ANALYSIS OF COUPLING MECHANISMS

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Introduction

The present study analyses kinematic and dynamic relations of transmission of rotary motion between two connected shafts, parallel or intersecting. In its course the following points will be investigated:

1. Degree of freedom necessary for coupling mechanisms;
2. Variations of angular velocities and angular acceleration of connected parallel shafts; moments and forces acting on the shafts;
3. Irregularities of rotation occurring at connected intersecting shafts; torques and bending moments acting on the shafts;
4. How shafts, parallel, intersecting or skew, can be connected so that the angular velocities of both shafts are equal at any instant;
5. Basic principles of design of synchronous drives.

Within the scope of the above the accuracy of approximate formulae s examined and the limits of their applicability are outlined.

1. Degree of freedom necessary for coupling mechanisms

Locate the two shafts to be connected, 1 and 2, in a system of coordinates x, y, z so that the central lines of the shafts coincide with the axis y (Fig. 1a).

The degree of freedom of a coupling mechanism denotes the number of independent relative motions allowed for the shafts by the coupling along or about the coordinate axes.

a) Shift shaft 2 parallelly with itself along the axis x to the distance s_x, and along the axis z to the distance s_z (Fig. 1b). As a result two parallel shafts at a distance of $s = \sqrt{s_x^2 + s_z^2}$ from each other have been obtained. Accordingly, the coupling connecting these parallel shafts should be such that they have relative motions along two directions perpendicular to each other.

b) Rotate shaft 2 about the axis x through an angle α_x, and about the axis z through an angle α_z (Fig. 1c). Two shafts located at an angle

$\alpha = \arccos(\cos \alpha_x \cos \alpha_z)$ relatively to each other have been produced. Accordingly, for their coupling a mechanism is needed which allows them to rotate about two axes perpendicular to each other.

Possibilities of securing the degree of freedom necessary for coupling mechanisms will be examined below.

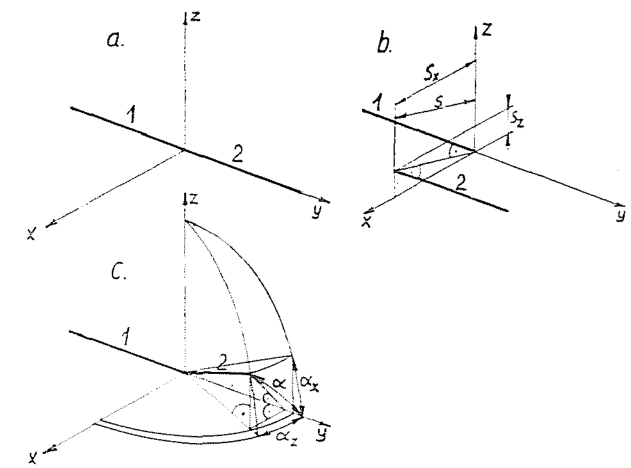

Fig. 1

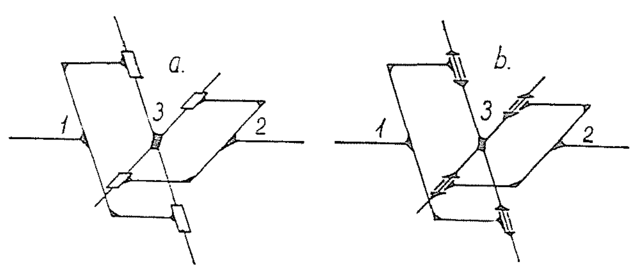

Fig. 2

For parallel shafts the two motions of translation can be produced by means a cross-pin inserted between the shafts which allows them relative motions in two directions perpendicular to each other (Fig. 2 a). Such is the mechanism of the Oldham coupling.

For shafts center lines of which intersect, the two rotations can be made possible through the insertion of a cross-pin similar to that mentioned above, with the difference that a joint mechanism is applied instead of the solution with a link (Fig. 2b). This is the way a cardan joint is developed.

