# CRITICAL SIZE AND FLUX DISTRIBUTION OF HOLLOW SPHERICAL AND CYLINDRICAL REACTORS

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# Introduction

The literature in reactor physics [1, 2, 3] generally deals only with the critical size computation of compact spherical and cylindrical reactors. Nevertheless, there may be some points of view, that could justify the building of reactors, which do not contain either fuel or moderator in their centre. That is, such reactors have an "empty" space in their middle. As a matter of fact this is one of the advantageous ways, that the building of spherical reactors for energetic purposes can be imagined, because the system of central inflow and radial outflow of the cooling medium necessitated by thermo-dynamic viewpoints [4], requires an "empty" space in the centre. In the case of cylindrical reactors the radial cooling pipe system [5] as well as the manner of cooling with twin stream [6] having a central cooling inlet may also require the building of a reactor with a central cavity.

In this paper we shall describe a simple as well as an approximative method for the computation of the critical sizes of hollow spherical and cylindrical nuclear reactors. The critical size can be calculated by the so called one-group diffusion method, under the assumed following simplifying conditions:

- we assume bare reactors, without a reflector,

— we consider the central cavity as completely empty from reactorphysical viewpoint as well,  $\Sigma_a = 0$  inside the hollow space ( $\Sigma_a$  is the macroscopic absorption cross section),

- the reactor is homogeneous in the whole active zone,

- we regard the geometrical dimensions of the reactor as equal to its extrapolation dimensions.

### 1. Hollow spherical reactor

Fig. 1 demonstrates the structure of a reactor with a central cavity. Concerning this inner hollow spherical space the following diffusion equation can be accepted:

$$\Delta \Phi - \frac{\Sigma_a}{D} \Phi = 0 \tag{1.1}$$

But since, as we assumed,  $\Sigma_a = 0$ , therefore

$$\varDelta \Phi = 0$$

that is

$$\Phi = \text{const} = \Phi_0 \tag{1.2}$$



Fig. 1. Structure of the spherical reactor with a central cavity

Concerning the active zone encompassed by spherical surfaces  $r_0$  and R the critical diffusion equation, the so-called wave equation can be used

$$\Delta \Phi + B_g^2 \Phi = 0$$

or in spherical coordinates:

$$\frac{d^2 \Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} + B_g^2 \Phi = 0$$
(1.3)

The following marginal conditions have to be statisfied in order to solve the equation:

1. If 
$$r = r_0$$
,  $\Phi = \Phi_0$   
2. If  $r = r_0$ ,  $\frac{d\Phi}{dr} = 0$ , from  $D_0 \frac{d\Phi_0}{dr} = D \frac{d\Phi}{dr}$ 

(as the neutron flux is constant inside the empty space).

3. If r = R,  $\Phi = 0$ .

The general solution of Equ. (1.3) is

$$\Phi(r) = C_1 \frac{\sin B_g r}{r} + C_2 \frac{\cos B_g r}{r}$$
(1.4)

From the first condition:

$$\Phi_0 = C_1 \frac{\sin B_g r_0}{r_0} + C_2 \frac{\cos B_g r_0}{r_0}$$
(1.5)

From the second condition:

$$C_{1}\left[B_{g}\cos B_{g}r_{0} - \frac{\sin B_{g}r_{0}}{r_{0}}\right] - C_{2}\left[B_{g}\sin B_{g}r_{0} + \frac{\cos B_{g}r_{0}}{r_{0}}\right] = 0 \quad (1.6)$$

and from the third condition:

$$C_1 \frac{\sin B_g R}{R} + C_2 \frac{\cos B_g R}{R} = 0$$
 (1.7)

From Equs. (1.6) and (1.7) the geometrical buckling can be expressed in the following implicit form:

$$\operatorname{tg} B_{g} R = \frac{\operatorname{tg} B_{g} r_{0} - B_{g} r_{0}}{B_{g} r_{0} \cdot \operatorname{tg} B_{g} r_{0} + 1} \,,$$

or in simpler form

$$B_g r_0 = tg \left( B_g r_0 - B_g R \right)$$
 (1.8)

