by substituting the function

$$g(x) = \frac{\gamma_\xi(x) - iq_\xi(x)}{1 + i\dfrac{d\Phi}{dx}(x)} \tag{6}$$

is analytical in the open interval $J(0 < x < x_2)$, if functions $q_\xi(x)$ and $\gamma_\xi(x)$ are analytical in the interval J and diverge to ∞ at the terminals maximum in the order of $1/x^\gamma$ where $\gamma < 1$. The characteristics of $\Phi(x)$, $q_\xi(x)$, and $\gamma_\xi(x)$ render a proper basis for the determination of region T_ζ in the neighbourhood of curve K where function (6) may be continued.

Since functions $\Phi(x)$, $q_\xi(x)$, and $\gamma_\xi(x)$are analytic, expanding in Taylor's series for real x values, then substituting x with the complex variable $z = x + iy$, the $\Phi(z)$, $q_\xi(z)$, and $\gamma_\xi(z)$ complex variable functions holomorphic in T_z will be obtained. Finally, Equation (5) will be replaced by the mapping function

$$\zeta = z + i\Phi(z) \tag{7}$$

whereby the $0 \leq x \leq x_2$ interval of the real axis of plane (z) and the region T_z would be mapped into the curve K of plane ζ and to region T_ζ, respectively.

Let us assume T_z in the complex plane (z) as a region

1. simple connected with function (7) singlevalued in T_z, and
2. $J \subset T_z$
3. $\Phi(z)$, $q_\xi(z)$, and $\gamma_\xi(z)$ are holomorphic in T_z and
4. $\dfrac{d\zeta}{dz} = 1 + i\,\dfrac{d\Phi}{dz}(z) \neq 0$ in T_z.

Due to the conditions the inverse function $z = f(\zeta)$ will be holomorphic in T_ζ.

Theorem I

If the region T_z meeting the requirements under 1—4 is mapped by function (7) onto the region T_ζ then, within this region T_ζ, function

$$g(\zeta) = \frac{\gamma_\xi(z(\zeta)) - iq_\xi(z(\zeta))}{1 + i\dfrac{d\Phi}{dz}(z(\zeta))} \tag{8}$$

is holomorphic, and would render the analytical extension of $g(x)$ given by expression (6).

It is readily understood that, if $z = x$, then expression (7) represents point $\zeta_k = x + i\Phi(x)$ of curve K and, after substitution, Equation (8) would pass into (6), thus Equation (8) actually represented the extension of Equation (6). Since, according to condition 3, the functions in Equation (8) as the functions of (z) are holomorphic and as the functions of (ζ), due to the holomorphic nature of the inverse function $z = f(\zeta)$, are similarly holomorphic and, finally, as the denominator of (8) does not equal zero owing to condition 4, the complex function (8) is also holomorphic thus, actually representing an analytical continuation.

Assuming curve S of Fig. 3 contained in such a region where function $g(\zeta)$ is imposed on curve K and characterized by distribution $q + i\gamma$ is holomorphic. In this case, the conjugate of the velocity induced at any ζ_0 point of the external region enclosed by curves K and S [4] can be determined by using the expression

$$\bar{c}_i(\zeta_0) = \bar{c}_{ik}(\zeta_0) = \frac{1}{2\pi}\int\limits_K \frac{q_k + i\gamma_k}{\zeta_0 - \zeta_k}|d\zeta_k| =$$

$$= \frac{1}{2\pi}\int\limits_S \frac{q_s + i\gamma_s}{\zeta_0 - \zeta_s}|d\zeta_s| = \bar{c}_{is}(\zeta_0) \tag{9}$$

In the internal region enclosed by the two curves, c_{ik} and c_{is} do not coincide any longer [4].

Theorem II

If curves K and S of plane ζ having coincident terminals are in such a region where function $g(\zeta)$ defined by Equation (8) is holomorphic then, considering any of these two curves as singularity carrier curves, the $c_i(\zeta)$ distribution produced by this curve according to Equation (1) may be analytically continued, via the other curve, to the singularity carrier curve.

Had $c_i(\zeta)$ exhibited singularity within the range between the selected singularity carrier curve and the other curve, Equation (9) could not hold true, that is, $g(\zeta)$ would not prove holomorphic within the internal region enclosed by the two curves which, however, would contradict the set preconditions.

Theorem III

If region T_z is simply connected and satisfies the requirements under 1—4, and z_p represents such a point at the boundary of T_z where $d\Phi/dz(z_p)$, $\gamma_\xi(z_p)$, and $q_\xi(z_p)$ are interpreted, and

$$1 + i\frac{d\Phi}{dz}(z_p) = 0$$

moreover $\gamma_\xi(z_p) \neq 0$ and $q_\xi(z_p) \neq 0$ then, in case when

$$|g(\zeta)| \rightarrow \infty .$$

The validity of Theorem III should be readily understood on grounds of Equation (8).