In both solutions a third member, viz. a cross-pin has been inserted between the shafts. The problem can also be solved in such a way that the two shafts secure the degree of freedom necessary through a direct contact (Fig. 3a and 3b). A special formation of the two contacting shaft ends

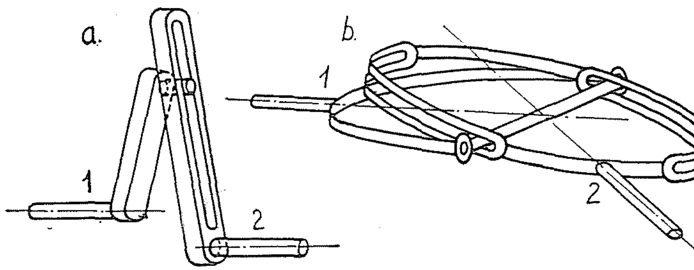

Fig. 3

is required for this purpose. The simplest ways of connecting two shafts, parallel or intersecting, directly to each other, will be investigated next.

2. Transmission of rotation between parallel shafts

a) Kinematical analysis

In the case of parallel shafts connected directly to each other, for the angle of rotation φ_1 of the driving shaft 1, angle of rotation φ_2 of the driven

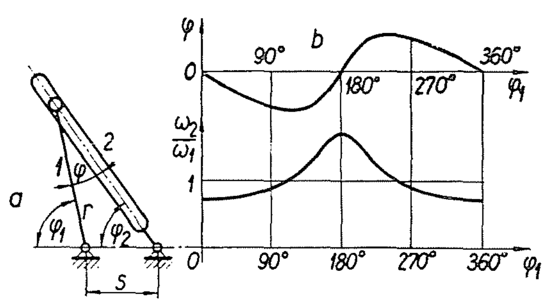

Fig. 4

shaft 2 and angular deviation $\varphi_2 - \varphi_1 = \varphi$, from Fig. 4a, the following relationships may be derived :

$$\operatorname{tg} \varphi_2 = \frac{r \cdot \sin \varphi_1}{s + r \cdot \cos \varphi_1} \tag{2.1}$$

$$\operatorname{tg}\varphi = -\frac{s\cdot\sin\varphi_1}{r+s\cdot\cos\varphi_1} \tag{2.2}$$

The maximum of angular displacement can be determined through differentiation of 2.2 with respect to φ_1, which gives

$$\operatorname{tg}\varphi_{\max} = -\frac{\frac{s}{r}}{\sqrt{1-\left(\frac{s}{r}\right)^2}} \tag{2.3}$$

This value appears at $\varphi_2 = 90°$.
The variation in angular velocity is

$$\delta = \frac{\omega_{2\max}-\omega_{2\min}}{\omega_1} = \frac{2\frac{s}{r}}{1-\left(\frac{s}{r}\right)^2} \tag{2.4}$$

and the approximate magnitude of the degree of irregularity

$$\delta' \cong 2\,\frac{s}{r} \tag{2.5}$$

Taking this into consideration and substituting tg $\varphi \cong \varphi$, Taylor's expansion of 2.2 gives

$$\varphi' \cong -\frac{\delta'}{2}\sin\varphi_1 + 0.5\left(\frac{\delta'}{2}\right)^2 \sin 2\,\varphi_1 \tag{2.6}$$

Relative error in 2.6 is

$$R = \left|\frac{\varphi-\varphi'}{\varphi}\right| \leqq 2\left(\frac{s}{r}\right)^2 + 1 - \frac{\varphi_{\max}}{\operatorname{tg}\varphi_{\max}} \tag{2.7}$$

which, for

$\frac{s}{r} = 0{,}10$ 0,15 and 0,20

with $\varphi_{\max} = 5{,}7°$ 8,6° and 11,5°

gives a relative error

of $R \leqq$ 2,5% 5,2% and 9,2% respectively.

As can be seen, the approximate expression 2.6 yields results of sufficient accuracy only in case of

$$\frac{s}{r} < 0.2 \tag{2.8}$$

provided an error of 9% may still be considered as tolerable.

