# SOME REFLECTIONS ON THE RELAXATION OF BIHARMONIC DIFFERENTIAL EQUATIONS IN POLAR COORDINATES

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## 1. The application of polar coordinates

In the plane problem of elasticity the Airy stress function must satisfy the following requirements:

*a)* In the multiply connected region $C$ bounded by the contours $L_1, L_2 \dots L_n$ the biharmonic differential equation

$$\nabla^4 \psi(x, y) = 0 \tag{1}$$

holds.

*b)* On the external boundary $L_1$ the boundary conditions

$$\psi(x, y) = K(x, y), \quad \frac{\partial \psi(x, y)}{\partial n} = k_n(x, y) \quad \text{hold,}$$

where $n$ denotes the outward directed normal.

These conditions can be written in the following form

$$\psi(x, y) = K(x, y), \quad \frac{\partial \psi(x, y)}{\partial x} = k_x(x, y), \quad \frac{\partial \psi(x, y)}{\partial y} = k_y(x, y) \tag{2}$$

too.

*c)* On each interior boundary $L_2 \dots L_i \dots L_n$ the boundary conditions of type

$$\begin{gathered} \psi(x,y) = K_i(x, y) + A_i x + B_i y + C_i \\ \frac{\partial \psi(x, y)}{\partial x} = k_{xi}(x, y) + A_i, \quad \frac{\partial \psi(x, y)}{\partial y} = k_{yi}(x, y) + B_i \end{gathered} \tag{3}$$

hold.

The constants $A_2, B_2, C_2 \dots A_i, B_i, C_i \dots A_n, B_n, C_n$ can be computed from the equations

$$\begin{gathered} \int\limits_{C_i} \nabla^4 \psi(x, y)\, dx\, dy = 0 \\ \int\limits_{C_i} x \nabla^4 \psi(x, y)\, dx\, dy = 0 \\ \int\limits_{C_i} y \nabla^4 \psi(x, y)\, dx\, dy = 0\,, \end{gathered} \tag{4}$$

where $C_i$ denotes the region bounded by contour $L_i$. To this purpose the function $\psi(x, y)$ must be continued across the boundary $L_i$ into the region $C_i$ in a manner that will satisfy the boundary conditions (3) and will possess continuous differential quotients up to the second order.

If the contours $L_1 \ldots L_i \ldots L_n$ are mainly composed of arcs of concentric circles and of their radii, and their origin does not belong to the region, then it may be convenient to use polar coordinates instead of cartesian ones, as has already been done in the case of harmonic differential equations [2]. I. pp. 44—45.

This problem can be treated in a very elegant manner using complex variables [3] pp. 192—195., but we prefer an elementary treatment more suitable for the method of relaxation.

We cover the region $C$ by a square net consisting of concentric circles and their radii. The angles between the radii are everywhere $\Delta\vartheta = h$. The con-

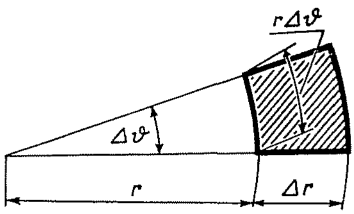


*Fig. 1*

dition that the net be square is $r\Delta\vartheta = \Delta r$, i. e. $\Delta\vartheta = \Delta \ln r$. Denoting $\xi = \ln r$, $\eta = \vartheta$ this condition can be expressed by $\Delta\xi = \Delta\eta = h$. (Fig. 1)

It is clear that in polar coordinates the sides of the squares are $r\,h$, so that $h$ is the side of the square where $r = 1$. Considering $\xi$, $\eta$ as a cartesian system of coordinates the originally curved net is transformed into a rectilinear one.

It is known that in a rectilinear square net

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0}{h^2},$$

where $h$ denotes the mesh size, and the numbering of the $\psi$-s is that in Fig. 2.

In polar coordinates the mesh size varies proportionally to $r$ along the radius and amounts to $r\,h$, therefore

$$\nabla^2 \psi(x, y) = \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0}{r^2 h^2} = \frac{1}{r^2}\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right) = e^{-2\xi}\, \nabla^2 \psi(\xi, \eta)$$

with the numbering in Fig. 3.

