DIRECT AND INVERSE METHOD OF CALCULATING ROTATING CASCADES WITH AN INFINITE NUMBER OF BLADES AND RADIAL FLOW

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1. The method of singularities is one of the most widespread methods for the calculation of rotating radial cascades. Numerous variants have been developed to suit the specific conditions in hand.

The calculations assume two-dimensional potential flow.

In dimensioning radial-flow fans with backward-curved impeller blades, incompressibility of the medium and infinitely thin blading may be assumed, while for the singularity carrier line the logarithmic spiral based on the leading and trailing edges of the blade which fairly resemble the blade curve, may be chosen by way of approximation [1], [2].

In the calculation of fans with radial and forward-curved blades, although the logarithmic spiral meeting the blade tips still continues to have importance, the carrier of singularity will be the blade curve itself [3].

The calculations are of iterative character. To accelerate convergence, preliminary calculations should be performed [4]. The present paper deals with the direct (design) and inverse methods of the calculation of radial fans, assuming infinite number of blades so as to enhance convergence.

2. A cascade with N number of blades lying in the plane z is transformed into a straight cascade of the spacing t, in the plane ζ, by the conformal mapping function:

$$\zeta = \frac{Nt}{2\pi} \ln z$$

(Fig. 1). In the course of mapping according to the known method, we obtained a set of straight lines from the circles of the radius r respectively, of the logarithmic spirals of the angle β, enclosing in the plane z the β angle in the plane ζ. The points of intersection P_1 and P_2 are the terminal points of the blade curve g_1 approximating the logarithmic spiral g_2, with the g_1' line section corresponding to the blade in the plane ζ.

The velocity pattern in the rotating radial blade casvade in the plane z is the sum of the following two components:

a) the velocity c_{0z} produced by the source of Q/b strength in relation to unity of impeller width placed at the zero point of the coordinate system and

b) the velocity c_{i_z} induced by the γ vortex distribution on the carrier line of singularities replacing the blades.

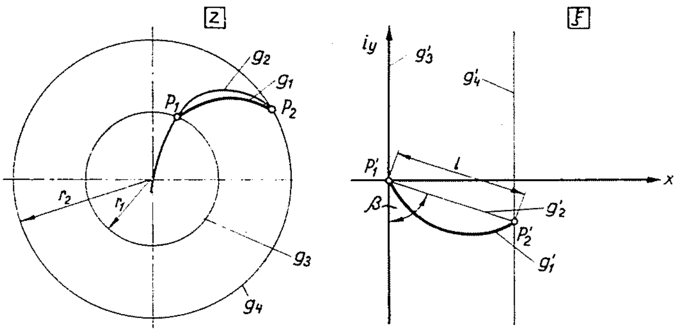

Fig. 1

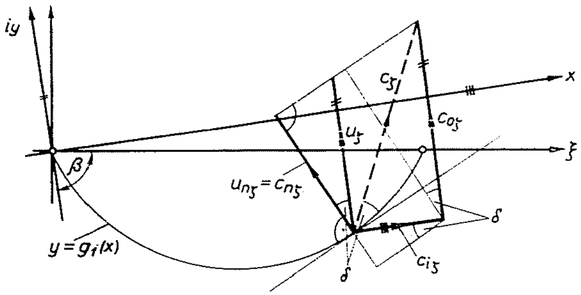

Fig. 2

To determine the velocities on the ξ plane z, the following may be written:

$$c_\zeta = c_z \left| \frac{\mathrm{d}z}{\mathrm{d}_\zeta} \right|$$

in which, by the transformation used:

$$\left. \frac{\zeta}{z} \right| = \frac{Nt}{2r\pi}$$

In determining the blade curve we started out on the kinematic assumption that no fluid flows through the blade line. According to Fig. 2 the following relation may be written for this assumption in the plane:

$$u_{n\zeta} = u_\zeta \cos\delta = c_{n\zeta}$$

viz. the absolute and the rotor velocity components normal to the blade tangent are equal. And since

$$c_{n\zeta} = c_{i\zeta} \cos \delta - c_{0\zeta} \sin \delta$$

the following may be written:

$$\operatorname{tg} \delta = \frac{c_{i\zeta} - u_{\zeta}}{c_{0\zeta}} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

The slope of the blade line is given by the differential quotient of the blade curve $y = g_1(x)$.

