Variables in dimensionless form

$\bar{x} = x/L$
$\bar{y} = y/L$
$\bar{\theta} = \theta/L$

1. Introduction

The contribution of skin friction on the drag of slender missile at supersonic or hypersonic Mach number is more than that of the pressure distribution. One of the main factors in missile design is the extreme surface temperature produced by the frictional heating of the missile. Knowledge of the boundary layer properties at high Mach number is essential to the missile designer because friction drag and heat transfer can then be determined.

At hypersonic speed the deceleration of the fluid by a shock wave or by viscous processes in a boundary layer generally produces very high temperatures. Due to high temperature, there is an increase in the thickness of the boundary layer over the thickness encountered at the same free stream Reynolds number at lower speeds, also various physical phenomena, like dissociation and ionization, diffusion of atoms and ions occur, which cause the air to depart from a perfect gas behaviour. The increased thickness of the boundary layer contributes to two types of interaction of the boundary layer with the inviscid flow field; (a) pressure interaction and (b) vorticity interaction. "Pressure" interaction is important for slender bodies, and "vorticity" interaction is important both for the slender as well as for blunt bodies.

The effect due to these interactions can be neglected and the air can be considered as a perfect gas if the Mach number is not very high ($M_\infty < 10$).

An approximate method for the calculation of the laminar compressible boundary layer on three dimensional bodies in axial flow, based on the momentum and the energy integral equations of the boundary layer, has been developed by ROTT and CRABTREE [1] under the following assumptions:

(a) The surface is heat insulated (no heat transfer).

(b) The Prandtl number, P, of the gas is unity.

(c) The viscosity, μ, varies proportionally to the absolute temperature, T. Further ROTT [2] developed a method which is applicable to adiabatic walls when $\mu \sim T$ across the boundary layer, but $\mu \sim T^n$ outside the boundary layer, and there is no restriction concerning the Prandtl number. But they did not take the effect due to the presence of shock wave into account.

HANTZSCHE and WENDT [3] have shown that the equations for a thin laminar boundary layer on a circular cone at zero angle of incidence in a supersonic stream with attached shock can be reduced by means of a simple transformation to equations of the same form as those for a flat plate.

They have obtained the characteristics of the boundary layer in terms of the flow condition just outside the boundary layer of the cone and not in front of fthe attached shock wave.

The present author applies the approximate method of ROTT for the calculation of the characteristics of the laminar compressible boundary layer on supersonic or hypersonic flow past unyawed infinite solid circular cone.

The basic assumptions are:

(a) The cone angle, α, and the free stream Mach number, M_∞, are such that the non viscous flow is conical and irrotational and the shock wave is attached to the vertex of the cone.

(b) There is no restriction on the Prandtl number, P.

(c) The viscosity, μ, is proportional to T across the boundary layer i. e., $\frac{\mu}{\mu_w} = \frac{T}{T_w}$, otherwise $\mu \sim T^n$.

(d) Boundary layer is thin compared to the shock layer and considered as separate from the inviscid flow field. "Pressure" interaction and "vorticity" interaction have been neglected.

(e) The air behind the shock is considered to be perfect.

The present method is simple in operation and gives momentum thickness as a simple quadrature. All the characteristics have been analytically expressed in terms of free stream quantities.

2. Characteristics of the inviscid flow in the shock layer

The non viscous flow outside the boundary layer is assumed to be unaffected by the presence of the thin layer and is conical. All physical quantities are constant along rays from the apex and along generators of the cone. Therefore, physical quantities at the outer edge of the boundary layer depend on α, but not on x. The various characteristics of the flow at the surface of the cone have been obtained as in [4] by assuming that the density in the shock layer is constant. The results are given below:

$$\frac{u_1}{u_\infty} = \frac{\operatorname{cosec}^2 \alpha}{K} \tag{1}$$

where

$$K = B + \ln\left(\frac{\sin \psi}{1 - \cos \psi}\right) - \sec \psi$$

$$B = -\ln\left(\frac{\sin \alpha}{1 - \cos \alpha}\right) - \frac{\cos \alpha}{\sin^2 \alpha}$$