It can be immediately verified that

$$\nabla^2 (UV) = (\nabla^2 U) V + 2 \text{ grad } U \cdot \text{grad } V + U(\nabla^2 V),$$

and thus

$$\nabla^4 \psi(x, y) = \nabla^2 [\nabla^2 \psi(x, y)] = e^{-2\xi} \nabla^2 [e^{-2\xi} \nabla^2 \psi(\xi, \eta)] =$$

$$= e^{-2\xi} \left[ 4 e^{-2\xi} \nabla^2 \psi(\xi, \eta) - 2 \cdot 2 e^{-2\xi} \frac{\partial}{\partial \xi} \nabla^2 \psi(\xi, \eta) + e^{-2\xi} \nabla^4 \psi(\xi, \eta) \right] =$$

$$= e^{-4\xi} \left[ 4 \nabla^2 \psi(\xi, \eta) - 4 \frac{\partial}{\partial \xi} \nabla^2 \psi(\xi, \eta) + \Delta^4 \psi(\xi, \eta) \right]$$

$$r^4 \nabla^4 \psi(x, y) = \nabla^4 \psi(\xi, \eta) - 4 \frac{\partial}{\partial \xi} \nabla^2 \psi(\xi, \eta) + 4 \nabla^2 \psi(\xi, \eta) = 0 . \tag{4}$$

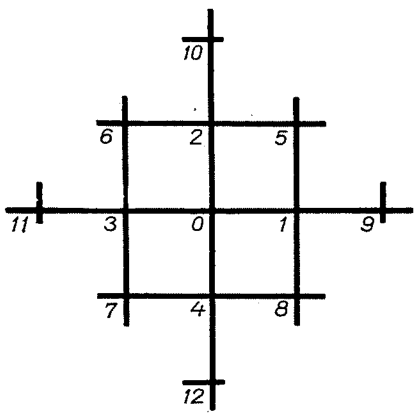

*Fig. 2*

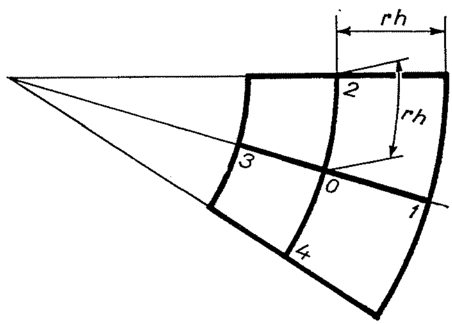

*Fig. 3*

Distorting the polar coordinates $\xi$, $\eta$ to rectilinear ones and using the numbering of Fig. 2. it ensues

$$\nabla^4 \psi(\xi, \eta) = \frac{-1}{h^4} [8(\psi_1 + \psi_2 + \psi_3 + \psi_4) - 2(\psi_5 + \psi_6 + \psi_7 + \psi_8) -$$

$$- (\psi_9 + \psi_{10} + \psi_{11} + {}_{12}) - 20 \psi_0]$$

$$\frac{\partial}{\partial \xi} \nabla^2 \psi(\xi, \eta) = \frac{1}{2h} \{[\nabla^2 \psi(\xi, \eta)]_1 - [\nabla^2 \psi(\xi, \eta)]_3\} =$$

$$- \frac{1}{2h^3} (\psi_9 + \psi_5 + \psi_8 - 4\psi_1 - \psi_6 - \psi_7 - \psi_{11} + 4\psi_3) \cdot$$

From these equations the formula results for the residuals $R_0$

$$\begin{aligned} R_0 = & (8 - 8h - 4h^2)\psi_1 + (8 + 8h - 4h^2)\psi_3 + \\ & + (8 - 4h^2)(\psi_2 + \psi_4) - (2 + 2h)(\psi_5 + \psi_8) - (2 - 2h)(\psi_6 + \psi_7) - \\ & - (1 + 2h)\psi_9 - (1 - 2h)\psi_{11} - \psi_{10} - \psi_{12} - (20 - 16h^2)\psi_0 \end{aligned} \tag{6}$$

and the relaxation pattern Fig. 4.

$$\begin{array}{ccccc} & & -1 & & \\ & -(2-2h) & 8-4h^2 & -(2+2h) & \\ -(1-2h) & 8-8h-4h^2 & -(20-16h^2) & 8+8h-4h^2 & -(1+2h) \\ & -(2-2h) & 8-4h^2 & -(2+2h) & \\ & & -1 & & \end{array}$$

*Fig. 4.*

On the boundary the equations

$$\frac{\partial\psi}{\partial r} = \frac{\psi_1 - \psi_3}{2rh} = \frac{1}{r}\frac{\partial\psi}{\partial\xi}, \quad \frac{\partial\psi}{\partial\vartheta} = \frac{\partial\psi}{\partial\eta}$$

subsist. In polar coordinates the components of grad $\psi$ are $\frac{\partial\psi}{\partial r}$ and $\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}$, in the cartesian coordinates they are $\frac{1}{r}\frac{\partial\psi}{\partial\xi}$ and $\frac{1}{r}\frac{\partial\psi}{\partial\eta}$, i. e.