It is preferable to carry out the calculations throughout with dimensionless quantities.

To render the velocities in the plane ζ dimensionless the $u_{2\zeta}$ rotor velocity relative to the radius r_2 while to make the lengths dimensionless, the blade chord length determined by the points P_1' and P_2' (Fig. 1) has been applied. Accordingly:

$$\operatorname{tg} \delta = \frac{\left(\mathrm{d}\,\frac{y}{l}\right)}{\left(\mathrm{d}\,\frac{x}{l}\right)} = \frac{\frac{c_{i\zeta}}{u_2} - \frac{u_{\zeta}}{u_{2\zeta}}}{\frac{c_{0\zeta}}{u_{2\zeta}}} \tag{1}$$

The vortex distribution γ should be assumed as the function of the blade chord length ξ/l, respectively, as the function of the ξ/l angle associated to the value ε as per Fig. 3, in the manner suggested by SCHLICHTING [5], in the form of the following Glauert series:

$$\frac{\gamma}{u_{2\zeta}} = \frac{t}{l}\,\frac{2\,\varphi^x}{\pi}\left(\frac{A_0}{2}\,\operatorname{cotg}\,\frac{\varepsilon}{2} + \sum_{1}^{n} A_n \sin n\varepsilon\right)^* \tag{2}$$

* The dimensionless power factors of the impeller are

a) the flow number:

$$\varphi^x = \frac{Q/b}{2\, r_2\, \pi\, u_2}$$

b) the ideal head number:

$$\psi_i = \frac{\Delta\, p_{\ddot{o}i}}{\frac{\varrho}{2}\, u_2^2}$$

where $\Delta p_{\ddot{o}i}$ denotes the ideal total pressure increase produced by the impeller and ϱ the density of the flow medium.

This assumption of the vortex distribution is justified for more points than one. First, the computation of the induced velocities is simple if the blade chord is the carrier of singularity, secondly this assumption enables the easy handling of the inverse problem.

The assumption of an infinite number of blades means a uniform spacing of the singularities over the bladed section of plane, in stripes with a width

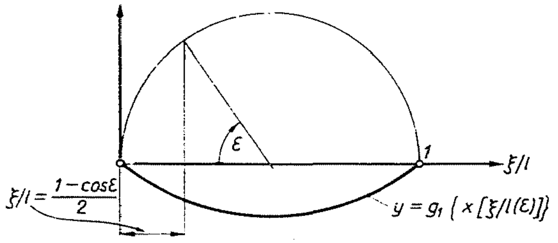

Fig. 3

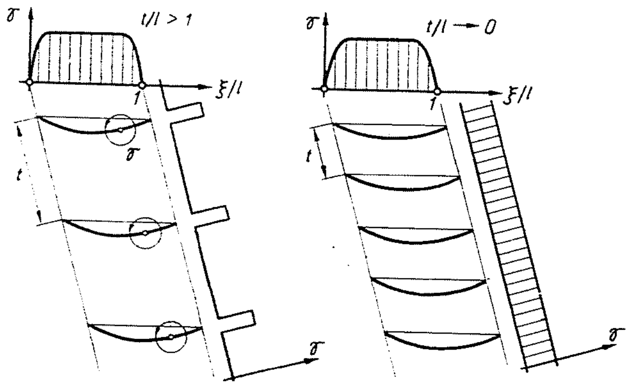

Fig. 4

of dr, respectively, d(ζ/l) (Fig. 4). According to this assumption the induced velocity can be computed in the following way:

$$\frac{c_{i\zeta}}{u_{2\zeta}} = \frac{l}{t} \int_0^{\xi/l} \frac{\gamma}{u_{2\zeta}} \, \mathrm{d}\,(\xi/l) \tag{3}$$

The velocities $c_{0\zeta}$ and u_ζ are determined by the following equations:

$$\frac{c_{0\zeta}}{u_{2\zeta}} = \varphi^x$$

$$\frac{u_\zeta}{u_{2\zeta}} = \left(\frac{r}{r_2}\right)^2$$

3. In the case of a concrete design we start out from the given values of φ_0^* and ψ_{i0} and r_2/r_1, taking the constant A_0 standing for the vortex distribution in the Glauert series to be zero. In this case the stagnation point falls on the blade edge; obviously ensuring the most favourable flow pattern around the blade.