In order to determine the least "eigenvalues" of the geometrical buckling typical for the critical reactor we start out from the inverse of Equ. (1.8)

$$B_g r_0 - B_g R = \operatorname{Arc} \operatorname{tg} B_g r_0 + n\pi \tag{1.9}$$

from which it is evident, if n = 0 the solution is trivial and the least "eigenvalue" of  $B_g$  derives from n = -1

$$B_g r_0 - B_g R = \operatorname{Arc} \operatorname{tg} B_g r_0 - \pi \tag{1.10}$$

From this equation we obtain the known result  $B_g=\pi/R$  for compact reactor  $(r_0=0)$  .

From Equ. (1.10) the geometrical buckling cannot be expressed in explicit form, here we have to adopt, e.g. the way of graphical solution. For this purpose let us transform Equ. (1.10)

 $B_g R\left(rac{r_0}{R}-1
ight)=\mathrm{Arc} \ \mathrm{tg} \ B_g Rrac{r_0}{R}-\pi$ 

$$B_g R = \frac{R}{R - r_0} \left[ \pi - \operatorname{Arc} \operatorname{tg} B_g R \frac{r_0}{R} \right]$$
(1.11)

or

Fig. 2 shows the graphical solution, by introducing the symbol X for  $B_g R$ and the symbol Y for  $\frac{R}{R-r_0} \left[ \pi - \operatorname{Arc} \operatorname{tg} B_g R \frac{r_0}{R} \right]$ . The dotted line, connecting the intersecting points of curves X and Y shows the products of multiplication  $B_g R$  of the critical reactor in function of  $r_0/R$  (in case of compact sphere  $B_g R = \pi$ ). The graph in Fig 2 in a different scale also shows the relation of



Fig. 2. Graphical determination of the geometrical buckling of the hollow spherical reactor

the geometrical buckling of the hollow and compact spherical reactors  $(B_g/B_{g_0})$  in case of an identical outer radius (R = const.). It is evident that in case of identical outer radius the geometrical buckling of the hollow spherical reactor is greater than that of the compact one.

Assuming constant outer radius and knowing the variations of the geometrical buckling the comparative relations of the critical volume of the hollow spherical reactor to that of the compact one can be determined. Assuming the same matrices for both types of reactors the material buckling can be taken as constant in all cases ( $B_m = \text{const.}$ ). For a critical compact spherical reactor  $B_{g_0} = \frac{\pi}{R_0} = B_m$  and the critical volume of the reactor is:

$$V_{0} = \frac{4\pi}{3} R_{0}^{3} = \frac{4\pi}{3} \left(\frac{\pi}{B_{m}}\right)^{3} \simeq 130 \frac{1}{B_{m}^{3}}$$
(1.12)

In case of critical hollow spherical reactor the values of the geometrical buckling must likewise be concordant with that of the material buckling  $(B_g = B_m)$ . At a given outer radius  $(R = R_0)$  according to Fig. 2, the geometrical buckling  $(B_g)$  of the hollow spherical reactor is greater than that of the compact one  $(B_{g_0})$ . In order to become critical the hollow spherical reactor must increase its inner and outer radius at the same  $B_g/B_{g_0}$  rate, i.e.



Fig. 3. Critical volume ratio of the hollow and compact spherical reactors, in the function of  $r_0/R$ 

Therefore, the critical volume of the hollow spherical reactor is

$$V = \frac{4\pi}{3} \left( R^{\prime 3} - r^{\prime 3}_{0} \right) = \frac{4\pi}{3} \left[ 1 - \left( \frac{r_{0}}{R} \right)^{3} \right] R^{\prime 3} =$$
$$= \frac{4\pi}{3} \left[ 1 - \left( \frac{r_{0}}{R} \right)^{3} \right] R_{0}^{3} \left[ \frac{B_{g}}{B_{g0}} \right]^{3}$$
(1.14)

The comparative relation of the critical volume of the hollow spherical reactor and that of the compact one is given by the quotient of Equs. (1.14) and (1.12)

$$\frac{V_0}{V} = \left[1 - \left(\frac{r_0}{R}\right)^3\right] \left[\frac{B_g}{B_{g0}}\right]^3 \tag{1.15}$$

In Fig. 3 the variations of the critical volume ratio in function of  $\frac{r_0}{R}$  are given.