$$\frac{\varrho_\infty}{\varrho_1} = \frac{\frac{2}{M_\infty^2 \sin^2 \psi} + (Y - 1)}{Y + 1} = \epsilon \tag{2}$$

$$\frac{p_1}{p_\infty} = \left[\frac{1 - \dfrac{\operatorname{cosec}^4 \alpha}{K^2}}{1 - \dfrac{\left(B + \ln \dfrac{\sin \psi}{1 - \cos \psi}\right)^2 + \operatorname{cosec}^2 \psi}{K^2}} \right] [1 + Y(1 - \epsilon) M_\infty^2 \sin^2 \psi] \tag{3}$$

$$\frac{T_1}{T_\infty} = \left[\frac{1 - \dfrac{\operatorname{cosec}^4 \alpha}{K^2}}{1 - \dfrac{\left(B + \ln \dfrac{\sin \psi}{1 - \cos \psi}\right)^2 + \operatorname{cosec}^2 \psi}{K^2}} \right] [\epsilon\{1 + Y(1 - \epsilon) M_\infty^2 \sin^2 \psi\}] \tag{4}$$

$$\frac{h_1}{h_\infty} = \left[\frac{1 - \dfrac{\operatorname{cosec}^4 \alpha}{K^2}}{1 - \dfrac{\left(B + \ln \dfrac{\sin \psi}{1 - \cos \psi}\right)^2 + \operatorname{cosec}^2 \psi}{K^2}} \right] \left[1 + \frac{Y-1}{2} M_\infty^2 (1 - \epsilon^2) \sin^2 \psi\right] \tag{5}$$

$$M_1^2 = \frac{M_\infty^2 \dfrac{\operatorname{cosec}^4 \alpha}{K^2}}{\dfrac{T_1}{T_\infty}} \tag{6}$$

Table I

$\alpha°$	20	20	30	30
M_∞	6	7.5	6	7.5
$\psi°$	24.5	23.6	35.1	34.25
$\frac{\varrho_1}{\varrho_\infty}$	3.3193	3.8592	4.2251	4.6852
$\frac{p_1}{p_\infty}$	7.4063	10.7003	14.0925	21.0376
$\frac{T_1}{T_\infty}$	2.2313	2.7727	3.3354	4.4902
M_1	3.6759	4.1432	2.7084	2.9413

Mach number, M_∞, and shock wave angle, ψ, are related by:

$$M_\infty = \operatorname{cosec}\psi \left[\frac{2K}{(Y+1)\left\{B + \ln\left(\frac{\sin\psi}{1-\cos\psi}\right) + \frac{\cos\psi}{\sin^2\psi}\right\} - (Y-1)K} \right]^{\frac{1}{2}} \quad (7)$$

they are given in Table I.

3. Velocity, temperature, density etc. in the boundary layer

Knowing the characteristics of the inviscid flow at the outer edge of the boundary layer, the various characteristics of the boundary layer can be easily obtained. According to the simplifying assumption of the von Kármán—Pohlhausen method, any velocity profile in the boundary layer may be represented as a member of a one-parameter family of profiles. Thus, the velocity profile may be written as:

$$\frac{u}{u_1} = l_i(\beta)\frac{y}{\theta} + \frac{1}{L^2} m_i(\beta)\left(\frac{y}{\theta}\right)^2 + \ldots \quad (8)$$

where β is a paramerer and $m_i = -\frac{du_1}{dx}\frac{\theta^2}{\nu_0}\left(\frac{T_1}{T_0}\right)^{\frac{3-2Y}{Y-1}}$ (9)

For the present case $m_i = 0$, as u_1 is independent of X. β can be made equal to m_i. According to the modification proposed by THWAITES [5], the universal function $l_i(m_i)$ can be calculated from Fig. 1 of [5], when $m_i = 0$. Since θ can be obtained from [18], $\frac{u}{u_1}$ can be known.

