HEAT TRANSFER IN COMPACT PLATE-FIN HEAT EXCHANGERS

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High-efficiency compact heat exchangers are an effictive means for the stepping up of the efficiency of gas turbine plants, nuclear power stations, compact nuclear drives, etc. Compact heat exchangers are characterized by a high heat output at a comparatively small input of circulating work, small temperature differences and still small bulk.

The present paper's purpose is to find solutions to one of the problems encountered in the dimensioning of compact heat exchangers.

Let us start out from the general definition of heat exchangers:

The heat exchanger is an equipment destined to establish contact between two streaming media, so as to make possible the flow of thermal energy from one medium into the other. The closer the contact between the two media is, the more compact is the heat exchanger.

Heat transfer takes place in three steps. In the first step thermal energy from the inside of the warmer fluid flows into the solid retaining wall; in the second, thermal energy passes by conductance through the wall which separates the two fluids; while in the third step thermal energy, from the wall-side boundary of the colder fluid, reaches the inside of the colder stream.

Since heat exchanger dimensions are in most cases determined by the heat transfer coefficient of the streaming media, heat flow can be accelerated either by increasing the mass turbulence or by the application of very thin fluid films which, by their small dimensions, will present only very low thermal resistance.

The latter approach, however, — due to constructional reasons — will in most cases expand the path the thermal energy must travel through conductance across the wall.

The consideration of these points of view led to the evolution of the so-called laminar-flow heat exchangers. One type of the laminar-flow heat exchanger embodies thin plate fins. Mass flow in such types takes place along the plate fins in a thin film-like laminar layer (Fig. 1).

The plate fins pick up heat from the fluid at T_k temperature, and conduct it to the wall where T_0 temperature obtains.

In determining the dimensions of the plate fins the thermodynamical problem is generally posed in the following forms.

The medium, with homogeneous temperature distribution ($T_k(x, 0) =$ $=$ constant) and representing a known time rate of heat capacity, enters the row of fins of known thermotechnical characteristics and known geometry. The question is: what is the quantity of heat dissipated by the fluid to the fins, provided the temperature in the fin base is T_0 (this temperature may vary in the direction of flow), or — less frequently: what will be the temperature profile of the fluid at the outlet.

The reply to questions of similar nature generally consists of introducing the concept of the fin efficiency and its calculation. But the neglections made

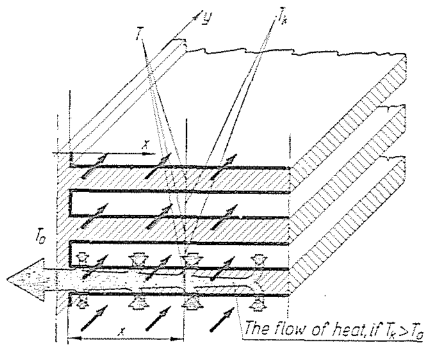

Fig. 1. Energy stream of medium flowing between plate fins

in the classical computation method are of considerable significance for us, namely in the classical computation method it is assumed that

1. the temperature of the fluid in its flow close to the fin remains constant.

This assumption is untenable in our case from more than one point of view. Namely, owing to the laminar flow, we must not neglect either the temperature difference which takes place in the flowing medium between points closer to, and more remote from, the fin base, or due to the great heat output the warming-up or cooling off of the fluid — i.e. its temperature change in the direction of flow.

2. It is assumed that thermal conductance in the fins takes place only in the direction normal to flow.

The high degree of warming up of the fluid in compact heat exchangers owing to the high thermal output, brings about a very considerable temperature difference in the fin, also in the direction of flow. Thus this temperature difference, compared with the temperature difference setting in normal to the flow, must not be neglected. Accordingly, in the differential equation of heat transfer, two components of the laplacian have to be taken into consideration (the influence of thermal conductance normal to the fin surface is, even in our case, negligible).

3. Completely steady state conditions and negligible conductivity of the fluid are generally assumed in the classical computation method.

These same assumptions are applied also in our treatise.

The mathematical problem as written down on the basis of the above considerations will yield a system of partial differential equations of the second order, to which a comparatively simple solution has been made possible by the method of operational calculus, as elaborated in recent years by MIKUSIŃSKY [1].

In what follows we shall briefly outline the solution to the problem, at the same time pointing at the exact mathematical process followed. Publications which will furnish more details and diagrams to facilitate eventual practical calculations are under preparation.

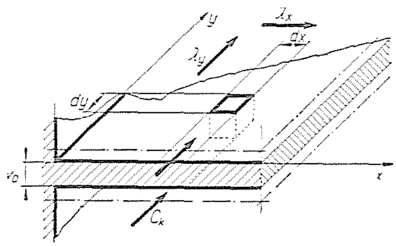

Fig. 2. Plate fin with one differential element fully drawn

Let us now examine one single plate and the mass rate of flow along it (Fig. 2).