$$\operatorname{grad}\psi(x, y) = \frac{1}{r}\operatorname{grad}\psi(\xi, \eta)\,. \tag{7}$$

## 2. The application of infinite blocks

A wedge shaped domain is bounded by two straight lines $\xi = \text{const.}$ in the $\xi, \eta$ representation. If the angle of the wedge equals $\pi$, the wedge degenerates to the infinite half plane. If the sides of the wedge are unloaded and the environment of the edge is loaded by forces in equilibrium, the boundary conditions $\psi = 0$, $\frac{\partial\psi}{\partial\eta} = 0$ subsist on the sides of the wedge. Fig. 5.

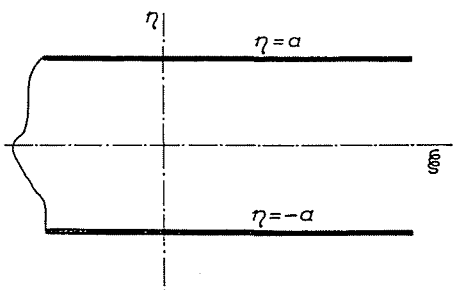


*Fig. 5*

In the $\xi$, $\eta$ plane this domain extends to infinity in the direction of increasing $\xi$-s and is bounded in the opposite direction, where no further requirement concerning the shape of the boundary and boundary conditions is postulated.

The linearity of the differential equation (5) and its constant coefficients suggest the idea to seek an asymptotic solution of the shape

$$\psi(\xi, \eta) = e^{\omega(\xi)}\, \varphi(\eta) \, .$$

Putting this into the equation (5)

$$\frac{d^4 \varphi}{d\eta^4} + (2\omega^2 - 4\omega + 4)\frac{d^2 \varphi}{d\eta^2} + \omega^4 - 4\omega^3 + 4\omega^2 = 0$$

follows.

The general solution of this differential equation is $\varphi = A \cos \omega\, \eta + B \sin \omega\, \eta + C \cos (\omega - 2)\, \eta + D \sin (\omega - 2)\, \eta$. The constants $A, B, C, D$ are to be computed from the boundary conditions $\varphi = 0$, $\dfrac{d\varphi}{d\eta} = 0$, valid at $\eta = \pm a$. After an easy transformation the boundary condition delivers the two pairs of equations

$$\begin{aligned} &A \cos \omega a + C \cos (\omega - 2)a = 0 \\ &\omega A \sin \omega a + (\omega - 2)\, C \sin (\omega - 2)a = 0 \, . \\ &B \sin \omega a + D \sin (\omega - 2)a = 0 \\ &\omega B \cos \omega a + (\omega - 2)\, D \cos (\omega - 2)a = 0 \, . \end{aligned}$$

It follows that $\varphi$ is identically zero, unless at least one of the two determinants

$$\begin{vmatrix} \cos \omega a & \cos (\omega - 2)a \\ \omega \sin \omega a & (\omega - 2) \sin (\omega - 2)a \end{vmatrix} \quad \text{and}$$

$$\begin{vmatrix} \sin \omega a & \sin (\omega - 2)a \\ \omega \cos \omega a & (\omega - 2) \cos (\omega - 2)a \end{vmatrix}$$

equals zero, *i. e.*, when

$$(1 - \omega) \sin 2a + \sin 2\,(1 - \omega)a = 0 \quad \text{or}$$
$$-\,(1 - \omega) \sin 2a + \sin 2\,(1 - \omega)a = 0 \, .$$

The second equation is fulfilled by $\omega = 0$, independently of $a$, but in this case $\varphi \equiv 0$, except when $a = \dfrac{\pi}{2}$ or $a = \pi$. For these values

$$\varphi = A(1 + \cos 2\eta)$$
$$\text{and} \qquad \varphi = A(1 - \cos 2\eta) \ \text{respectively.}$$

Inversely, if $a = \frac{\pi}{2}$, $\omega$ must be an integer, if $a = \pi$, $\omega$ must be an integer multiple of one half. The values $\omega > 0$ are excluded by the principle of de Saint-Venant, the asymptotic values of $\psi$ for $\omega < 0$ tend to zero, $a = \frac{\pi}{2}$ corresponds with the infinite half plane, $a = \pi$ with the half split whole plane.

