DESIGN OF STRAIGHT CASCADES OF SLIGHTLY CURVED BLADINGS BY MEANS OF SINGULARITY CARRIER AUXILIARY CURVE

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Introduction

Literature [1], [2] presents existence theorems permitting the employment of a calculation technically advantageous of physically unfeasible singularity carrier curve partially outside of the profile, instead of the physically feasible one entirely within the profile in case of any profile thickness.

In the simplest case the chord of the physically feasible singularity carrier curve (S) is used as a singularity carrier auxiliary curve. In this sense, the employment of the chord is theoretically identical to that shown by the methods found in the literature [3], [4] since the chord may be considered a full value singularity carrier curve here instead of being regarded as an approximation of curve (S). The present paper explains the application of the singularity carrier straight line for the calculation of a straight plain cascade of blades. Calculations are restricted here to low-arc thin blades. Rotary systems and high-arc thick blades will be dealt with in subsequent papers.

Let us assume analytical $\gamma_x(x)$ and $q_x(x)$ distributions in the interval $J(0 < x < x_2)$ illustrated by Fig. 1. With the substitution $x = z$, the complex functions $\gamma_x(z)$ and $q_x(z)$ would be holomorphic within the circle drawn around the straight section J, as a diameter. This means that the complex function

$$g(z) = \gamma_x(z) - iq_x(z)$$

is, within the circle, similarly holomorphic and assumes, in the interval J, the values

$$g(x) = \left[\gamma_x(x) - iq_x(x)\right] \mid \mathrm{d}x \mid / \, \mathrm{d}x = \gamma_x(x) - iq_x(x)$$

Consequently, as far as the curve (S) intersecting points $x = 0$ and $x = x_2$ is concerned, the conjugate of the velocity induced within the closed external region bounded by the chord of curve (S) and by this curve proper, is

$$c_i(z_0) = \frac{1}{2\pi} \int_{s=0}^{s_2} \frac{q_s(s) + i\gamma_s(s)}{z_0 - z_s} \, ds = \frac{1}{2\pi i} \int_{s=0}^{s_2} \frac{g(z_s)}{z_s - z_0} \, dz_s =$$

$$= \frac{1}{2\pi} \int_{x=0}^{x_2} \frac{q_x(x) + i\gamma_x(x)}{z_0 - x} \, dx.$$

Thus the chord will actually replace curve (S) which will have to fit into the circle drawn around the chord, as a diameter, with terminals represented by points $x = 0$ and $x = x_2$, respectively.

Accordingly, the field of velocity may be continued through curve (S) to the chord or through the chord to curve (S), in an analytical manner.

Principal symbols

ξ, η	=	co-ordinates normal and parallel, respectively, to the cascade of blades
x, y	=	co-ordinates along the direction of/and normal to the chord, respectively
$c_{\infty\xi}$, $c_{\infty\eta}$	=	velocity components of the undisturbed flow in direction ξ and η, respectively
q_s	=	source distribution along curve (S)
γ_s	=	circulation distribution along curve (S)
q_x	=	source distribution along the chord
γ_x	=	circulation distribution along the chord
q^*_ξ	=	dimensionless auxiliary function for the production of q_x
γ^*_ξ	=	dimensionless auxiliary function for the production of γ_x
v	=	velocity along the chord apart from the local difference
$c_{L\xi}$, $c_{L\eta}$	=	ξ and η-direction, respectively, components of the velocity induced
β_0	=	chord angle
v_x, v_y	=	x and y-direction, respectively, components of (v)
Γ	=	blade circulation
y_{kn}, y_{ks}	=	blade contour co-ordinates
y_s	=	co-ordinate of curve (S)

Co-ordination of the singularity carrier auxiliary curve and the physically feasible singularity carrier curve

Let the duty of the t-spacing cascade of blades to transform the velocity c_0 existing upstream of the cascade to c_3 (see Fig. 1). The circulation produced by a single blade is thus

$$\Gamma = t\,(c_{3\eta} - c_{0\eta}) \tag{1}$$

Since $c_{\infty\xi} = c_{3\xi}$, the undisturbed flow having the flow induced by the blades superimposed might be characterized by components

$$c_{\infty\xi} = c_{3\xi} \tag{2a}$$

and

$$c_{\infty\eta} = \frac{c_{3\eta} - c_{0\eta}}{2} \tag{2b}$$

The induced flow may be considered as produced by distributions $\gamma_s(s)$ and $q_s(s)$ located over the singularity carrier curves (S) within the profiles. The in-