The approximate value of the variation of relative angular velocity may be obtained through differentiation of Eq. 2.6 with respect to time, and division by ω_1:

$$\frac{\Delta\omega}{\omega_1} \cong -\frac{\delta'}{2}\cos\varphi_1 + \left(\frac{\delta'}{2}\right)^2 \cos 2\varphi_1 \tag{2.9}$$

where

$$\Delta\omega = \omega_2 - \omega_1 \,.$$

Angular acceleration of the shaft 2 is

$$\varepsilon_2 = \omega_1^2 \left[\frac{\delta'}{2}\sin\varphi_1 - 2\left(\frac{\delta'}{2}\right)^2 \sin 2\varphi_1\right] \tag{2.10}$$

Variations of the angular deviation and angular velocity are shown in Fig. 4b.

The result 2.8 obtained by the investigation of the accuracy of Formula 2.6 holds true, of course, also for Relationship 2.9 and 2.10.

b) *Dynamical analysis*

As is seen in Fig. 5a, on both shaft ends an only force P is exerted, loading then for torsion and bending.

The torque acting on the shaft 1 is

$$M_{cs_1} = M = P_0 \cdot r = P \cdot r \cdot \cos\varphi \tag{2.11}$$

On the basis of the figure the torque affecting shaft 2 may be written as

$$M_{cs_2} = P\,[r \cdot \cos\varphi + s \cdot \cos\varphi_2] =$$
$$= M\left[1 + \frac{s}{r}\,\frac{\cos\varphi_2}{\cos\varphi}\right] \tag{2.12}$$

Substituting the values 2.1 and 2.2 in 2.12, the relative variation in the torque

$$\frac{\Delta M_{cs}}{M} = \frac{s + r \cdot \cos\varphi_1}{r + s \cdot \cos\varphi_1} \cdot \frac{s}{r} \tag{2.13}$$

where

$$\Delta M_{cs} = M_{cs_2} - M \cdot$$

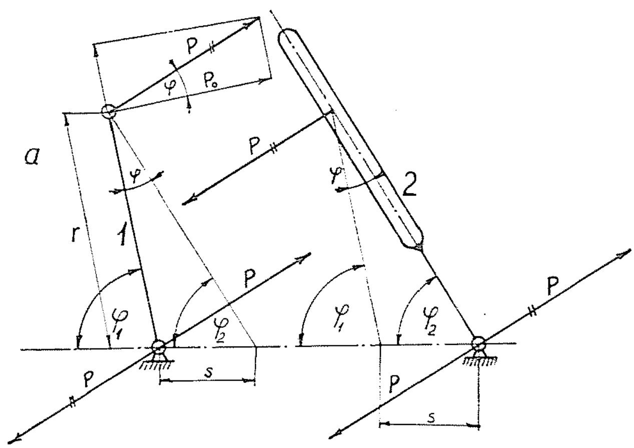

Fig. 5a

In addition to the torque, the force P acts on both shaft ends. Its magnitude is according to the figure

$$P = \frac{P_0}{\cos\varphi} = \frac{M}{r \cdot \cos\varphi} \tag{2.14}$$

Substititung the value of 2.2 in 2.14 we get

$$P = \frac{M}{r} \frac{\sqrt{r^2 + s^2 + 2 \cdot r \cdot s \cdot \cos\varphi_1}}{r + s \cdot \cos\varphi_1} \tag{2.15}$$

The way of construction and the variation in time of the relative torque are shown in Fig. 5b, and the bending force P can be constructed according to Fig. 5c.