The question arises how this reasoning is to be modified if issuing from equation (6) of finite differences instead of the differential equation (5). The values of $\psi$ can be regarded asymptotically constant along each row. Denoting them downwards from above by $\psi_{10}, \psi_2, \psi_0, \psi_4, \psi_{12}$, they satisfy the asymptotic equation

$$(4 - 4h^2)(\psi_2 + \psi_4) - (\psi_{10} + \psi_{12}) - (6 - 8h^2)\,\psi_0 = 0$$

Moreover the boundary condition $\psi = 0$ must hold and the fictitious value of $\psi$ just outside the boundary must be equal to the value just inside of it.

Numbering the $\psi$-values provisionally consecutive from the boundary, the following equations must hold

$$\begin{aligned}
(-7 + 8h^2)\,\psi_1 + (4 - 4h^2)\,\psi_2 - \psi_3 &= 0\\
(4 - 4h^2)\,\psi_1 + (-6 + 8h^2)\,\psi_2 + (4 - 4h^2)\,\psi_3 - \psi_4 &= 0\\
-\psi_1 + (4 - 4h^2)\,\psi_2 + (-6 + 8h^2)\,\psi_3 + (4 - 4h^2)\,\psi_4 - \psi_5 &= 0 \qquad (9)\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
-\psi_{n-2} + (4 - 4h^2)\,\psi_{n-1} + (-7 + 8h^2)\,\psi_n &= 0.
\end{aligned}$$

There are as many equations as unknown $\psi$-values. This system of homogeneous linear equations has a non-trivial solution only if the equation

$$\begin{vmatrix}
-7+8h^2 & 4-4h^2 & -1 & & & & \\
4-4h^2 & -6+8h^2 & 4-4h^2 & -1 & & & \\
-1 & 4-4h^2 & -6+8h^2 & 4-4h^2 & -1 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & -1 & 4-4h^2 & -6+8h^2 & 4-4h^2 \\
 & & & & -1 & 4-4h^2 & -7+8h^2
\end{vmatrix} = 0 \qquad (10)$$

hold.

For the first moment it seems that $h = \frac{\pi}{n+1}$ could fulfil this equation, but this is not the case. As a matter of fact it is necessary to distort $h$ for the use in the formulae and to compute its distorted value from this equation (10). Introducing the following matrices:

$$\mathbf{A}=\begin{bmatrix} -7 & 4 & -1 & & & \\ 4 & -6 & 4 & -1 & & \\ -1 & 4 & -6 & 4 & -1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & -1 & 4 & -6 & 4 \\ & & & -1 & 4 & -7 \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix} 8 & -4 & & & & \\ -4 & 8 & -4 & & & \\ & -4 & 8 & -4 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & -4 & 8 & -4 \\ & & & & -4 & 8 \end{bmatrix}, \quad \Psi=\begin{bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \psi_n \end{bmatrix},$$

the equations (9) and (10), respectively, can be written

$$(\mathbf{A}+\mathbf{B}\,h^2)\,\Psi=0 \qquad |\mathbf{A}+\mathbf{B}\,h^2|=0$$

respectively.

Good approximative values of the $\psi$-s are known: as

$$\eta_k=\frac{\pi}{2}-\frac{k\pi}{n+1},$$

they are

$$\psi_k=A\left(1-\cos\frac{2k\pi}{n+1}\right) \qquad (k=0,1\ldots n)\,. \tag{11}$$

This approximation is the better, the denser the net is, *i. e.*, the larger $n$ is. With its help Rayleigh's formula can be applied:

$$h=\sqrt{\frac{-\Psi^*\,\mathbf{A}\,\Psi}{\Psi^*\,\mathbf{B}\,\Psi}}\,. \tag{12}$$

This value of $h$ is to be put into equation (6) and into the relaxation pattern (Fig. 4.).