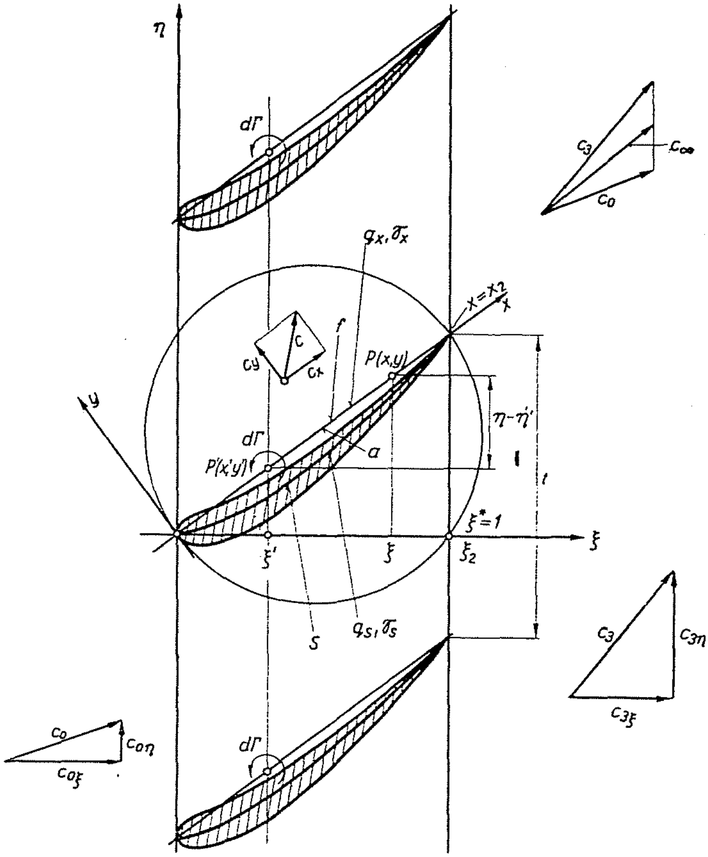

Fig. 1

duced flow within the enclosed external region bounded by curve (S) and its chord will be calculated by making use of distributions $\gamma_x(x)$ and $q_x(x)$ imposed on the chord. Thus, at any point of the chord, when calculating in the manner known from literature [5], the components of the induced velocity as indicated by the symbols of Fig. 1, and apart from the local velocity difference would be

$$c_{L\xi}(x) = \frac{\Gamma}{t}\int_0^1 \left| \gamma_\xi^*(\xi^{*\prime})\Phi\left(\frac{\xi-\xi'}{t}, \frac{\eta-\eta'}{t}\right) + q_\xi^*(\xi^{*\prime})\Psi\left(\frac{\xi-\xi'}{t}, \frac{\eta-\eta'}{t}\right)\right| d\xi^{*\prime} \tag{3a}$$

and

$$c_{L\eta}(x) = \frac{\Gamma}{t}\int_0^1 \gamma_\xi^*(\xi^{*\prime})\,\Psi\left(\frac{\xi-\xi'}{t},\frac{\eta-\eta'}{t}\right) - q_\xi^*(\xi^{*\prime})\,\Phi\left(\frac{\xi-\xi'}{t},\frac{\eta-\eta'}{t}\right) d\xi^{*\prime} \quad (3b)$$

with the dimensionless quantities $\xi^* = \xi/\xi_2$ and

$$\gamma_\xi^*(\xi^*) = \frac{\xi_2\sqrt{1+\tan^2\beta_0}}{\Gamma}\,\gamma_x(\xi^*) \quad (4a)$$

and

$$q_\xi^*(\xi^*) = \frac{\xi_2\sqrt{1+\tan^2\beta_0}}{\Gamma}\,q_x(\xi^*). \quad (4b)$$

In the integrals (3a) and (3b), as is well known, the Cauchy value should be reckoned with [2].