The blade circulation can be calculated according to the following relationship:

$$\Gamma = u_{2\zeta} l \int_0^1 \frac{\gamma}{u_2} \mathrm{d}\,(\xi/l) \tag{4}$$

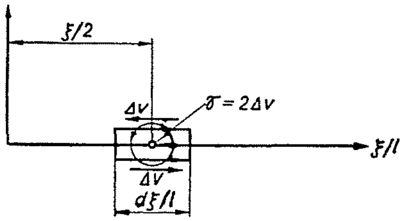

Fig. 5

It shows that the magnitude of the total circulation around the blade is determined by the constant A_1. This constant can be derived from the basic design data ψ_{0i} and φ_0^* in the following way:

$$A_1 = \frac{\psi_{i0}}{\varphi_0^x}$$

This leads to the vortex distribution:

$$\frac{\gamma}{u_{2\zeta}} = \frac{t}{l}\,\frac{2\,\psi_{i0}}{\pi}\left(\sin\varepsilon + \sum_2^n \frac{A_n}{A_1}\sin n\varepsilon\right) \tag{5}$$

Turning to dimensionless quantities with

$$\Delta v = u_{2\zeta}\frac{t}{l}\,\frac{\psi_{i0}}{4}$$

shown on Fig. 5, we get

$$\frac{\gamma}{2\Delta\,v} = \frac{4}{\pi}\left(\sin\varepsilon + \sum_2^n \frac{A_n}{A_1}\sin n\varepsilon\right)$$

For the vortex distribution we may write

$$\int_0^1 \frac{\gamma}{2\Delta\,v}\,\mathrm{d}\,(\xi/l) = 1.$$

In the Glauert series the (n) number of members and the ratio A_n/A_1 of the constants are arbitrarily chosen, to suit the case in hand.

To compute the induced velocity, the relationship for the vortex distribution (5) is substituted into equation (3), so, after integration we obtain the following relationship:

$$\frac{c_{i\zeta}}{u_{2\zeta}} = \frac{\psi_{i0}}{2\pi}\left[\varepsilon - \frac{\sin 2\varepsilon}{2} + \sum_{2}^{n}\frac{A_n}{2A_1}\left(\frac{\sin(n-1)\varepsilon}{n-1} - \frac{\sin(n+1)\varepsilon}{n+1}\right)\right]$$

Substituting the velocity components into equation (1):

$$\frac{\mathrm{d}(y/l)}{\mathrm{d}(x/l)} = \frac{\psi_{i0}}{2\pi\varphi_0^x}\left[\varepsilon - \frac{\sin 2\varepsilon}{2} + \sum_{2}^{n}\frac{A_n}{A_1}\left(\frac{\sin(n-1)\varepsilon}{n-1} - \frac{\sin(n+1)\varepsilon}{n+1}\right)\right] - \frac{1}{\varphi_0^x}\left(\frac{r}{r_2}\right)^2$$

Substituting now from the above equation

$$\mathrm{d}\left(\frac{x}{l}\right) = \sin\beta\,\mathrm{d}(\xi/l)$$

the blade curve can be computed:

$$\frac{y/l}{\sin\beta} = \frac{\psi_{i0}}{2\pi\varphi_0^x}\int_0^{\xi/l}\left[\varepsilon - \frac{\sin 2\varepsilon}{2} + \sum_{2}^{n}\frac{A_n}{A_1}\left(\frac{\sin(\mathrm{n}-1)\varepsilon}{n-1} - \frac{\sin(n+1)\varepsilon}{n+1}\right)\right]\mathrm{d}(\xi/l) - \frac{1}{\varphi_0^x}\int_0^{\xi/l}\left(\frac{r}{r_2}\right)^2\mathrm{d}(\xi/l)$$