In order to determine the flux distribution in the hollow spherical reactor, the values of  $C_1$  and  $C_2$  can be obtained from Equs. (1.5) and (1.6). Putting these values into Equ. (1.4) we get the following form for the flux distribution:

$$\frac{\Phi(r)}{\Phi_0} = \frac{r_0}{\sin B_g r_0 - \operatorname{tg} B_g R \cdot \cos B_g r_0} \left[ \frac{\sin B_g r}{r} - \operatorname{tg} B_g R \frac{\cos B_g r}{r} \right] \quad (1.16)$$

Fig. 4. Flux distribution in compact and hollow spherical reactors  $(r_0/R = 0,2)$ 

for which the  $B_g$  values can be taken from Fig. 2 Equ. (1.16) if  $r \rightarrow 0$  gives as a result of the flux distribution of the compact spherical reactor.

$$\frac{\Phi(r)}{\Phi_0} = \frac{1}{B_g} \frac{\sin B_g r}{r} = \frac{R}{\pi} \frac{\sin \frac{\pi}{R} r}{r}$$
(1.17)

Fig. 4. shows the flux distribution in the case of  $r_0 = 0.2$  R.

#### 2. Cylindrical reactor with longitudinal cavity

Fig. 5 shows the line diagram of the cylindrical reactor with a longitudinal cavity. In the empty axial cylinder of  $r_0$  radius we consider the neutron flux as constant in the *r* direction but varying in the *z* direction, according to the value of the flux the  $r_0$  surface.

Concerning the cylindrical collet form active zone the wave equation can be expressed in cylindrical coordinates:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} + B_g^2 \Phi = 0$$
(2.1)

Separating the variables r and z in Equ. (2.1)  $[\Phi(r, z) = \Theta(r) \cdot Z(z)]$  and dividing with  $\Theta \cdot Z$  we obtain

$$\frac{1}{\Theta} \left[ \frac{d^2 \Theta}{dr^2} + \frac{1}{r} \frac{d\Theta}{dr} \right] + \frac{1}{Z} \left[ \frac{d^2 Z}{dz^2} \right] + B_g^2 = 0$$
(2.2)

Fig. 5. Structure of the cylindrical reactor with a longitudinal cavity

where the parts depending on r and z can each be equalized by a constant

$$\frac{1}{\Theta} \left[ \frac{d^2 \Theta}{dr^2} + \frac{1}{r} \frac{d\Theta}{dr} \right] = -a^2$$
(2.3)

and

$$\frac{1}{Z} \left[ \frac{d^2 Z}{dz^2} \right] = -\beta^2 \tag{2.4}$$

furthermore it can be demonstrated, that  $\alpha$  and  $\beta$  are positives.

The general solution of Equ. (2.3) is

$$\Theta = C_1 J_0(ar) + C_2 Y_0(ar) \tag{2.5}$$

where  $J_0$  and  $Y_0$  are the first and second Bessel functions, zero order. Marginal conditions in the z = 0 plane:

1. If  $r = r_0$ ,  $\Theta = \Theta_0 (\Phi = \Phi_0)$ 2. If  $r = r_0$ ,  $\frac{d\Theta}{dr} = 0$ 3. If r = R,  $\Theta = 0$  According to the first condition:

$$\Theta_0 = C_1 J_0(ar_0) + C_2 Y_0(ar_0) = \Phi_0$$
(2.6)

from the second condition

$$\frac{d\Theta}{dr} = a \left[ -C_1 J_1 (ar_0) - C_2 Y_1 (ar_0) \right] = 0$$

$$C_1 J_1 (ar_0) + C_2 Y_1 (ar_0) = 0$$
(2.7)