Theorem IV

If curve S in region T_ζ where $g(\zeta)$ is holomorphic tends to point ζ_p defined by Theorem III then at the point of curve S tending to ζ_p

$$|c_{is}(\zeta)| \rightarrow \infty .$$

The validity of this Theorem can be understood on grounds of Theorem III and Equation (2).

2. Application of the auxiliary singularity carrier curves in calculating straight plane cascades of blades

Assuming at any point within the cascade of blades of (t) pitch, as shown by *Fig. 4*, a singularity carrier curve S is entirely within the profile regardless of blade thickness. Accordingly, the magnitude of the velocity component normal to S, can be but as $q_s/2$.

Let us assume, furthermore, that K as a curve having terminals coinciding with those of curve S, where the complex singularity distribution is defined according to (6), which is continued in the neighbourhood of K according to (8), and where S is entirely in the region where $g(\zeta)$ appears holomorphic.

Using the symbols of Fig. 4 k and s are the arch lengths as parameters of K and S, respectively, the precondition of a closed profile contour may be expressed by the following formulae:

$$\int_0^{k_2} q_k \, dk = 0 \quad \text{and} \quad \int_0^{s_2} q_s \, ds = 0 . \tag{10}$$

The terminals of curves K and S will coincide, if along the f-side of curve K

$$\int_0^{k_2} c_{fn} \, dk = 0 \tag{11}$$

that is, after substituting $c_{fn} = v_n + q_k/2$ and taking (10) into consideration

$$\int_0^{k_2} v_n \, dk = 0 \tag{12}$$

where, if the equation of curve K is $\eta_k = \eta(\xi)$, then

$$v_n = v_\xi \frac{-\eta_k'(\xi)}{\sqrt{1 + \eta_k'(\xi)^2}} + v_\eta \frac{1}{\sqrt{1 + \eta_k'(\xi)^2}} . \tag{12/a}$$

Since after substituting $v_\xi = c_{\infty\xi} + c_{ic\xi}$ and $v_\eta = c_{\infty\eta} + c_{ic\eta}$ (where $c_{ic\xi}$ and $c_{ic\eta}$ represent components of the velocity given by the integral of Equa-

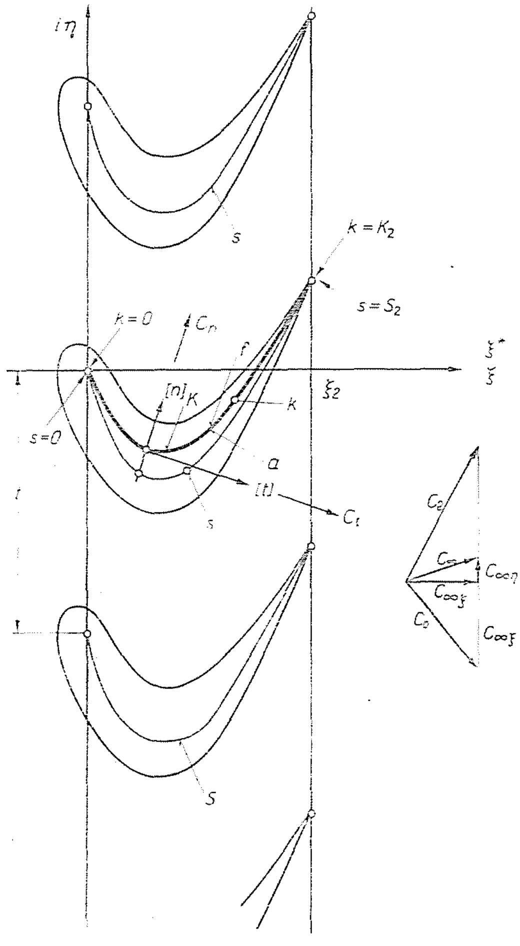

Fig. 4

tion (2) with the lattice arrangement taken into account), then

$$c_{\infty\xi} = \frac{\int\limits_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left[c_{\infty\eta} + c_{ic\eta} - \eta_k'(\xi)\, c_{ic\xi}\right] dk}{\int\limits_0^{k_2} \frac{\eta_k'(\xi)}{\sqrt{1+\eta_k'(\xi)^2}}\, dk} \tag{13}$$

if

$$\int_0^{k_2} \frac{\eta_k'(\xi)}{\sqrt{1+\eta_k'(\xi)^2}}\, dk \neq 0.$$

Moreover

$$c_{\infty\eta} = \frac{\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left[\eta_k'(\xi)\,(c_{\infty\xi} + c_{ic\xi}) - c_{ic\eta}\right] dk}{\int_0^{k_2} \frac{dk}{\sqrt{1+\eta_k'(\xi)^2}}} \tag{14}$$