Denoting with C_k the time rate of heat capacity referred to the length, measured in x direction (as to the geometrical dimensions and directions we shall rely on the designations of Fig. 2), and distinguishing between plate conductance in the direction of x and y (which will not entail any difficulty whatsoever mathematically and might be useful in certain cases), the conservation of energy may be expressed by the following two equations:

$$\left.\begin{aligned} 2\alpha (T_k - T)\, dx \cdot dy &= \frac{\partial}{\partial y}\left(-\lambda_y \frac{\partial T}{\partial y} v_0\, dx\right) dy + \frac{\partial}{\partial x}\left(-\lambda_x v_0\, dy\right) dx, \text{ and} \\ 2\alpha (T_k - T)\, dx\, dy &= -C_k\, dx \frac{\partial T_k}{\partial y}\, dy. \end{aligned}\right\} \quad (1)$$

In the first equation it has been expressed that, due to the stationary nature of temperature distribution, thermal energy flowing — through the effect of temperature difference arising between wall and medium (T_k — T) — from anyone of the plate fin elements into the medium, is equal to the surplus heat gained through conductance.

The second equation illustrates the equality of the heat transferred by the fluid and the thermal loss of the fluid.

Summing up what has been so far mentioned in our differential equation system the dependent variables are: $T = T(x, y)$, the plate temperature and $T_k = T_k(x, y)$ the fluid temperature; the independent variables are the room coordinates x and y; while α, λ_x, λ_y, v_0 and C_k are constant.

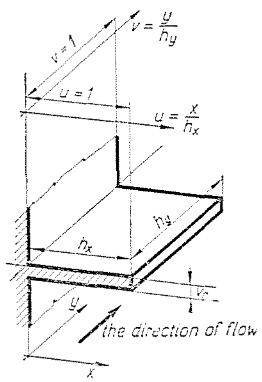

Fig. 3. The plate fin dimensions

Since in the further calculations the equations will be made dimensionless, we wish to call attention already at this point to the fact that the same system of dimensions will have to be substituted throughout.

Let us introduce the following new denotations (see Fig. 3):

h_x for the fin dimension in the direction of x (the distance between fin base and that plane in the fin in which the heat flow normal to the stream is equal to zero), and

h_y for the fin dimension in the direction of x (that is, in flow direction).

Let us further write

$$\left.\begin{aligned} \Delta T &= T - T_k, \\ \Delta T_k &= T(0,0) - T_k, \\ \Delta T_f &= T(0,0) - T. \end{aligned}\right\} \tag{2}$$

The temperature difference should be referred to:

$$\Delta T_0 = T(0,0) - T_k(0,0). \tag{3}$$

We shall now define the following quantities:

$$\left.\begin{aligned} \Phi_k &= \frac{\Delta T_k}{\Delta T_0} = \frac{T(0,0)}{\Delta T_0} - \frac{T_k}{\Delta T_0}, \\ \Phi &= \frac{\Delta T_f}{\Delta T_0} = \frac{T(0,0)}{\Delta T_0} - \frac{T}{\Delta T_0}. \end{aligned}\right\} \quad (4)$$

Whence

$$\left.\begin{aligned} &\Delta\Phi = \Phi - \Phi_k = -\frac{\Delta T}{\Delta T_0}; \text{ resp.} \\ &dT_k = -\Delta T_0\, d\Phi_k \quad \text{and} \quad dT = -\Delta T_0\, d\Phi. \end{aligned}\right\} \quad (5)$$

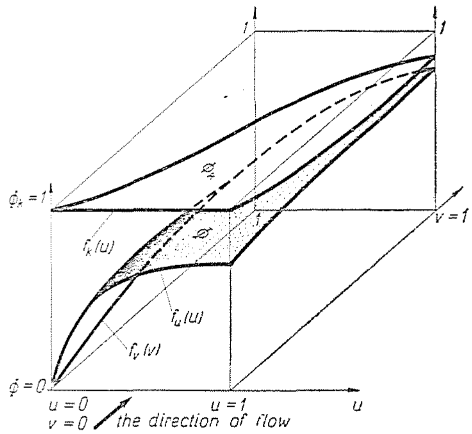

Fig. 4. Qualitative illustration of the $\Phi(u, v)$ and $\Phi_k(u, v)$ surfaces

Let us, instead of x, y, introduce the following dimensionless room coordinates:

$$u = \frac{x}{h_x} \quad \text{and} \quad v = \frac{y}{h_y}. \quad (6)$$

Our system of equations (1) will assume the following form:

$$\Delta\Phi = \frac{\lambda_y}{2a}\frac{v_0}{h_y^2}\frac{\partial^2 \Phi}{\partial v^2} + \frac{\lambda_x}{2a}\frac{v_0}{h_x^2}\frac{\partial^2 \Phi}{\partial u^2}$$

and

$$\Delta\Phi = \frac{C_k}{2ah_y}\frac{\partial \Phi_k}{\partial v}.$$

The constants combine to give:

$$A_v = \frac{\lambda_y}{2a}\frac{v_0}{h_y^2}, \quad A_u = \frac{\lambda_x}{2a}\frac{v_0}{h_x^2} \quad \text{and} \quad C = \frac{C_k}{2ah_y}. \quad (7)$$

Thus:

$$\begin{aligned} \Delta\Phi &= A_v \frac{\partial^2 \Phi}{\partial v^2} + A_u \frac{\partial^2 \Phi}{\partial u^2} \text{ and} \\ \Delta\Phi &= C \frac{\partial \Phi_k}{\partial v} . \end{aligned} \qquad (8)$$

Fig. 4. shows the newly introduced denotations and the character of the surfaces arrived at.