Fig. 6. Graphical determination of the product  $\alpha R$  in case of cylindrical reactor with longitudinal cavity

from the third condition

$$\Theta = C_1 J_0(aR) + C_2 Y(aR) = 0 \tag{2.8}$$

From Equs. (2.7) and (2.8)

$$\frac{J_1(ar_0)}{Y_1(ar_0)} = \frac{J_0(aR)}{Y_0(aR)}$$
(2.9)

The least eigenvalues of the product aR in function of  $r_0/R$  can be determined from Equ. (2.9) in a graphical way. Introducing the symbols

$$X = rac{J_1(ar_0)}{Y_1(ar_0)} \qquad ext{and} \qquad Y = rac{J_0(aR)}{Y_0(aR)}$$

Fig. 6 shows the graphical solution. The results of the solution are given in Fig. 7, where the diagram on the one hand shows the products aR in the

function of  $r_0/R$  and on the other hand the comparative relation of the  $\alpha$  factors of the hollow cylindrical reactor and that of the compact one.

The values of  $C_1$  and  $C_2$  can be determined by Equs. (2.6) and (2.7), by inserting these values into Equ. (2.5) we obtain the radial flux distribution in z = o plane (Z = 1):

$$\frac{\Phi(r)}{\Phi_{0}} = \frac{1}{J_{0}(ar_{0}) - \frac{J_{0}(aR)}{Y_{0}(aR)}Y_{0}(ar_{0})} \left[J_{0}(ar) - \frac{J_{0}(aR)}{Y_{0}(aR)}Y_{0}(ar)\right] = \frac{1}{J_{0}(ar_{0}) - \frac{J_{1}(ar_{0})}{Y_{1}(ar_{0})}Y_{0}(ar_{0})} \left[J_{0}(ar) - \frac{J_{1}(ar_{0})}{Y_{1}(ar_{0})}Y_{0}(ar)\right] \quad (2.10)$$

Fig. 8 shows the radial flux distribution of  $r_0 = 0.2R$ .

The solution of Equ. (2.4) containing the z variable is identical in the cases of both the hollow cylindrical reactor and of the compact one, i.e.

$$\beta^2 = \left(\frac{\pi}{H}\right)^2 \tag{2.11}$$

For the flux distribution of the whole active zone it can be stated:

$$\frac{\Phi(rz)}{\Phi_{n}} = \frac{1}{J_{0}(ar_{0}) - \frac{J_{0}(aR)}{Y_{0}(aR)} Y_{0}(ar_{0})} \left[ J_{0}(ar) - \frac{J_{0}(aR)}{Y_{0}(aR)} Y_{0}(ar) \right] \cdot \cos\frac{\pi}{H} z$$
(2.12)

The geometrical buckling of the critical hollow reactor can be determined by the equation:

$$P_g^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{aR}{R}\right)^2 \tag{2.13}$$

where the aR values in function of  $r_0/R$  are taken from Fig. 7. The critical volume of the hollow cylindrical reactor can be determined with the knowledge of the material buckling ( $B_g = B_m$ ). From Equ. (2.13)

$$R^{2} = \frac{H^{2} (aR)^{2}}{H^{2} B_{m}^{2} - \pi^{2}}$$
(2.14)

and so the critical volume is

$$V = \left[1 - \left(\frac{r_0}{R}\right)^2\right] R^2 \pi H = \left[1 - \left(\frac{r_0}{R}\right)^2\right] \frac{H^3 (\alpha R)^2 \pi}{H^2 B_m^2 - \pi^2}$$
(2.15)

The condition of the minimal critical volume:

$$\frac{dV}{dH} = 0 \tag{2.16}$$

As  $B_g^2$  too is a function of H it is complicated to solve Equ. (2.16).