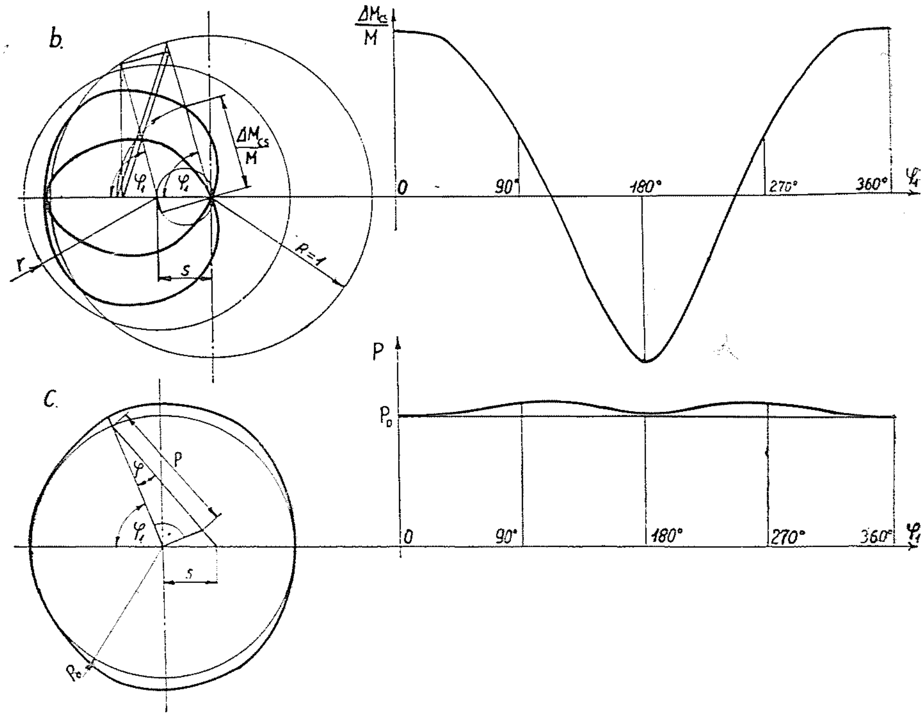

Fig. 5b, c

3. Transmission of rotation between shafts intersecting

a) Kinematical analysis

For shafts center lines which intersect, between the angles of rotation φ_1 and φ_2 of the driving shaft 1 and driven shaft 2, respectively, and shaft angle α from Fig. 6a, the following relationship may be derived :

$$\operatorname{tg} \varphi_2 = \operatorname{tg} \varphi_1 \cdot \cos \alpha \tag{3.1}$$

Hence the angular deviation $\varphi = \varphi_2 - \varphi_1$ may be determined as

$$\operatorname{tg} \varphi = \frac{\operatorname{tg} \varphi_1 (\cos \alpha - 1)}{1 + \operatorname{tg}^2 \varphi_1 \cdot \cos \alpha} \tag{3.2}$$

The angular deviation has its largest value at

$$\operatorname{tg} \varphi_1 = \frac{1}{\sqrt{\cos \alpha}} \tag{3.3}$$

this largest value is

$$\operatorname{tg} \varphi_{\max} = \frac{\cos \alpha - 1}{2\sqrt{\cos \alpha}} \tag{3.4}$$

The variation in angular velocity

$$\delta = \frac{\omega_{2\max} - \omega_{2\min}}{\omega_1} = \frac{1}{\cos \alpha} - \cos \alpha \tag{3.5}$$