Integration will yield the following result:

$$\frac{y/l}{\sin\beta} = \frac{\psi_{i0}}{4\pi\varphi_0^x}\left\{\frac{3}{4}\sin\varepsilon + \frac{1}{12}\sin 3\varepsilon - \varepsilon\cos\varepsilon + \frac{A_2}{2A_1}\left(\varepsilon - \frac{2}{3}\sin 2\varepsilon + \frac{1}{12}\sin 4\varepsilon\right) + \sum_{3}^{n}\frac{A_n}{A}\left[\frac{\sin(n-2)\varepsilon}{2(n-2)n-1} - \frac{\sin n\varepsilon}{(n+1)(n-1)} + \frac{\sin(n+2)\varepsilon}{2(n+2)(n+1)}\right]\right\} - \frac{\left(\frac{r}{r_1}\right)^2 - 1}{2\varphi_0^x\left(\frac{r_2}{r_1}\right)^2\ln\frac{r_2}{r_1}} \qquad (6)$$

With the values at the blade tip $\varepsilon = \pi$ and $r = r_2$:

$$\frac{y_0/l}{\sin\beta} = \frac{\psi_{i0}}{4\,\varphi_0^x}\left(1 + \frac{A_2}{2\,A_1}\right) - \frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{2\,\varphi_0^x\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}}$$

By the expression $y_0/l = -\cos\beta$, the angle of the logarithmic spiral can be determined in the following manner:

$$\frac{1}{tg\,\beta} = \frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{2\,\varphi_0^x\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}} - \frac{\psi_{i0}}{4\,\varphi_0^x}\left(1 + \frac{A_2}{2\,A_1}\right) \tag{7}$$

which makes it evident that the β angle of the logarithmic spiral crossing the blade tips, in addition to the φ_0^* and ψ_{i0} power factors and the diameter ratio $\frac{r_2}{r_1}$, depends on the value of the constants A_2 and A_1 of the series describing the distribution of vorticity.

The calculations performed on the assumption of infinite number of blades can be utilised in the calculations assuming a finite number of blades, for the approximate determination of the β angle of the logarithmic spiral crossing the blade tips and the close-to-accurate determination of the blade curve.

The vortex distributions most frequently used in design are

a) elliptic circulation distribution:

$$\frac{\gamma}{2\,\Delta v} = \frac{4}{\pi}\sin\varepsilon$$

b) constant circulation distribution in the plane ζ:

$$\frac{\gamma}{2\,\Delta v} = 1$$

Should the vortex distribution be given in the form of the function $\gamma = f(\zeta/l)$, the constants of the Fourier series can be determined according to the following relationship:

$$A_n = \frac{1}{2}\int_0^\pi \sin n\,\varepsilon \cdot f(\zeta/l)\,d\varepsilon$$

With constant vortex distribution $A_n = \frac{1}{n}$, however, the values of n are, of necessity, odd numbers $n = 1, 3, 5, 7, \ldots$ Consequently, vortex distribution can be written in the following form:

$$\frac{\gamma}{2\,\Delta v} = \frac{4}{\pi}\left[\sin\varepsilon + \frac{1}{3}\sin 3\,\varepsilon + \frac{1}{5}\sin 5\,\varepsilon + \ldots\right]$$

With vortex distribution mentioned in paras *a)* and *b)* the β angle is easily calculable:

$$\frac{1}{\operatorname{tg}\beta} = \frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{2\,\varphi_0^x\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}} - \frac{\psi_{i0}}{4\,\varphi_0^x} \tag{8}$$

From (7) it can be established that the β angle can be ascertained from equation (8) for symmetric vortex distribution.

c) Constant distribution of the circulation in the plane z can be expressed in the following form:

$$\frac{\gamma}{2\,\Delta v} = \frac{\ln\frac{r_2}{r_1}}{\frac{r_2}{r_1} - 1}\,\frac{r}{r_1} = \frac{\ln\frac{r_2}{r_1}}{\frac{r_2}{r_1} - 1}\, e^{\xi/l \ln\frac{r_2}{r_1}}$$

Hence:

$$A_n = \frac{\ln\frac{r_2}{r_1}}{2\left(\frac{r_2}{r_1} - 1\right)}\int_0^{\pi} \sin n\,\varepsilon\, e^{\xi/l \ln\frac{r_2}{r_1}}\, d\,\varepsilon\,.$$