Let us now determine the boundary conditions generally encountered in technical practice.

The temperature of the fluid at the cross section of inlet ($y = 0$, resp. $v = 0$) shall be given in the function of u: $T_k = T_k(u, 0)$. This permits the determination of $\Phi_k(u, 0) = f_k(u)$.

At the point of $u = 0$, $\Phi_k(0, 0) = f_k(0) = 1$.*

Examining also the temperature of the plate in the same cross section ($v = 0$) beside the wall ($x = 0$, resp. $u = 0$), it will be found that T is equal to $T(0, 0)$, that is ΔT_f is equal to $T(0, 0) - T(0, 0)$ and thence Φ is equal to zero.

We assume a given temperature distribution for the fin base also, i.e. we assume that $\Phi(0, v)$ is equal to $f_v(v)$ provided u is equal to zero, respectively, in the simplest case f_v is equal to 0 (no change takes place in the temperature of the fin base in flow direction).

Further boundary conditions are set by the fact that — with the sole exception of the fin base — heat flow normal to the enclosing surfaces of the fin must be equal to zero. Thus, if v is equal to 0 or to 1, $\partial\Phi/\partial v$ is equal to 0; and should u be equal to 1, then $\partial\Phi/\partial u$ is equal to 0.

Summing up:

if v is equal to 0, then Φ_k is equal to $f_k(u)$ and $\partial\Phi/\partial v$ is equal to 0,
if v is equal to 1, then $\partial\Phi/\partial v$ is equal to 0,
if u is equal to 0, then Φ is equal to $f_v(v)$ finally
if u is equal to 1, then $\partial\Phi/\partial u$ is equal to 0.

The last condition is determined either by symmetry — viz. by the fact that the plane at a distance of h_x from the fin base ($u = 1$) constitutes the plane of symmetry of the temperature distribution, or by the fact that in the $u = 1$ the fin is enclosed by an insulated frontal plane.

The first step in approaching the problem will be "to remove" the "v" variable through the operational method.

* Or at least at the $(0 + du, 0)$ point.

According to the general formula of the operational calculus, the following may be written:

$$\left\{\frac{\partial \Phi}{\partial v}\right\} = s\bar{\Phi} - \Phi(u,0)$$

$$\left\{\frac{\partial^2 \Phi}{\partial v^2}\right\} = s^2 \bar{\Phi} - s\Phi(u,0) - \left.\frac{\partial \Phi}{\partial v}\right|_{v=0} \quad \text{and}$$

$$\left\{\frac{\partial \Phi_k}{\partial v}\right\} = s\bar{\Phi}_k - \Phi_k(u,0).$$

Note:

In all formulae and throughout the present paper a horizontal line above any character* will denote the operator, while the letter s will stand for the differential operator.

If we introduce according to what was stated above for the $\Phi(u, 0)$, $\Phi(0, v)$ and $\Phi_k(u, 0)$ functions, the following denotations:

$$\Phi(u,0) = f_u(u), \quad \Phi(0,v) = f_v(v) \quad \text{and} \quad \Phi_k(u,0) = f_k(u)\,, \tag{10}$$

then, taking also the boundary conditions into consideration, the (8) system of equations will assume the following form

$$\left.\begin{aligned} \Delta\bar{\Phi} &= A_v(s^2\,\bar{\Phi} - sf_u) + A_u \frac{\partial^2 \bar{\Phi}}{\partial u^2} \quad \text{and} \\ \Delta\bar{\Phi} &= Cs\bar{\Phi}_k - Cf_k\,. \end{aligned}\right\} \tag{11}$$

Since, on the other hand $\Delta\bar{\Phi} = \bar{\Phi} - \bar{\Phi}_k$, the last relationship will give:

$$\bar{\Phi} - \bar{\Phi}_k = C \cdot s \cdot \bar{\Phi}_k - Cf_k\,,$$

respectively,

$$\bar{\Phi}_k = \frac{\bar{\Phi} + Cf_k}{1 + Cs}\,.$$

Substituting it into (11), we arrive at

$$\bar{\Phi}\left(1 - \frac{1}{1 + Cs}\right) = A_v\,(s^2\bar{\Phi} - sf_u) + A_u \frac{d^2\bar{\Phi}}{du^2} + Cf_k \frac{1}{1 + Cs}\,.$$