In point (P) of the chord, the average velocities along and normal to the chord, respectively, with only the velocities given by integrals (3) are taken into consideration, and can be calculated from the expressions

$$v_x = \frac{1}{\sqrt{1+\tan^2\beta_0}}\left[c_{\infty\xi} + c_{L\xi} + \tan\beta_0\,(c_{\infty\eta} + c_{L\eta})\right] \quad (5)$$

and

$$v_y = \frac{1}{\sqrt{1+\tan^2\beta_0}}\left[c_{\infty\eta} + c_{L\eta} - \tan\beta_0\,(c_{\infty\xi} + c_{L\xi})\right] \quad (6)$$

if the angle of chord is β_0. Consequently, the velocities on the "f" and "a" side respectively, of the chord (see Fig. 1) are

$$c_{xf} = v_x - \frac{\gamma_x}{2}; \quad c_{xa} = v_x + \frac{\gamma_x}{2} \quad (7a)$$

and

$$c_{yf} = v_y + \frac{q_x}{2}; \quad c_{ya} = v_y - \frac{q_x}{2} \quad (7b)$$

The terminals of curve (S) and its chord must coincide, therefore the condition

$$\int_0^{x_2}\left(v_y + \frac{q_x}{2}\right) dx = 0 \quad (8)$$

must exist, that is, since the condition of a closed profile contour is

$$\int_0^{x_2} q_x \, dx = 0$$

on the basis of $dx = \xi_2 \sqrt{1+\tan^2\beta_0}\, d\xi^*$

$$\int_0^1 v_y \, d\xi^* = 0 \tag{9}$$

In order to ensure the condition referred to, let us introduce distribution

$$\gamma_\xi^* = \gamma_{\xi 1}^* + \varkappa\, \gamma_{\xi 2}^* \tag{10}$$

with the assumptions

$$\int_0^1 \gamma_{\xi 1}^* \, d\xi^* = 1 \tag{11a}$$

and

$$\int_0^1 \gamma_{\xi 2}^* \, d\xi^* = 0 \tag{11b}$$

and let us assume, according to Equations (3), that

$$c_{L\xi 1}(\xi) = \frac{\Gamma}{t}\int_0^1 [\gamma_{\xi 1}^* \Phi + q_\xi^* \Psi]\, d\xi^{*\prime} \tag{12a}$$

and

$$c_{L\xi 2}(\xi) = \frac{\Gamma}{t}\int_0^1 \gamma_{\xi 2}^* \Phi \, d\xi^{*\prime} \tag{12b}$$

that is

$$c_{L\eta 1}(\xi) = \frac{\Gamma}{t}\int_0^1 [\gamma_\xi^* \Psi - q_{\xi 1}^* \Phi]\, d\xi^{*\prime} \tag{12c}$$

and

$$c_{L\eta 2}(\xi) = \frac{\Gamma}{t}\int_0^1 \gamma_{\xi 2}^* \Psi \, d\xi^{*\prime} \tag{12d}$$

finally, $c_{L\xi} = c_{L\xi 1} + \varkappa c_{L\xi 2}$ and

$$c_{L\eta} = c_{L\eta 1} + \varkappa c_{L\eta 2} \tag{13}$$

By making use of distribution (10), the condition may be written as

$$\varkappa = \frac{\int\limits_0^1 \left[c_{\infty\eta} + c_{L\eta 1} - \tan\beta_0 \left(c_{\infty\xi} + c_{L\xi 1}\right)\right] \mathrm{d}\xi^*}{\int\limits_0^1 \left(\tan\beta_0\, c_{L\xi 2} - c_{L\eta 2}\right) \mathrm{d}\xi^*} \tag{14}$$

and there are no difficulties to be found in its computation.