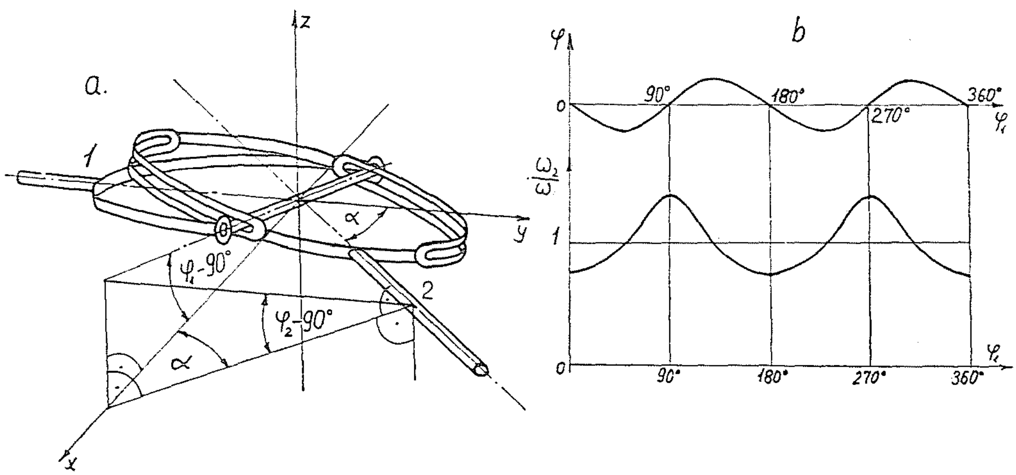

Fig. 6

and the approximate value of the degree of irregularity

$$\delta' \cong 2\ (1 - \cos \alpha) \tag{3.6}$$

Taking this into consideration and substituting tg $\varphi \cong \varphi$, Taylor's expansion of 3.2 gives

$$\varphi' \cong -\frac{\delta'}{4}\left(1 + \frac{\delta'}{4}\right) \sin 2\varphi_1 + 0.5\left(\frac{\delta'}{4}\right)^2 \sin 4\varphi_1 \tag{3.7}$$

Relative error in 3.7 is

$$R = \left|\frac{\varphi - \varphi'}{\varphi}\right| \leq$$

$$\leq (1 - \cos \alpha)^2 + \frac{1 - \cos \alpha}{1 + \cos \alpha} + 1 - \frac{\varphi_{\max}}{\operatorname{tg} \varphi_{\max}} \tag{3.8}$$

which, for

$$\alpha = 20^\circ \qquad 25^\circ \quad \text{and} \quad 30^\circ$$

$$\text{with } \varphi_{\max} = 1{,}9^\circ \qquad 2{,}8^\circ \quad \text{and} \quad 4{,}2^\circ$$

gives a relative error of $R \leqq 3{,}5\% \qquad 5{,}9\%$ and $9{,}1\%$ respectively.

Thus the approximate formula 3.7 has a satisfactory accuracy only in case of

$$\alpha < 30^\circ \tag{3,9}$$

again provided an error of 9% may still be tolerated.

The approximate variation of angular velocity is given by the first derivative with respect to time of Eq. 3.7 as

$$\frac{\Delta\omega}{\omega_1} \cong -\frac{\delta'}{2}\left(1+\frac{\delta'}{4}\right)\cos 2\varphi_1 + 0{,}5\left(\frac{\delta'}{2}\right)^2 \cos 4\varphi_1 \tag{3.10}$$

and the approximate variation in the angular velocity of shaft 2

$$\varepsilon_2 \cong \omega_1^2\left[\delta'\left(1+\frac{\delta'}{4}\right)\sin 2\varphi_1 - 0{,}5\,\delta'^2 \sin 4\varphi_1\right] \tag{3.11}$$

The variations of angular deviation and relative angular velocity are shown in Fig. 6b.

b) *Dynamical analysis*

Moment is transmitted from shaft 1 to shaft 2 through a couple of forces perpendicular to shaft 2. As is seen in Fig. 7a, torques and bending moments acting on the shafts are

$$M_{cs_1} = M = P \cdot a \tag{3.12}$$

$$M_{h_1} = (P \cdot \operatorname{tg}\vartheta)\, a = M \cdot \operatorname{tg}\vartheta \tag{3.13}$$

$$M_{cs_2} = \frac{P}{\cos\vartheta}\, a \cdot \sin\delta = M\,\frac{\sin\delta}{\cos\vartheta} \tag{3.14}$$

$$M_{h_2} = \frac{P}{\cos\vartheta}\, a \cdot \cos\delta = M\,\frac{\cos\delta}{\cos\vartheta} \tag{3.15}$$

The angle ϑ denotes the angle of inclination of the two forks' planes: According to Fig. 7a, its magnitude is

$$\operatorname{tg} \vartheta = \frac{\sin \alpha \sin (\varphi_1 - 90°)}{\cos \alpha} = - \operatorname{tg} \alpha \cos \varphi_1 \tag{3.16}$$

The magnitude of angle δ, again according to Fig. 7a is
$$\cos \delta = \sin \alpha \cos (\varphi_1 - 90°) = \sin \alpha \sin \varphi_1 \tag{3.17}$$