Since $\bar{\Phi}$ is the function of u only, our equation can now be written in the following, more simple form:

$$\bar{\Phi}'' - \frac{A_v}{A_u}\left(\frac{1}{A_v}\,\frac{Cs}{1 + Cs} - s^2\right)\bar{\Phi} = \frac{A_v}{A_u}\,sf_u - \frac{1}{A_u}\,\frac{f_k}{s + 1/C}\,.$$

* Respectively the braces: { }

Introducing the following denotations:

$$\left.\begin{aligned} \bar{L} &= \beta\left(\frac{1}{A_v}\frac{s}{s+1/c} - s^2\right), \quad \beta = A_v/A_u \quad \text{and} \\ \overline{w} &= \beta s f_u^{\bullet} - \frac{1}{A_u}\frac{f_k}{s+1/C} = s\bar{k}, \end{aligned}\right\} \tag{12}$$

the equation will appear in its final form

$$\overline{\Phi}'' - \bar{L}\overline{\Phi} = \overline{w}\,, \tag{13}$$

where $\overline{\Phi}, \bar{L}$ and $\overline{w}$ are operators.

The solutions of the (13) operational equation will be operators — v-functions — in the parameter of u which, by the general formulae of the operational calculus, will satisfy two of the boundary conditions of the original differential equation, namely: if v is equal to 0, $\partial\Phi/\partial v$ will be equal to 0 and $\Phi_k(u, 0)$ will be equal to f_k.

The resultant functions will also satisfy a third condition, viz. provided v is equal to 0, $\Phi(u, 0)$ will be equal to f_u.

Since, however, this fact can only very infrequently be applied as a boundary condition, further investigations are required to adapt the solutions to the actually encountered boundary conditions (9) — in the course of which the f_u function is also to be determined.

The (13) equation is an inhomogeneous differential equation with the following right side: $\overline{w} = \overline{w}(u)$. The conventional method of seeking the general solution of inhomogeneous equations consists of the superposition of the solution of the homogeneous equation on an independent particular solution of the inhomogeneous equation. This method cannot be applied in our case to the fact that the formal solutions of the homogeneous operational equation pertaining to (13) are mostly non operators, thus the only solution of the homogeneous equation is the trivial one [2].

It should be borne in mind, however, that this would hold good only if $\sqrt{A_v/A_u} \neq 0$. Should, namely, e.g. $A_v = 0$, then we might obtain a function as the solution of the homogeneous equation. This would represent the case when plate conductivity in the direction of flow is negligible.

Since our investigation's purpose is to clarify the more general aspects of the problem, we take plate conductivity in both directions into consideration. Thus, we cannot apply the conventional method and must resort to the particular solution of (13) — which in our case yields, at the same time, the complete solution.

The search for such a solution will in the majority of cases not present any appreciable difficulties, the problem essentially being that although the function f_u in (13) differential equation may be set theoretically for one of the boundary conditions (it is the value of Φ in case $v = 0$), in actual practice this function is generally unknown. For this reason it has been omitted from the system (9) of boundary conditions.

Since the system of boundary conditions given under (9) is, in the majority of cases, actually known, the solution of the differential equation must satisfy it.

On the ot herhand, if we expect the solution to satisfy the system (9) of boundary conditions, f_u cannot be pre-determined any longer, but will

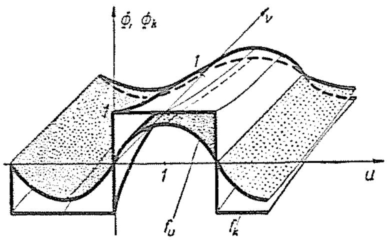

Fig. 5. The qualitative chart of Φ and Φ_k in case f_k is an odd square function (f_k having a period equal to f_u)

follow from the solution. Consequently we shall have to seek for such a form on the right side of (13) inhomogeneous differential equation (f_u figured in $\overline{w}$), with which the solution will satisfy the conditions as set in the (9).

A substitution will readily prove (2) that one of the particular solutions to (13) equation may always be obtained by the following series (provided it is convergent and can be evolved):

$$\overline{\Phi} = -\frac{s}{\overline{L}} \sum_{n=0}^{\mu} \frac{\overline{k}^{(2n)}}{\overline{L}^{n}}, \tag{14}$$

where (12 equation) the $\overline{k}$ operator is a parametric function with u as the parameter. In the summation μ is defined by the requirement being the last derivative function of k (derived according to u) which will not be analogously clear, just the $(2\mu + 1)$th derivative. Provided the infinite series is a convergent one, it may be that $\mu \to \infty$.

The solution in this form will not be suitable for further calculations, unless f_u and f_k viz. $\overline{k}$ are given.

Since, however, this case is very rarely met with in practice, we must find the solution in a different form.