Hence,
$$\frac{u}{u_\infty} = l_i(0)\,\frac{y}{\theta}\,\frac{\operatorname{cosec}^2\alpha}{K} \quad (10)$$

As in [2],

$$\frac{h_w}{h_1} = \frac{\varrho_1}{\varrho_w} = \frac{T_w}{T_1} = 1 + \sqrt{P}\,\frac{Y-1}{2} M_1^2 \quad (11)$$

$$\frac{h}{h_1} = \frac{\varrho_1}{\varrho} = \frac{T}{T_1} = 1 + \sqrt{P}\,\frac{Y-1}{2} M_1^2 \left\{1 - \left(\frac{u}{u_1}\right)^2\right\} \quad (12)$$

$$\frac{T_w}{T_\infty} = \sqrt{P}\,\frac{Y-1}{2} M_\infty^2\,\frac{\operatorname{cosec}^4\alpha}{K^2} + \left[\frac{1 - \frac{\operatorname{cosec}^4\alpha}{K^2}}{1 - \frac{\left(B + \ln\frac{\sin\psi}{1-\cos\psi}\right)^2 + \operatorname{cosec}^2\psi}{K^2}}\right] \cdot \left[\epsilon\left\{1 + Y M_\infty^2 (1-\epsilon)\sin^2\psi\right\}\right] \quad (13)$$

$$\frac{T_0}{T_w} = \frac{1 + \frac{Y-1}{2} M_1^2}{1 + \sqrt{P} \frac{Y-1}{2} M_1^2} \tag{14}$$

$$\frac{\varrho_w}{\varrho_\infty} = \frac{1}{\epsilon \left[1 + \sqrt{P} \frac{Y-1}{2} M_1^2\right]} \tag{15}$$

$$\frac{h_w}{h_\infty} = \left[1 + \sqrt{P} \frac{Y-1}{2} M_1^2\right]\left[1 + \frac{Y-1}{2} M_\infty^2 (1-\epsilon^2) \sin^2 \psi\right]$$

$$\left[\frac{1 - \frac{\operatorname{cosec}^4 \alpha}{K^2}}{1 - \frac{\left(B + \ln \frac{\sin \psi}{1 - \cos \psi}\right)^2 + \operatorname{cosec}^2 \psi}{K^2}}\right]$$

The pressure inside the boundary layer is the same as the pressure on the surface in the inviscid flow.

$T\omega/T_1, \frac{\varrho_w}{\varrho_1}$ etc. are given in Table II.

Table II

$\alpha°$	20	20	20	20	30	30	30	30
M_∞	6	6	7.5	7.5	6	6	7.5	7.5
P	0.737	1	0.737	1	0.737	1	0.737	1
$\frac{h_\omega}{h_1} = \frac{T_\omega}{T_1}$	3.3199	3.7026	3.9471	4.4332	2.2593	2.4670	2.4853	2.7303
$\frac{\varrho_\omega}{\varrho_1}$	0.3012	0.2701	0.2533	0.2255	0.4426	0.4053	0.4024	0.3662
$\frac{\varrho_\omega}{\varrho_\infty}$	0.9998	0.8965	0.9777	0.8705	1.8701	1.7126	1.8852	1.7160
$\frac{T_\omega}{T_\infty}$	7.4076	8.2615	10.9439	12.2919	7.5357	8.2286	11.1595	12.2596
$\frac{T_0}{T_w}$	1.1153	1	1.1232	1	1.0919	1	1.0986	1
H	10.9230	12.2974	13.1756	14.9215	7.1141	7.8602	7.9257	8.8056