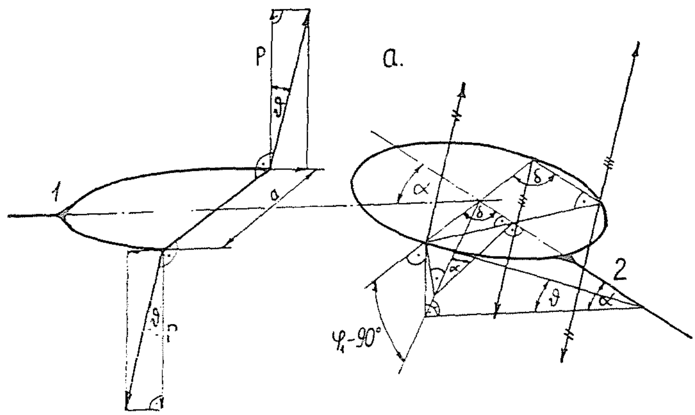

Fig. 7a

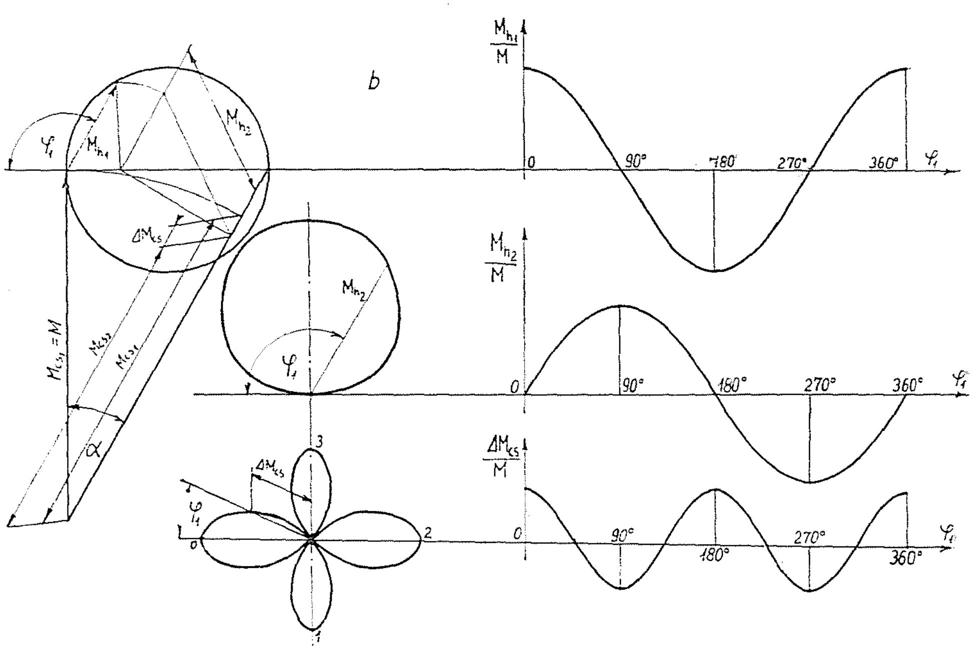

Fig. 7b

By substituting 3.16 and 3.17 in 3.13, 3.14 and 3.15 the variations of magnitude of the moments with respect to φ_1 are obtained

$$M_{h_1} = M \cdot \mathrm{tg}\ \alpha\ \cos\ \varphi_1 \tag{3.18}$$

$$M_{cs_2} = M \cdot \cos\ \alpha\ (1 + \mathrm{tg}^2\ \alpha\ \cos^2\ \varphi_1) \tag{3.19}$$

$$M_{h_2} = M \cdot \sin\ \alpha\ \sin\ \varphi_1 \sqrt{1 + \mathrm{tg}^2\ \alpha\ \cos^2\ \varphi_1} \tag{3.20}$$

The method of construction for each moment as well as their variations are shown in Fig. 7b.