To evolve f_u and f_k in the form of a Fourier series seems expedient. This will naturally presuppose that both are periodic functions, but right at the outset it will be evident that such proviso will substantially facilitate the satisfaction of the (9) boundary conditions.

Namely, if we choose the period of f_k and f_u in such a way that the planes $u = -1$; $u = +1$; $u = +3$, etc., are planes of symmetry, then in these planes the derivatives of Φ in the direction of u will disappear and a further boundary condition will be satisfied. Whereafter from the last boundary condition ($\partial\Phi/\partial v = 0$, if v is equal to 1), in the knowledge of the required coefficients of the f_k function, the Fourier coefficients of f_u can be readily determined.

In that which follows we shall restrict ourselves to the treatment of that specific case when $f_v = 0$, viz. wall temperature in the direction of flow remains unchanged. This boundary condition is automatically met with by selecting the series of f_k so as to realize the function seen in Fig. 5. (We refer here to the fact that by the proper choice of f_k and f_u, f_v can be shaped to meet to the full all practical requirements.)

Choosing equal periods for f_u and f_k, due to symmetry — as outlined previously — f_u and f_k, f_u with unknown coefficients for the time being, may be written with the following Fourier series:

$$f_k = \frac{4}{\pi}\left(\sin\frac{\pi}{2}u + \frac{1}{3}\sin\frac{3\pi}{2}u + \dots\right)$$

and (15)

$$f_u = b_1 \sin\frac{\pi}{2}u + b_2\sin\frac{3\pi}{2}u + \dots,$$

while according to the (12) equation:

$$\bar{k} = \frac{\bar{w}}{s} = \left(\beta b_1 - \frac{4}{A_u\pi}\,\frac{1}{s(s+1/C)}\right)\sin\frac{\pi}{2}u +$$

$$+\left(\beta b_2 - \frac{4}{3A_u\pi}\,\frac{1}{s(s+1/C)}\right)\sin\frac{3\pi}{2}u + \dots =$$

$$= B_1\sin\frac{\pi}{2}u + B_2\sin\frac{3\pi}{2}u + \dots,$$

where

$$B_n = \beta b_n - \frac{1}{(2n-1)\pi A_u}\cdot\frac{1}{s(s+1/C)}. \qquad (16)$$

Finally:

$$\bar{k} = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi}{2}u. \qquad (17)$$

A simple substitution will at once show that if $\bar{k} = \bar{k}_1 + \bar{k}_2$ and $\bar{\Phi}_1$, further $\bar{\Phi}_2$ are the roots of the $\frac{\bar{\Phi}''}{s} - \bar{L}\,\frac{\bar{\Phi}}{s} = \bar{k}_1$ respectively of the $\frac{\bar{\Phi}''}{s} -$

$-\bar{L}\dfrac{\bar{\Phi}}{s}=\bar{k}_2$ equation, the root of the $\dfrac{\bar{\Phi}''}{s}-\bar{L}\dfrac{\bar{\Phi}}{s}=\bar{k}$ equation will be $\bar{\Phi}=$ $=\bar{\Phi}_1+\bar{\Phi}_2$. Thence it follows, that solving the

$$\frac{\bar{\Phi}''}{s}-\bar{L}\frac{\bar{\Phi}}{s}=B_n\sin\frac{(2n-1)\pi}{2}u$$

equation, the $\bar{\Phi}$ function sought for can be evolved from its $\bar{\Phi}_n$ roots in the following way:

$$\bar{\Phi}=\sum_{n=1}^{\infty}\bar{\Phi}_n. \tag{18}$$

Let us now compute the value of $\bar{\Phi}_n$, making use of the expression of $\bar{\Phi}$, obtained from the (18) equation. It is obvious, namely, that

$$\frac{d^{2k}}{du^{2k}}\left(B_n\sin\frac{(2n-1)\pi}{2}u\right)=(-1)^k\left(\frac{(2n-1)\pi}{2}\right)^{2k}B_n\sin\frac{(2n-1)\pi}{2}u,$$

whence, according to (14) equation:

$$\bar{\Phi}_n=-B_n\sin\left(\frac{2n-1}{2}\pi u\right)\frac{s}{\bar{L}}\sum_{k=0}^{\infty}\left(-\frac{\left(\frac{(2n-1)\pi}{2}\right)^2}{\bar{L}}\right)^k=$$

$$=-B_n\sin\frac{(2n-1)\pi}{2}u\cdot\frac{s}{\bar{L}}\cdot\frac{1}{1+\dfrac{\left(\frac{2n-1}{2}\pi\right)^2}{\bar{L}}}=$$

$$=-B_n\frac{s}{\bar{L}+\left(\frac{2n-1}{2}\pi\right)^2}\sin\frac{(2n-1)\pi}{2}u.$$