Fig. 7. Variation of  $\alpha R$  in function of  $r_0/R$  for cylindrical reactor with a longitudinal cavity



Fig. 8. Flux distribution in the compact cylindrical reactor and in the cylindrical reactor with longitudinal cavity  $(r_{\eta}/R = 0.2)$ 

As an approximation we assume even in the case of the hollow cylindrical reactor the optimal H = 1,847 R ratio valid only in case of the compact cylindrical reactor. Consequently, using Equ. (2.14)

$$H = 1,847 \left| \sqrt{\frac{H^2 (aR)^2}{H^2 B_m^2 - \pi^2}}, \right|$$

$$H = \left| \sqrt{\frac{1,847^2 (aR)^2 + \pi^2}{B_m^2}}, \right|$$

$$(2.17)$$

$$\frac{25}{\frac{V}{V_0}}$$

$$\frac{15}{100} + \frac{100}{100} +$$

Fig. 9. Comparative relations of the critical volumes of the cylindrical reactor with a longitudinal cavity and of the compact one, in function of  $r_0/R$  (if H = 1.847 R)

Putting this into Equ. (2.15) the critical volume of the hollow cylindrical reactor is:

$$V = \left[1 - \left(\frac{r_0}{R}\right)^2\right] \frac{1,847 \pi \left[\left(\frac{\pi}{1,847}\right)^2 + (\alpha R)^2\right]^{3/2}}{B_m^3}$$
(2.18)

As an illustration let us compare the critical volume of the hollow cylindrical reactor with that of the compact one  $(V_0 = 148/B_m^2)$ 

$$\frac{V}{V_0} = \left[1 - \left(\frac{r_0}{R}\right)^2\right] \frac{1,847^2 \pi \left[\left(\frac{\pi}{1,847}\right)^2 + (aR)^2\right]^{3/2}}{148}$$
(2.19)

Fig. 9 shows the comparative relation of the critical volumes as computed from Equ. (2.19). It is to be noted that the minimal critical size of the

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and from this:

hollow cylindrical reactor is somewhat smaller than that shown in Fig. 9, as we have not taken into consideration the optimal R/H ratio peculiar of the hollow cylindrical reactors.

#### 3. Cylindrical reactor with a transverse cavity

Fig. 10 shows the structure of the cylindrical reactor having a transverse cavity, i.e. two cylindrical zones. Inside the hollow space of  $h_0$  height and of



Fig. 10. Structure of the reactor with transverse cavity

R radius the neutron flux can be considered as constant in Z direction and varying in r direction, according to the values of the flux on the  $z = \frac{h_0}{2}$  plane surface.

For the dual cylindrical active zone the same wave equation can be used as in the case of Equ. (2.1), the unchanged form can also be used in Equs. (2.2), (2.3) and (2.4).

The solution of Equ. (2.3) is the same in this case as in that of the compact cylindrical reactor, i.e.

$$a^2 = \left(\frac{2,405}{R}\right)^2$$
(3.1)

The general solution of Equ. (2.4) in this case is

$$Z = C_1 \sin \beta z + C_2 \cos \beta z \tag{3.2}$$

Marginal conditions in r = 0 axis

1. If 
$$z = h_0/2$$
,  $Z = Z_0 (\Phi = \Phi_0)$   
2. If  $z = h_0/2$ ,  $\frac{dZ}{dz} = 0$   
3. If  $z = H/2$ ,  $Z = 0$ 

According to the first condition

$$Z_0 = C_1 \sin \frac{\beta h_0}{2} + C_2 \cos \frac{\beta h_0}{2} = \Phi_0, \qquad (3.3)$$

from the second condition

$$C_1 \cos \frac{\beta h_0}{2} - C_2 \sin \frac{\beta h_0}{2} = 0 \tag{3.4}$$



Fig. 11. Variation of  $\beta H$  in function of  $h_{i}/H$  in the case of cylindrical reactor with transverse cavity

from the third condition

$$C_1 \sin \frac{\beta H}{2} + C_2 \cos \frac{\beta H}{2} = 0$$
 (3.5)

The function of  $\beta H$  can be formulated from Equs. (3.4) and (3.5)

$$\operatorname{tg}\frac{\beta h_0}{2} = -\frac{1}{\operatorname{tg}\frac{\beta H}{2}} = \operatorname{tg}\left(\frac{\beta H}{2} - \frac{\pi}{2}\right)$$

from which

$$\beta H = \frac{H}{H - h_0} \pi \tag{3.6}$$

Fig. 11 shows the values of the  $\beta H$  products and the ratio of factors  $\beta H/(\beta H)_0$  of the hollow cylindrical reactor and that of the compact one in function of  $h_0/H$ .