The constant required for our calculations is:

$$A_2 = \frac{1}{2}\left[\frac{1 + \frac{r_2}{r_1}}{1 - \frac{r_2}{r_1}} + \frac{2}{\ln\frac{r_2}{r_1}}\right]$$

whereby the β angle is as follows:

$$\frac{1}{\operatorname{tg}\beta} = \frac{\left(\left(\frac{r_2}{r_1}\right)^2 - 1\right)}{2\,\varphi_0^x\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}} - \frac{\psi_{i0}}{2\,\varphi_0^x}\left(\frac{1}{\ln\frac{r_2}{r_1}} - \frac{1}{\frac{r_2}{r_1} - 1}\right) \tag{9}$$

4. In the inverse problem the optimum power factors have to be determined (ψ_{i0}, φ_0^x) and the ideal characteristics, starting from the known dimensions of the rotating cascade.

Induced velocity in such a case should be calculated so as to have the full Glauert series (2) in the relationship (3). Accordingly:

$$\frac{c_{i\zeta}}{u_{2\zeta}} = \frac{\varphi^x}{2\pi}\left[A_0(\varepsilon + \sin\varepsilon) + A_1\left(\varepsilon - \frac{\sin 2\varepsilon}{2}\right) + \right.$$
$$\left. + \sum_2^n A_n\left(\frac{\sin(n-1)\varepsilon}{n-1} - \frac{\sin(n+1)\varepsilon}{(n+1)}\right)\right]$$

For the blade curve, starting from equation (1) and applying the following symbols:

$$f_0(\varepsilon) = \sin\varepsilon - \varepsilon\cos\varepsilon + \frac{\varepsilon}{2} - \frac{1}{4}\sin 2\varepsilon$$

$$f_1(\varepsilon) = \frac{3}{4}\sin\varepsilon - \varepsilon\cos\varepsilon + \frac{1}{12}\sin 3\varepsilon$$

$$f_2(\varepsilon) = \frac{1}{2}\left(\varepsilon - \frac{2}{3}\sin 2\varepsilon + \frac{1}{12}\sin 4\varepsilon\right)$$

$$f_n(\varepsilon) = \frac{\sin(n-2)\varepsilon}{2(n-2)(n-1)} - \frac{\sin n\varepsilon}{(n+1)(n-1)} + \frac{\sin(n+2)\varepsilon}{2(n+2)(n+1)}$$

it may be written that:

$$\frac{y/l}{\sin\beta} = \frac{1}{4\pi}\left[A_0 f_0(\varepsilon) + A_1 f_1(\varepsilon) + A_2 f_2(\varepsilon) + \sum_3^n A_n f_n(\varepsilon)\right] -$$
$$- \frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{2\varphi_0^x\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}} \tag{10}$$

After rearrangement:

$$A_0 f_0(\varepsilon) + A_1 f_1(\varepsilon) + A_2 f_2(\varepsilon) + \sum_3^n A_n f_n(\varepsilon) =$$
$$= 4\pi\frac{y/l}{\sin\beta} + \frac{2\pi}{\varphi^x}\frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}} \tag{11}$$

From the blade curve (Fig. 6) determined with the related values $\frac{r}{r_1}$ and α, y/l, β, ε can be calculated:

$$\frac{y/l}{\sin\beta} = \frac{\alpha}{\ln\frac{r_2}{r_1}}$$

and

$$\cos\varepsilon = 1 - 2\frac{\ln\frac{r_2}{r_1}}{\ln\frac{r_2}{r_1}}$$

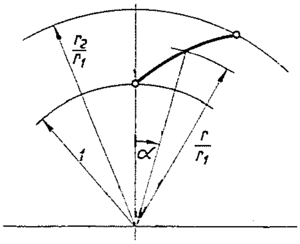

Fig. 6

The constants A_n and their factors $f_n[\varepsilon(r)]$ at a given point of the blade are fixed. Producing the Glauert series for the vortex distribution for n number of members and writing the relationship (11) for n points of the blades, a linear inhomogeneous system of equations of the nth order is obtained whose result yields the values of the constants A_n.