The complete solution, on the basis of (18) equation will be obtained as follows:

$$\bar{\Phi}=-\sum_{n=1}^{\infty}\frac{sB_n}{\bar{L}+\left(\frac{2n-1}{2}\pi\right)^2}\sin\frac{(2n-1)\pi}{2}u. \tag{19}$$

Denoting $\dfrac{2n-1}{2}\pi$ with ω:

$$\frac{2n-1}{2}\pi=\omega, \tag{20}$$

and substituting the value of B_n ((16) equation), the nth coefficient to the trigonometric series of $\bar{\Phi}$ may be set up in the following form:

$$\frac{\beta s b_n}{\bar{L}+\omega^2} - \frac{2}{A_u \omega} \frac{1}{(s+1/C)(\bar{L}+\omega^2)} . \tag{21}$$

β being A_v/A_u (12),* the demoninator may be written as:

$$-\beta\left[s^3+\frac{1}{C}s^2-\frac{1+\omega^2 A_u}{A_v}s-\frac{A_u\omega^2}{A_v C}\right],$$

whence the nth coefficient becomes:

$$-\frac{b_n s^2+\frac{b_n}{C}s-\frac{2}{A_v\omega}}{s^3+\frac{1}{C}s^2-\frac{1+A_u\omega^2}{A_v}s-\frac{A_u\omega^2}{A_v C}} . \tag{22}$$

The roots of the denominator shall be ε_{1n}, ε_{2n} and ε_{3n} [2]. With these roots the nth coefficient may be decomposed with simple fractions in the following form:

$$-\left[\frac{D_{1n}}{s-\varepsilon_{1n}}+\frac{D_{2n}}{s-\varepsilon_{2n}}+\frac{D_{3n}}{s-\varepsilon_{3n}}\right]. \tag{23}$$

Taking into consideration also the relationships between the ε_n roots, the following system of equations will be available to determine

$$D_{1n}+D_{2n}+D_{3n}=b_n , \tag{24}$$

$$\varepsilon_{1n}D_{1n}+\varepsilon_{2n}D_{2n}+\varepsilon_{3n}D_{3n}=0 \text{ and} \tag{25}$$

$$\frac{1}{\varepsilon_{1n}}D_{1n}+\frac{1}{\varepsilon_{2n}}D_{2n}+\frac{1}{\varepsilon_{3n}}D_{3n}=-\frac{2C}{A_u\omega^3} . \tag{26}$$

The literature referred to under [2] describes the thorough examinations carried on into the case when one of the roots of the cubic equation is equal to zero. In such cases, namely, our equation system cannot be applied in this form.

There remains now to determine the b_n Fourier coefficients of the unknown f_u function. This must be done in such a way as to make the derivative of the Φ function, with respect to v, equal to zero at the $v = 1$ position.

* Substituting also $\bar{L}$.

Substituting the operational form with the conventional form of the Φ function [(19) and (23) equations]:

$$\Phi = \sum_{n=1}^{\infty} \left(D_{1n}\, e^{\varepsilon_{1n} v} + D_{2n}\, e^{\varepsilon_{2n} v} + D_{3n}\, e^{\varepsilon_{3n} v}\right) \sin \omega u \,. \tag{27}$$

The partial derivative in respect to v at the $v = 1$ position is

$$\left.\frac{\partial \Phi}{\partial v}\right|_{v=1} = \sum_{n=1}^{\infty} \left(D_{1n}\, \varepsilon_{1n}\, e^{\varepsilon_{1n}} + D_{2n}\, \varepsilon_{2n}\, e^{\varepsilon_{2n}} + D_{3n}\, \varepsilon_{3n}\, e^{\varepsilon_{3n}}\right) \sin \omega u \,.$$

To render this function at an arbitrary u equal to zero, the following fourth equation will present itself:

$$D_{1n}\, \varepsilon_{1n}\, e^{\varepsilon_{1n}} + D_{2n}\, \varepsilon_{2n}\, e^{\varepsilon_{2n}} + D_{3n}\, \varepsilon_{3n}\, e^{\varepsilon_{3n}} = 0 \,. \tag{28}$$

Our task has been fulfilled, viz. with the (27) equation the Φ function satisfying the pre-set boundary conditions in the form of a Fourier series, has been set up.

Let us now sum up the sequence of the process followed.

The nth coefficient of the Fourier series of Φ was obtained in the following manner:

1. Computing the value of ω from n (20) we have used it to determine the roots of the denominator of the nth operator coefficient (22);

2. solving the system of inhomogeneous linear equations (25), (26) and (28), we have determined the value of D_{1n}, D_{2n} and D_{3n}.

With the so obtained valuesthe n th member of the Fourier series of Φ may be determined on the basis of (27).

This method is, of course, applicable for the determination of any optional member of the series of Φ.

Technical practice, however, is interested in the volume of the transferred heat much rather than in the developing temperature pattern. For this reason, we shall compute the heat transferred under conditions given in the presentation of the problem.