4. Momentum thickness and skin friction coefficient

The momentum thickness, θ, of the compressible boundary layer is given by:

$$\theta^2 = 0.45\, \nu_0 \left(\frac{T_0}{T_1}\right)^{\frac{2}{Y-1} - 2\sqrt{P}} \frac{u_1^{-6}}{\gamma^2} \int_0^x \left(\frac{T_1}{T_0}\right)^{\frac{Y}{Y-1} - 2\sqrt{P}} \left(\frac{T_0}{T_w}\right)^{1-n} \gamma^2 u_1^5 \, dx \qquad (17)$$

where

$$\gamma = x \sin \alpha$$

Hence,

$$\bar{\theta}^2 = \frac{0.15 \in \left(\frac{T_1}{T_\infty}\right)^n \left(\frac{T_0}{T_w}\right)^{1-n}}{Re \frac{\operatorname{cosec}^2 \alpha}{K}} \bar{x} \qquad (18)$$

If

$$P = 1, \frac{T_0}{T_w} = 1$$

$$\bar{\delta} = \frac{315}{37} \bar{\theta} = 8.5135 \left[\frac{\in \left(\frac{T_1}{T_\infty}\right)^n \left(\frac{T_0}{T_w}\right)^{1-n}}{Re \frac{\operatorname{cosec}^2 \alpha}{K}} \bar{x} \right]^{\frac{1}{2}} \qquad (19)$$

$$H = \left(\frac{T_w}{T_1}\right) H_i + \frac{T_w}{T_1} - 1 = \left[1 + \sqrt{P}\, \frac{Y-1}{2} M_1^2\right] [H_i + 1] - 1 \qquad (20)$$

where H_i can be calculated from Table I of [5].

$$c_0 = \frac{\zeta}{\frac{1}{2}\varrho_\infty u_\infty^2} = \frac{2\, l_i}{\sqrt{Re\, x}} \; \frac{\left(\frac{T_i}{T_\infty}\right)^{\frac{n}{2}} \left(\frac{\operatorname{cosec}^2 \alpha}{K}\right)^{\frac{3}{2}}}{\left(\frac{T_0}{T_w}\right)^{\frac{1-n}{2}} (0.15 \in)^{\frac{1}{2}}} \qquad (21)$$

where $\frac{T_0}{T_\infty}$ can be obtained from [4], $\frac{T_0}{T_w}$ from [14], M_1 from [6] and l_i from Fig. 1 of [5]. θ, c_0 etc. are given in Table III.

Table III

α°	20	20	20	20	20	20	20	20
M_∞	6	6	6	6	7.5	7.5	7.5	7.5
P	0.737	1	0.737	1	0.737	1	0.737	1
n	0.5	0.5	0.8	0.8	0.5	0.5	0.8	0.8
$\dfrac{\bar{\theta}\sqrt{Re}}{\sqrt{x}}$	2.819	2.745	3.379	3.339	3.714	3.599	4.476	4.425
$C_D\sqrt{Rex}$	0.307	0.315	0.256	0.258	0.255	0.262	0.207	0.209
$\bar{\delta}^*\sqrt{\dfrac{Re}{x}}$	30.792	33.755	36.909	41.058	48.932	53.700	58.971	66.025
$\bar{\delta}\sqrt{\dfrac{Re}{x}}$	23.998	23.368	28.765	28.424	31.607	30.638	38.104	37.670

α°	30	30	30	30	30	30	30	30
M_∞	6	6	6	6	7.5	7.5	7.5	7.5
P	0.737	1	0.737	1	0.737	1	0.737	1
n	0.5	0.5	0.8	0.8	0.5	0.5	0.8	0.8
$\bar{\theta}\sqrt{\dfrac{Re}{x}}$	1.527	1.494	1.781	1.716	1.788	1.747	2.048	2.029
$C_D\sqrt{Rex}$	0.462	0.471	0.410	0.412	0.417	0.426	0.363	0.366
$\varrho^*\sqrt{\dfrac{Re}{x}}$	10.863	11.743	12.314	13.487	14.171	15.382	16.232	17.865
$\bar{\delta}\sqrt{\dfrac{Re}{x}}$	12.999	12.718	14.736	14.608	15.221	14.872	17.434	17.273

5. Results of the numerical calculations

The numerical results are given in Tables I to III. From the results, it is possible to know the effects of Prandtl number, P, Mach number, M_∞, semi-vertical angle of the cone, α, Reynolds number, Re and index of viscosity temperature law, n, on the various characteristics of the boundary layer.