Determination of the physically feasible singularity carrier curve

Producing the vortex distribution as resolved according to (10) with the stipulations of Equations (11), Equation (14) permits the calculation of $\varkappa$, providing for the coincidence of the terminals of both singularity carrier curves [2]. The value of $\varkappa$ should preferably be as low as possible. According to the experiences gathered, with numerical calculations satisfactory results can be achieved if, following the determination of the infinitely dense blading for the calculation of $\tan\beta_0$, [5], the chord of the former is considered as the singularity carrier auxiliary curve then, by calculating the velocities induced from Equations (12), the value of $\varkappa$ is determined from Equation (14). Employing the vortex distribution according to (10), the straight section of $\tan\beta_0$ gradient will represent the chord of the physically acceptable singularity carrier curve (S) yet unknown.

Determination of curve (S) is arrived at through the following train of thought:

According to Fig. 2, the analytical continuation of the flow through curve (S) to the chord, and expressing the continuity equation for curve I, would give

$$\int\limits_0^x \left(-v_y + \frac{q_x}{2}\right) \mathrm{d}x - \int\limits_0^s \frac{q_s}{2}\, \mathrm{d}s = \int\limits_{y_s}^0 c_x(x, y)\, \mathrm{d}y =$$

$$= -\left[c_{xa}(x, 0)\, y_s + \frac{\partial c_{xa}}{\partial y}(x, 0)\frac{y_s^2}{2!} + \dots\right] \tag{15}$$

then the analytical continuation of the flow through the chord to curve (S) would give for curve II

$$\int_0^x \left(- v_y - \frac{q_x}{2}\right) dx + \int_0^s \frac{q_s}{2} ds = \int_{y_s}^0 c_x (x, y) dy =$$

$$= - \left[c_{xf} (x, 0) y_s + \frac{\partial c_{xf}}{\partial y} (x, 0) \frac{y_s^2}{2!} + \ldots \right] \tag{16}$$

where indices "a" and "f" mean the sides of the chord according to Fig. 1.

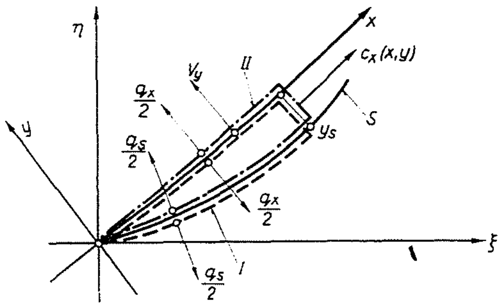

Fig. 2

With respect to the source and vortex-free nature of the flow,

$$\frac{\partial c_x}{\partial x} + \frac{\partial c_y}{\partial y} = 0 \tag{17a}$$

and

$$\frac{\partial c_y}{\partial x} - \frac{\partial c_x}{\partial y} = 0 \tag{17b}$$

Equations (15) and (16) can be transformed by making use of Equations (17) and (7). Thus

$$\int_0^x \left(- v_y + \frac{q_x}{2}\right) dx - \int_0^s \frac{q_s}{2} ds = - \left(v_x + \frac{\gamma_x}{2}\right) y_s - \frac{\partial}{\partial x} \left(v_y - \frac{q_x}{2}\right) \frac{y_s^2}{2!} +$$

$$+ \frac{\partial^2}{\partial x^2} \left(v_x + \frac{\gamma_x}{2}\right) \frac{y_s^3}{3!} - \ldots \tag{18}$$

may be written for curve I, and

$$\int_0^x \left(-v_y - \frac{q_x}{2}\right) \mathrm{d}x + \int_0^s \frac{q_s}{2}\,\mathrm{d}s = -\left(v_x - \frac{\gamma_x}{2}\right) y_s - \frac{\partial}{\partial x}\left(v_y + \frac{q_x}{2}\right)\frac{y_s^2}{2!} + \frac{\partial^2}{\partial x^2}\left(v_x - \frac{\gamma_x}{2}\right)\frac{y_s^3}{3\,!} - \ldots \tag{19}$$

for curve II.