The heat extracted from the wall through the fin can be readily established by determining the temperature gradient in the fin adjacent to, and normal to, the wall

$$dQ = -\lambda_x \left.\frac{\partial T}{\partial x}\right|_{x=0} v_0 \cdot dx \,,$$

or, integrating along the whole length of the fin:

$$Q = -\lambda_x\, v_0 \int_0^{h_y} \left.\frac{\partial T}{\partial x}\right|_{x=0} dy \,.$$

This will show the heat extracted from the wall by each of the fins.

Let us now compute the same heat quantity, applying a factor denoted with ε_L, similar to the fin efficiency. (The subscript L serves to call attention to the fact that this factor, although similar, is not identical with fin efficiency.) The following will give the definition of this factor:

$$Q = 2\, h_x h_y \varepsilon_L\, a \Delta T_0 . \tag{29}$$

Collating the two equations of Q and applying the denotations used previously (see (6) and (7) equations):

$$\varepsilon_L = A_u \int_0^1 \left.\frac{\partial \Phi}{\partial u}\right|_{u=0} dv . \tag{29a}$$

$\left.\frac{\partial \Phi}{\partial u}\right|_{u=0}$ may, however, be calculated from (27):

$$\left.\frac{\partial \Phi}{\partial u}\right|_{u=0} = \sum_{n=1}^{\infty} \omega \left[D_{1n} e^{\varepsilon_{1n} v} + D_{2n} e^{\varepsilon_{2n} v} + D_{3n} e^{\varepsilon_{3n} v}\right].$$

After integration:

$$\int_0^1 \left.\frac{\partial \Phi}{\partial u}\right|_{u=0} dv = \sum_{n=1}^{\infty} \left[\omega \left(\frac{D_{1n}}{\varepsilon_{1n}} e^{\varepsilon_{1n}} + \frac{D_{2n}}{\varepsilon_{2n}} e^{\varepsilon_{2n}} + \frac{D_{3n}}{\varepsilon_{3n}} e^{\varepsilon_{3n}}\right) - \omega \left(\frac{D_{1n}}{\varepsilon_{1n}} + \frac{D_{2n}}{\varepsilon_{2n}} + \frac{D_{3n}}{\varepsilon_{3n}}\right)\right].$$

Upon substitution of (26):

$$\int_0^1 \left.\frac{\partial \Phi}{\partial u}\right|_{u=0} dv = \sum_{n=1}^{\infty} \left[\omega \left(\frac{D_{1n}}{\varepsilon_{1n}} e^{\varepsilon_{1n}} + \frac{D_{2n}}{\varepsilon_{2n}} e^{\varepsilon_{2n}} + \frac{D_{3n}}{\varepsilon_{3n}} e^{\varepsilon_{3n}}\right) + \omega \frac{2C}{A_u\, \omega^3}\right].$$

Substituting ω from (20), the second member after the Σ sign can be easily computed:

$$\sum_{n=1}^{\infty} \frac{2C}{A_u\, \omega^2} = \frac{8C}{A_u \pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{C}{A_u} .$$

Substituting the expression of ω and applying the last equation, the value of ε_L becomes:

$$\varepsilon_L = C + \frac{A_u \pi}{2} \sum_{n=1}^{\infty} (2n-1) \left[\frac{D_{1n}}{\varepsilon_{1n}} e^{\varepsilon_{1n}} + \frac{D_{2n}}{\varepsilon_{2n}} e^{\varepsilon_{2n}} + \frac{D_{3n}}{\varepsilon_{3n}} e^{\varepsilon_{3n}}\right]. \tag{30}$$

In many instances it is not convenient to summate the reciprocal quadratic series. For these cases the relationship obtained for the value of ε_L will be as follows:

$$\varepsilon_L = A_u \sum_{n=1}^{\infty} \omega \left[D_{1n} \frac{e^{\varepsilon_{1n}} - 1}{\varepsilon_{1n}} + D_{2n} \frac{e^{\varepsilon_{2n}} - 1}{\varepsilon_{2n}} + D_{3n} \frac{e^{\varepsilon_{3n}} - 1}{\varepsilon_{3n}} \right]. \tag{31}$$

Solving the system of equations in three unknown quantities [2], we arrive at the following correlation ultimately:

$$\frac{\varepsilon_L}{C} = 1 - \sum_{n=1}^{\infty} \frac{2e^{\varepsilon_{3n}}}{\omega^2} \frac{1-\left(\frac{\varepsilon_{3n}}{\varepsilon_{2n}}\right)^2 - e^{\varepsilon_{1n}-\varepsilon_{2n}}\left[1-\left(\frac{\varepsilon_{3n}}{\varepsilon_{1n}}\right)^2\right] + e^{\varepsilon_{1n}-\varepsilon_{3n}}\left[\left(\frac{\varepsilon_{3n}}{\varepsilon_{2n}}\right)^2 - \left(\frac{\varepsilon_{3n}}{\varepsilon_{1n}}\right)^2\right]}{1-\left(\frac{\varepsilon_{3n}}{\varepsilon_{1n}}\right)^2 - e^{\varepsilon_{1n}-\varepsilon_{2n}}\left[1-\left(\frac{\varepsilon_{3n}}{\varepsilon_{2n}}\right)^2\right] - e^{\varepsilon_{3n}-\varepsilon_{2n}}\left[\left(\frac{\varepsilon_{3n}}{\varepsilon_{2n}}\right)^2 - \left(\frac{\varepsilon_{3n}}{\varepsilon_{1n}}\right)^2\right]}. \tag{32}$$