The system of equations can be written in matrix form, as follows:

$$\overline{\overline{E}}\,\overline{\overline{a}} = \alpha + \frac{1}{\varphi^x}\overline{\overline{r}} \tag{12}$$

in which the following symbols were used:

$$h(r) = 2\pi\frac{\left(\frac{r_2}{r_1}\right)^2 - 1}{\left(\frac{r_2}{r_1}\right)^2 \ln\frac{r_2}{r_1}}$$

$$\overline{\overline{E}} = \begin{bmatrix} f_0(\varepsilon_0) & f_1(\varepsilon_0) & fn(\varepsilon_n) \\ f_0(\varepsilon_1) & \cdot & \cdot \\ \cdot & \cdot & f_n(\varepsilon_{n-1}) \\ f_0(\varepsilon_n) & \cdot & f_n(\varepsilon_n) \end{bmatrix}$$

$$\bar{\bar{a}} = \begin{bmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} \qquad \bar{\bar{\alpha}} = \frac{4\pi}{\ln \frac{r_2}{r_1}} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \qquad \bar{\bar{r}} = \begin{bmatrix} h(r_0) \\ h(r_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h(r_n) \end{bmatrix}$$

As suggested by SCHLICHTING, it seems expedient to split the system of equations into one portion depending on φ^x and one independent of it. This will give the following form of the equation (12):

$$\bar{\bar{E}}\left(\bar{\bar{a}}_0 + \frac{1}{\varphi^x}\bar{\bar{a}}\,\varphi\right) = \bar{\bar{\alpha}} + \frac{1}{\varphi^x}\bar{\bar{r}} \tag{13}$$

in which

$$\bar{\bar{E}}\,\bar{\bar{a}}_0 = \bar{\bar{\alpha}}$$

and

$$\bar{\bar{E}}\,\bar{\bar{a}}\varphi = \bar{\bar{r}}$$

What was stated above proves that all Glauert series are constant and consist of the two following parts:

$$A_n = A_{n0} + \frac{1}{\varphi^x} A_n$$

The resolution can be found by the inversion of matrix $\bar{\bar{E}}$:

$$\bar{\bar{a}}_0 = \bar{\bar{E}}^{-1}\bar{\bar{\alpha}}$$

respectively:

$$\bar{\bar{a}}\,\varphi = \bar{\bar{E}}^{-1}\bar{\bar{r}}$$

With optimum flow conditions $A_0 = 0$ the head and flow numbers of the fan may be regarded as "design" numbers, representing the values ψ_{i0} and φ_0^x.

From the criterion $A_0 = 0$, φ_0^x can be determined in the following manner:

$$\varphi_0^x = -\frac{A_0}{A_{00}} \tag{14}$$

To obtain the ideal characteristics we may start from the blade circulation according to equation (4), which leads to the following result:

$$\frac{\psi_i}{\varphi^x} = A_0 + A_1$$

Substituting the split values of A_0 and A_1, we have the relationship:

$$\psi_i = A_0 \varphi + A_1 \varphi + \varphi^x (A_{00} + A_{10}) \tag{15}$$

which yields the ideal characteristics in the form of a straight line (Fig. 7).

Accordingly, the value of ψ_{i0} can be calculated from the following relationship:

$$\psi_{i0} = A_1 \varphi - \frac{A_0 \varphi \cdot A_{10}}{A_{00}} \tag{16}$$

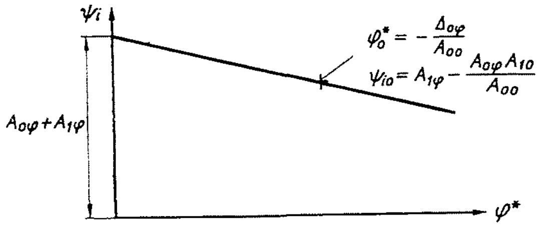

Fig. 7

By the inverse problem, assuming infinite number of blades, we have a fast process of approximation to determine the most favourable head and flow numbers and the ideal characteristics.

Summary

Calculating rotating blade cascades, mostly a finite number of blades is assumed while in design, the β angle of the logarithmic spiral should be assumed empirically or on the basis of approximation. Practice has shown that the calculation of the β angle on the assumption of an infinite number of blades helps in reducing the number of iteration steps.

The approximative processes used so far in the determination of the β angle ([4], [2]), yield information only on some special cases of vortex distribution. The present paper gives a clue to the approximation of the angle for an arbitrary distribution of vortex which enables development into Fourier series.

Writing the vortex distribution in the form of a Glauert series enables a ready and fast check on any given cascade.

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