Adding Equation (19) to (18) gives

$$\int_0^x v_y\,\mathrm{d}x = v_x y_s + \frac{\partial v_y}{\partial x}\frac{y_s^2}{2!} - \frac{\partial^2 v_x}{\partial x^2}\frac{y_s^3}{3!} + \ldots \tag{20}$$

Experiences with numerical calculations show that the sum of the right-hand side series is remarkably well approximated by the sum of the first three members, moreover, the first member alone represents a satisfactory approximation in many cases. Interrupting the right-hand side series after the third member would make the points of curve (S) represented by the minimum absolute value roots of cubic equations

$$\frac{1}{6}\frac{\partial^2 v_x}{\partial x^2}(x)\,y_s^3 - \frac{1}{2}\frac{\partial v_y}{\partial x}(x)\,y_s^2 - v_x(x)\,y_s - \int_0^x v_y(x)\,\mathrm{d}x = 0 \tag{21}$$

in the series of which no staggers are permissible. If there were any stagger encountered in the root series, the approximation of third degree could not be employed but a lower or higher order approximation should be used instead. The approximation of third degree can be used if

$$y_s \leqq y_k \tag{22}$$

where y_k is the lower absolute value real root of the quadratic equation

$$\frac{1}{2}\frac{\partial^2 v_x}{\partial x^2}(x)\,y^2 - \frac{\partial v_y}{\partial x}(x)\,y - v_x(x) = 0 \tag{23}$$

If there is no real root, the approximation of third degree may be used.

The points of curve (S) can thus be determined by means of Equation (21) if the coefficients of the equation are known. Their determination is, in turn, affected by an important factor not discussed so far. Taking the difference of Equations (18) and (19), and interrupting the right-hand side series again, would give equation

$$\int_0^x q_x\,\mathrm{d}x = \int_0^s q_s\,\mathrm{d}s - \gamma_x y_s + \frac{\partial q_x}{\partial x}\frac{y_s^2}{2!} + \frac{\partial^2 \gamma_x}{\partial x^2}\frac{y_s^3}{3!} \tag{24}$$

Distribution $q_x(x)$ and, according to Equations (4b), distribution $q_\xi^*(\xi^*)$ may not be selected arbitrarily but they must satisfy the requirements of Equation (24). Consequently, in case of a given distribution of

$$F(x) = \int_0^s q_s \, \mathrm{d}s \tag{25}$$

the approximation

$$\int_0^x q_x \, \mathrm{d}x \cong F(x) \tag{26}$$

is permissible only for arcs of very small camber, that is, it might be considered as a 0th approximation starting from which an iteration process could be constructed to satisfy (24). The fundamental idea underlying this process is as follows: the velocities induced and curve (S) are calculated with approximation (26) accepted. Knowing curve (S), a better approximation of the source distribution is calculated from (24), followed by the re-calculation of curve (S). This is repeated until recalculation results in the recovery of curve (S). According to the experiences, with numerical calculations this process is of an extremely rapid rate convergence.

If approximation (26) is accepted, curve (S) can be directly calculated without iteration, and this corresponds to the classical utilization of the chord [3] when calculating slightly curved blades. Making use of the iteration, and satisfying Equation (24) with distribution $q_x(x)$, will render the process equivalent to that method where the singularity distribution would have been imposed to curve (S), [6], but the required calculation is much less here than with the methods generally used, since the shape of the carrier curve does not change in the course of the iteration and, for example, the influence functions of intregrals (3) must be determined only once.

In case of highly cambered thick blades, the advantages of using the singularity carrier auxiliary curve are increasingly manifested. This will be dealt with by a subsequent paper.

Determination of the profile contour

The curve (S) definable on the basis of the previous chapter is, due to the conditions and stipulations described, entirely within the profile. An infinitely thin blade curve might be represented by curve (S) itself.

In the neighbourhood of the chord, through the expansion in series employed for expressing Equations (15) and (16), the rate of flow between curve

(S) and the contour would be, as expressed by the symbols of Fig. 3 and, for the approximation accepted earlier, by the symbols of Equation (25)

$$\int_0^s \frac{q_s}{2}\,ds = \frac{1}{2}F(x) \cong \left(v_x - \frac{\gamma_x}{2}\right)(y_{kn} - y_s) + \frac{1}{2}\frac{\partial}{\partial x}\left(v_y + \frac{q_x}{2}\right)\cdot(y_{kn}^2 - y_s^2) -$$
$$- \frac{1}{6}\frac{\partial^2}{\partial x^2}\left(v_x - \frac{\gamma_x}{2}\right)(y_{kn}^3 - y_s^3) \tag{27}$$