Our experience has shown, that the accuracy of $h$ and of the $\psi$-s computed from (11) is perfectly sufficient. By their help it is possible to construct a block effecting already the asymptotic distribution of the $\psi$ values defined by equation (11).

$$\begin{array}{cccccccc}
 & & -1 & -1 & -1 & -1 & \dots \\
 & -(2-2h) & 6+2h-4h^2 & 4-4h^2 & 4-4h^2 & 4-4h^2 & \dots \\
-1-2h & 7-6h-4h^2 & -13-6h+12h^2 & -5+2h+8h^2 & -6+8h^2 & -6+8h^2 & \dots \\
 & -(2-2h) & 6+2h-4h^2 & 4-4h^2 & 4-4h^2 & 4-4h^2 & \dots \\
 & & -1 & -1 & -1 & -1 & \dots
\end{array}$$

*Fig. 6*

First of all we construct a line block, extending on one side to infinity. (Fig. 6.) Employing it, it is easy to obtain the wanted block by the following procedure:

The $\psi$ values must be altered simultaneously in each row proportional to the $\psi_n$-s. The block obtained in this way has numbers differing from zero in its first four columns only. Applying this block we are able to remove residuals easily and definitively, therefore we gave it the name wonder block.

It is interesting, to compute the sum of residuals removed by a single application of this wonder block. Adding up the columns of the line block operator Fig. 6., the one-dimensional block Fig. 7. is yielded.

$$-(1-2h) \quad 3-2h-4h^2 \quad -3-2h+4h^2 \quad 1+2h \quad 0 \quad 0 \quad \dots$$

*Fig. 7.*

This block, too, alters the sum of the residuals in the four first columns only, their total being left invariable, for this reason it suffices to examine only the line blocks adjacent to the boundary and only the first four columns.

As

$$\eta_1 = \frac{\pi}{2} - h, \ \eta_2 = \frac{\pi}{2} - 2h,$$

the equations

$$\psi_1 = 2Ah^2, \qquad \psi_2 = 2^2 \cdot 2Ah^2, \dots$$

are approximatively valid for small values of $h$. Suppressing the common factor $2Ah^2$, the first four columns of the line block adjacent to the boundary remove the sum $8 + 4h - 4h^2$ of residuals equal to the total of the second row; those of the subsequent line block remove the sum $2^2\,(-2)$ equal to the fourfold total of the first row.

On the whole the approximate sum of residuals $2 \cdot 2Ah \cdot 4h = 16\,Ah^3$ is removed on the two boundaries.

The question arises where the front of the wonder block is to be put. To answer this question each column of the residuals to be relaxed are added together and so are the columns of the wonder block. The table of residuals made in this way one dimensional can now be relaxed by this one-dimensional wonder block. The result shows where the front of the original two-dimensional wonder block is to be put and what multiplicator is to be applied. These operations having been made, the sum of residuals in each column becomes zero, the residuals themselves become of alternating signs and can be removed by point relaxation or by small blocks. It may happen that some residuals steal back and the whole series of operations must be repeated, but with much smaller residuals now.

## 3. Biharmonic relaxation decomposed into twofold harmonic relaxation

In cartesian coordinates this procedure is already known [2]. t. 2. pp. 254—261 and 274—277.

In polar coordinates the differential equation

$$\nabla^2 \frac{1}{r^2} \nabla^2 \psi(\xi, \eta) = 0$$

can be decomposed into the differential equations

$$\nabla^2 w(\xi, \eta) = 0 \quad \text{and} \quad \nabla^2 \psi(\xi, \eta) = r^2 w(\xi, \eta) .$$

Apart from the factor $r^2$ these are the same equations as for cartesian coordinates. The advancement to a finer net is performed in a similar way as in cartesian coordinates.

We were concerned with problems for a doubly connected domain. The second and third of equations (4) are fulfilled by symmetry. The first equation can be rewritten to finite differences in the following way:

$$\iint_{C_2} \nabla^4 \psi(x, y)\, dx\, dy = \iint_{C_2} \frac{1}{r^2} \nabla^2 \left[ \frac{1}{r^2} \nabla^2 \psi(\xi, \eta) \right] r^2\, d\xi\, d\eta =$$

$$= -\frac{1}{h^4} \sum_{C_2} \frac{R}{r^4} r^2 h^2 = \frac{1}{h^2} \sum_{C_2} \frac{R}{r^2} = 0 .$$

Thus the equation $\sum \frac{R}{r^2} = 0$ must hold for the region interior to the inner boundary. Both at the biharmonic and at the second harmonic relaxation this equation serves for the computation of the residual of the block consisting of region $C_2$.

## Summary

In the first part of the paper the formula for the relaxation of the biharmonic differential equation is computed for the case, that instead of cartesian coordinates polar ones are used.

In the second part the application of blocks extending to infinity is shown. Circumstances necessitate thereby a slight distortion of the mesh size.

In the third part the method of dissolving the biharmonic relaxation into two successive harmonic ones is used in polar coordinates. The difficulty arising from an interior boundary is overcome.


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