This will have served to illustrate the sequence followed in solving the thermotechnical problem in hand. In equations (30), (31) and (32), in the form of infinite series, even a final formula had been worked out and presented for cases in which wall temperature, adjacent to the fin base in the direction of flow, is constant.

We must reiterate here that the method given is easily and efficiently applicable also to other functions of the wall temperature.

The given infinite series has already been applied in a number of calculations for actual problems. The calculations have shown that under the conditions and material characteristics generally encountered in practice, the series will rapidly converge and the establishment of four-five members will yield sufficient accuracy.

Summary

The paper, by investigations into the quantity of transferred heat and extending also to fin conductivity in flow direction and un-uniform temperature rise in the fluid along the fin, deals with the fundamental requirements of *accurate dimensioning of all kinds of plate fin type heat exchangers*. The conclusions drawn hold good mainly for the dimensioning of compact laminar-flow heat exchangers.

The treatment of the problem has led to a system of partial differential equations of the second order. The solution and its adaptation to the boundary conditions are based partly on the operational calculus as evolved by Mikusiński, partly on the expansion of the boundary conditions into Fourier series.

For the frequently occurring case when fin base temperature in the direction of flow is constant and the temperature distribution of the entering fluid homogeneous, the final result was obtained in the form of a rapidly converging series. (Under conventional material characteristics the convergence was particularly rapid.)

The calculation method is readily applicable to various boundary conditions.

Nomenclature

b_n the nth coefficient of the Fourier series of the f_u function:
f_k $f_k = \Phi_k(u, o)$
f_n $f_u = \Phi(u, o)$
f_v $f_v = \Phi(o, v)$
h_x the longitudinal dimension of the fin, normal to the flow
h_y the longitudinal dimension of the fin in the direction of flow
$\bar{k}$ $\bar{k} = \beta f_u - 1/A_u \cdot 1/s \cdot f_k/(s + 1/C)$
s the differential operator
u dimensionless coordinate normal to the flow, $u = x/h_x$
v dimensionless coordinate in flow direction, $v = y/h_y$
v_0 fin thickness
$\bar{w}$ $\bar{w} = s\bar{k}$
x room-coordinate normal to flow
y room-coordinate in flow direction
A_u dimensionless number $A_u = \lambda_x/2a \cdot v_0/h_x^2$
A_v dimensionless number $A_v = \lambda_y/2a \cdot v_0/h_y^2$
B_n the nth coefficient of the Fourier series of $\bar{k} : \bar{k} = \sum_{n=1}^{\infty}{}' B_n \sin \omega u$
C dimensionless number $C = C_k/2ah_y$
C_k the heat capacity of the rate of mass flow referred to the length, measured in the direction of x
the full rate of heat capacity for each fin is: $\int_0^{h_x} C_k d_x$
D_{1n}, D_{2n}, D_{3n} coefficients (23 equation)
$\bar{L}$ $\bar{L} = \beta\,[1/A_v \cdot s/(s+1/C) - s^2]$
Q heat transferred per time unit
T fin temperature
T_k fluid temperature
T_0 fin base temperature $T_0 = T_0(o, v)$
ΔT $\Delta T = T - T_k$
ΔT_k $\Delta T_k = T(0, 0) - T_k$
ΔT_0 $\Delta T_0 = T(0, 0) - T_k(0, 0)$
ΔT_f $\Delta T_f = T(0, 0) - T$
a heat transfer coefficient between fin and fluid
β $\beta = A_v/A_u$
ε_L dimensionless number, similar to fin efficiency. $\varepsilon_L = Q/2h_x h_y a\Delta T_0$
$\varepsilon_{1n}, \varepsilon_{2n}, \varepsilon_{3n}$ exponents (23 equation!)
λ_x, λ_y fin conductivity in x, respectively, in y direction
ω $\omega = (2n-1)\pi/2$
Φ variable, dimensionless, expressing the changes in the temperature of plate $\Phi = \Delta T_f/\Delta T_0$
Φ_k variable, dimensionless, expressing the changes in the fluid temperature $\Phi_k = \Delta T_k/\Delta T_0$
$\Delta\Phi$ $\Delta\Phi = \Phi - \Phi_k$
$\bar{z}$ resp. $\{z\}$ operator

Note

with respect to dimensions:

Physical equations have been applied throughout, consequently any consistent units may be employed.

Literature

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