(a) Effect of Prandtl number: Temperature at the wall, T_w, H, c_0 and δ^* increase, but $\bar{\theta}$, $\bar{\delta}$ and ϱ_w decrease as P increases.

(b) Effect of Mach number: Temperature, T_w, H, θ, δ and δ^* increase, but ϱ_w and c_0 decrease as M_∞ increases.

(c) Effect of Reynolds number: Reynolds number affects only $\bar{\theta}$, $\bar{\delta}$, c_0 and $\bar{\delta}^*$. They decrease as Re increases.

(d) Effect of the index of the viscosity-temperature law: $\bar{\theta}$, c_0, $\bar{\delta}^*$ and $\bar{\delta}$ depend upon n, others are independent of it. They increase as n increases.

(e) Effect of semi-vertical angle of the cone: $\bar{\theta}$, $\bar{\delta}$, H, δ^* and P_w decrease, but c_0 increases as α increases.

6. Conclusions

The momentum thickness, $\bar{\Theta}$, and boundary layer thickness, $\bar{\delta}$, the displacement thickness, $\bar{\delta}^*$, H, and T_w increase considerably as Mach number is increased but c_0 decreases as $M\infty$ increases. The effect of Prandtl number on the momentum thickness, $\bar{\Theta}$, the skin friction coefficient, Cff, and the boundary layer thickness, $\bar{\delta}$, is small, whereas its effect on wall temperature, T_w, wall density, ϱ_w, H and δ^* is considerably large. n has little effect on $\bar{\theta}$, Cf, $\bar{\delta}^*$ and $\bar{\delta}$, others are independent of it. The effect of α on the boundary layer characteristics is rather appreciable.

7. Summary

In the present paper, the approximate method of Rott has been applied for the calculation of the steady compressible laminar boundary layer characteristics under certain assumptions when the flow past an unyawed semi-infinite solid circular cone is supersonic or hypersonic. The assumptions are: (a) The cone semivertical angle, α, and the free-stream Mach number, M_∞, are such that the non viscous flow is conical and isentropic, the shock wave is straight and attached to the vortex of the cone. (b) The boundary layer is thin compared to the shock layer and considered as separate from the inviscid flow field. Pressure interaction and vorticity interaction have been neglected. (c) The air behind the shock is considered as perfect. (d) There is no restriction on Prandtl number, P, and is considered to be invariant with temperature. (e) The viscosity, μ, is proportional to T across the boundary layer, otherwise $\mu \sim T^n$ and the body is insulated i.e., there is no heat transfer. The present method is simple in operation. The characteristics of the boundary layer have been analytically expressed in terms of free stream quantities.

References

1. ROTT, N. and CRABTREE, L. F.: Simplified laminar boundary layer calculations for bodies of revolution and for yawed wings. Jour. Aero. Sci. **19**, 553—558 (1952).
2. ROTT, N.: Compressible laminar boundary layer on a heat insulated body. Readers' Forum. Jour. Aero. Sci. **20**, 67—68 (1953).
3. HANTZSCHE, W. and WENDT, H.: The laminar boundary layer on a circular cone at zero incidence in a supersonic stream. Rep. and Trans. No. 276, British M. A. P., Aug. 1946.
4. NATH, G.: Hypersonic flow past unyawed circular cones. Journal of the Indian Mathematical Society **28**, 7—24 (1964).
5. THWAITES, B.: Approximate calculation of the laminar boundary layer. Aero. Quar. **1**, 245—280 (1949).

Girishwar NATH, P. and T. Colony, Qr. No. 66, Kidwai Puri, Patna-1, India