The longitudinal flux distribution along the r = 0 axis ( $\Theta = 1$ ) can be defined by expressing the values of  $C_1$  and  $C_2$  from Equs. (3.3) and (3.4) and by putting them into Equ. (3.2)



Fig. 12. Flux distribution in the compact cylindrical reactor and in the cylindrical reactor with transverse cavity

Fig. 12 shows the longitudinal flux distribution of the hollow cylindrical reactor when the height of the cavity is  $h_0=0,2\;H$ .

The flux distribution of the whole active zone is

$$\frac{\Phi(rz)}{\Phi_0} = J_0 \left(\frac{2,405}{R} r\right) \cos\left[\frac{\pi}{H - h_0} \left(z - \frac{h_0}{2}\right)\right]$$
(3.8)

The geometrical buckling of the cylindrical reactor with a transverse cavity can be obtained from the equation:

$$B_{g}^{2} = \left(\frac{\beta H}{H}\right)^{2} + \left(\frac{2,405}{R}\right)^{2}$$
(3.9)

The critical volume can be computed in the same way as in the case of the cylindrical reactor with a longitudinal cavity. On the strength of the H = 1,847 R ratio, characterising the critical volume of the compact cylindrical reactor we can use the analogy of Equ. (2.19)



Fig. 13. Comparative relations of the critical volume of the cylindrical reactor with transverse cavity and of the compact one in function of  $h_0/H$  (if H = 1,847 R)

Fig. 13 shows the critical volume ratio as computed from Equ. (3.10). The graph as in the former case does not show the minimal critical volume, as we have not taken the R/H ratio into consideration, optimal for the hollow reactor.

# 4. Cylindrical reactor with longitudinal and transverse cavity

Fig. 14 shows the structure of the reactor with a longitudinal and a transverse cavity. For this type of reactor the equations given in Sec. 2 and 3 can be made valid. The geometrical buckling can be computed from the equation

$$B_g^2 = \left(\frac{aR}{R}\right)^2 + \left(\frac{\beta H}{H}\right)^2 \tag{4.1}$$

where the values of  $\alpha R$  can be taken from Fig. 7, and those of  $\beta H$  from Fig. 11.

For the flux distribution of the whole active zone we can write

$$\frac{\Phi(rz)}{\Phi_{0}} = \frac{1}{J_{0}(ar_{0}) - \frac{J_{0}(aR)}{Y_{0}(aR)}} Y_{0}(ar_{0})} \left[ J_{0}(ar) - \frac{J_{0}(aR)}{Y_{0}(aR)} Y_{0}(ar) \right] \cdot \cos\left[\frac{\pi}{H - h_{0}} \left(z - \frac{h_{0}}{2}\right)\right]$$
(4.2)



Fig. 14. Structure of the reactor with longitudinal and transverse cavities

Furthermore the comparative relation of the critical volume of a hollow reactor of this type to that of a compact cylindrical reactor — of H = 1.847R - is

$$\frac{V}{V_0} = \left[1 - \frac{h_0}{H}\right] \left[1 - \left(\frac{r_0}{R}\right)^2\right] \frac{1.847^2 \pi \left[\left(\frac{\beta H}{1.847}\right)^2 + (aR)^2\right]^{3/2}}{148}$$
(4.3)

#### Summary

In general, the literature of reactor physics deals with the computation of the flux and of the critical size of compact spherical and cylindrical reactors. This paper defines the geometrical buckling, flux distribution and critical size of hollow reactors with the one group diffusion method. The procedure is approximative because only by using the transport theory as well as the multigroup diffusion method could the exact values be determined. (The extension of the computation in this direction is planned). The investigation deals with spherical reactors having a central cavity, as well as with cylindrical reactors having longitudinal and transverse cavities. In each case the results are demonstrated by graphs.

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