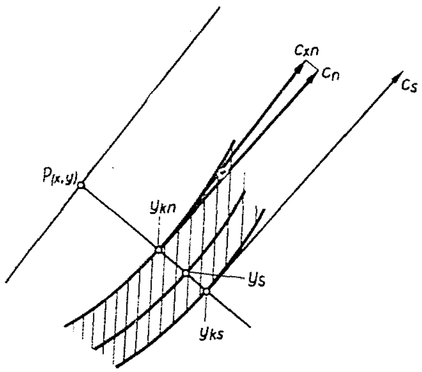

Fig. 3

and

$$\frac{1}{2}F(x) \cong \left(v_x + \frac{\gamma_x}{2}\right)(y_s - y_{ks}) + \frac{1}{2}\frac{\partial}{\partial x}\left(v_y - \frac{q_x}{2}\right)\cdot(y_s^2 - y_{ks}^2) -$$
$$- \frac{1}{6}\frac{\partial^2}{\partial x^2}\left(v_x + \frac{\gamma_x}{2}\right)(y_s^3 - y_{ks}^3) \tag{28}$$

The contour points of the pressure side are thus given by the solution of Equations (27) while those of the suction side by the solution of Equations (28), since only y_{kn} and y_{ks} are unknown therein. And since there are thin blades which are being dealt with, the nose point of the profile is considered identical to the initial point of the singularity carrier curve (S). Profile nose shapes in case of high-arc thick blades will be discussed in a subsequent paper.

Calculation of the velocity distribution along the contour

On grounds of the foregoing, the contour point of an y_{kn} co-ordinate reveals that

$$c_{xn} \cong \left(v_x - \frac{\gamma_x}{2}\right) + \frac{\partial}{\partial x}\left(v_y + \frac{q_x}{2}\right)y_{kn} -$$
$$- \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(v_x - \frac{\gamma_x}{2}\right)y_{kn}^2 \tag{29}$$

and point y_{ks} that

$$c_{xs} \cong \left(v_x + \frac{\gamma_x}{2}\right) + \frac{\partial}{\partial x}\left(v_y - \frac{q_x}{2}\right) y_{ks} -$$

$$- \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(v_x + \frac{\gamma_x}{2}\right) y_{ks}^2 \tag{30}$$

Since there is no velocity component normal to the contour, the pressure side velocity is obviously

$$c_n \cong \sqrt{1 + y'_{kn}(x)^2}\, c_{xn} \tag{31}$$

while the suction side velocity appears to be

$$c_s \cong \sqrt{1 + y'_{ks}(x)^2}\, c_{xs} \tag{32}$$

Summary

Instead of the application of a physically feasible singularity carrier curve section, the paper presents that of a singularity carrier auxiliary curve. Selecting the chord of the physically feasible curve section as the singularity carrier auxiliary curve gives a relatively simple method for the calculation of low-arc straight plain cascades of blades. The source distribution required over the chord can be determined by iteration, whereas the profile contour points are given as solutions of cubic equations.

References

1. FEINDT, E. G.: Beitrag zu den Grundlagen des Singularitätenverfahrens der Strömungsmechanik. Z. für Flugwissenschaft **10**, (1962).
2. FŰZY, O.: The employment of singularity carrier auxiliary curves for blade profile design, Per. Polytechnica **10**, 223 (1966).
3. BIRNBAUM, W.: Die Tragende Wirbelfläche als Hilfsmittel zur Behandlung des ebenen Problems der Tragflügeltheorie ZAMM, 1923.
4. SCHOLZ, N.: Strömungsuntersuchungen an Schaufelgittern. VDI Forschungsheft, 442.
5. FŰZY, O.: Design of mixed flow impeller. Per. Polytechnica **VI**, 299 (1963).
6. CZIBERE, T.: Berechnungsverfahren zum Entwurfe gerader Flügelgitter mit stark gewölbten Profilschaufeln. Acta Technica Ac. Sci. Hung. **XXVIII**, 241 (1960).

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