if

$$\int_0^{k_2} \frac{dk}{\sqrt{1+\eta_k'(\xi)^2}} \neq 0.$$

Denoting:

$$c_{ic} = \frac{\Gamma\, c_{ic}^*}{t}$$

where

$$\Gamma = t(c_{3\eta} - c_{0\eta})$$

would render

$$\Gamma = \frac{\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left(c_{\infty\eta} - \eta_k'(\xi)\, c_{\infty\xi}\right) dk}{\frac{1}{t}\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left(\eta_k'(\xi)\, c_{ic\xi}^* - c_{ic\eta}^*\right) dk} \tag{15}$$

where the denominator cannot equal zero.

Finally, be the vortex distribution along K characterized by $\gamma_\xi + \gamma_{\xi 0}$, where the introduction of a non-dimensional $\gamma_{\xi 0}^*$ would lead to $\gamma_{\xi 0} = \varkappa \gamma_{\xi 0}^*$ and $\int_0^{k_2} \gamma_{\xi 0}^*\, dk = 0$. With the part given by $\gamma_{\xi 0}^*$ of the integral in Equation (2) indicated by c_{ic0},

$$\varkappa = \frac{\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left[c_{\infty\eta} + c_{ic\eta} - \eta_k'(\xi)\,(c_{\infty\xi} + c_{ic\xi})\right] dk}{\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left[\eta_k'(\xi)\, c_{ic0\xi} - c_{ic0\eta}\right] dk} \tag{16}$$

where the denominator does not equal zero.

On grounds of the aforesaid theorems, coincidence of the terminals of the auxiliary singularity carrier curve and of singularity carrier curve S physically feasible and entirely within the blade in every case, this may be provided for in more than one way:

by the appropriate selection of the initial flow (c_∞)

by the appropriate selection of blade circulation, and

by the appropriate selection of circulation distribution.

To summarize

α) Let us assume K as curve in plane ζ produced by using the complex function $\zeta = z + i\Phi(z)$ through substituting $z = x$,

β) Assuming the conditions under 1—4 in the previous section as satisfied, and

γ) Assuming singularity carrier curve S entirely in region T_ζ where $g(\zeta)$ as defined by Equation (8) making use of curve K appears holomorphic.

Theorem V

With the conditions α, γ, and γ satisfied, curves K and S of coincident initial points will exhibit similarly coincident terminals, if either $c_{\infty\xi}$ satisfies Equation (13) and

$$\int_0^{k_2} \frac{\eta_k'(\xi)}{\sqrt{1+\eta_k'(\xi)^2}}\, dk \neq 0$$

or $c_{\infty\eta}$ satisfies Equation (14) and

$$\int_0^{k_2} \frac{dk}{\sqrt{1+\eta_k'(\xi)^2}} \neq 0$$

The validity of this Theorem is readily understood with the explanation of Equations (12), (13), and (14) in mind.

Theorem VI

With the conditions under α, β, and γ satisfied, curves K and S of coincident initial points will have coincident terminals as well, if the blade circulation satisfies the requirements of Equation (15), and

$$\int_0^{k_2} \frac{1}{\sqrt{1+\eta_k'(\xi)^2}} \left[\eta_k'(\xi)\, c_{ic\xi}^* - c_{ic\eta}^*\right] dk \neq 0 \,.$$

The validity of this Theorem can be directly understood on grounds of the explanation of (12) and (15).

Theorem VII

With the conditions under α, β, γ satisfied, curves K and S of coincident initial points will have coincident terminals as well, if in the distribution $\gamma'_\xi = \gamma_\xi + \varkappa\gamma^*_{\xi 0}$ along K

$$\int_0^{k_2} \gamma^*_{\xi 0}\, dk = 0$$

and $\varkappa$ satisfies the requirements of Equation (16), and if

$$\int_0^{k_2} \frac{1}{\sqrt{1 + \eta'_k(\xi)^2}} \left[\eta'_k(\xi)\, c_{ic0\xi} + c_{ic0\eta}\right] dk \neq 0\,.$$

The validity of this Theorem is understood on the basis of Equations (12) and (16).

With any of Theorems V—VII satisfied, curve K and the associated distribution $g_k(\zeta)$ may be used to substitute S in blade calculations.

Summary

The paper studies the possibilities of making use of singularity carrier auxiliary curves physically not feasible. In the course of this study some existence theorems are composed, a procedure conforming which would make the employment of the physically unfeasible singularity carrier auxiliary curve equivalent to that of the physically feasible singularity carrier always entirely within the profile in case of arbitrary profile thickness.

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