4. How to obtain a uniform transmission of rotation

A common objectionable feature of the couplings described in the paragraphs above is that the angular velocity of the shaft is variable, even

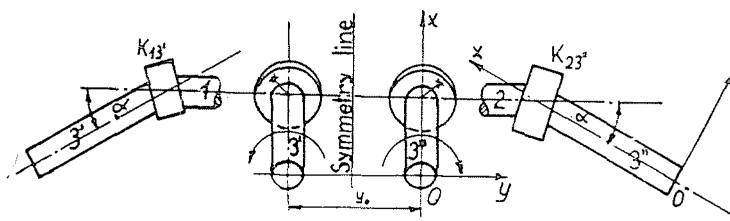

Fig. 8

if the driving shaft 1 has a constant angular velocity. This state is disadvantageous in regard to the inertia forces. Next will be shown how these variabilities can be done away with for connected shafts

a) parallel,
b) intersecting and
c) skew.

Put the two shafts (1 and 2) side by side parallelly and connect a third shaft, 3′ to 1 through a coupling $K_{13'}$ of any chosen design provided it leaves two degrees of freedom. Let the shaft angle of 1 and 3′ be α. Connect a shaft 3″ to 2 through a coupling $K_{13''}$ which is the opposite-hand view of $K_{13'}$. Thus the systems 1—3′ and 2—3″ form a symmetrical layout (Fig. 8).

Rotate shaft 3′ through an angle $\varphi_{3''}$ and shaft 3″ through an angle $\varphi_{3''} = - \varphi_{3'}$ about their own center lines. As a consequence of the symmetrical layout, the angular displacements of shafts 1 and 2 will be

$$\varphi_1 = - \varphi_2 \tag{4.1}$$

a) Let a space coordinate system as shown in Fig. 8 be taken. Rotate the system 2—3″ about the axis y through 180°, then shift it parallelly in the direction $-y$ to a distance y_0. Connecting the two shaft ends in this new position to each other rigidly (Fig. 9), the angular displacements of the parallel shafts 1 and 2 will be equal, viz.

$$\varphi_1 = \varphi_2 \tag{4.2}$$

As is seen a uniform transmission of rotation between parallel shafts can be obtained in case the layout is symmetrical in respect to some center O.

Let us enlarge the link mechanism shown in Fig. 4a by using symmetry with respect to some center (Fig. 10a). The disadvantage with this device

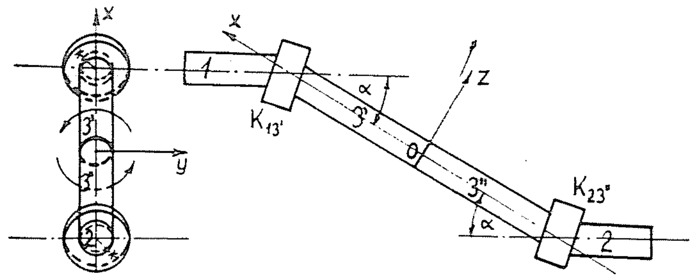

Fig. 9

is that also the link 3 has to be mounted in bearings. In order to avoid this, a link mechanism with

$$s = r \tag{4.3}$$

will be examined.

From Fig. 10b

$$\varphi_1 = \varphi_2 = 2\,\varphi_3 \tag{4.4}$$

i. e. link 3 rotates with an angular velocity of $\omega_3 = \dfrac{\omega_1}{2}$. Complete the link mechanism O_1ABO_2 through a mechanism $O_1A'B'O_2$ similar to the former but advancing relative to it by 180° (Fig. 10c). The two links will keep a constant advance of 90° relative to each other. As a consequence they can be replaced by a rigid cross member. The bearing support at point O has by this become superfluous.

By connecting points A' and B instead of O_1 and O_2 to the frame and dropping the members 1, 4, 2 and 5′ the Oldham coupling as shown in Fig. 2a will be arrived at (Fig. 10d).

The Oldham coupling is the simplest means for securing a true uniform transmission of rotation between parallel shafts. But because of constructional features it can be used only in such cases when the distances of the two shafts are not too large.

If the distance of the two shafts is somewhat large, the device shown in Fig. 9 will be used in practice. $K_{13'}$ and $K_{13''}$ for instance may be cardan joints.

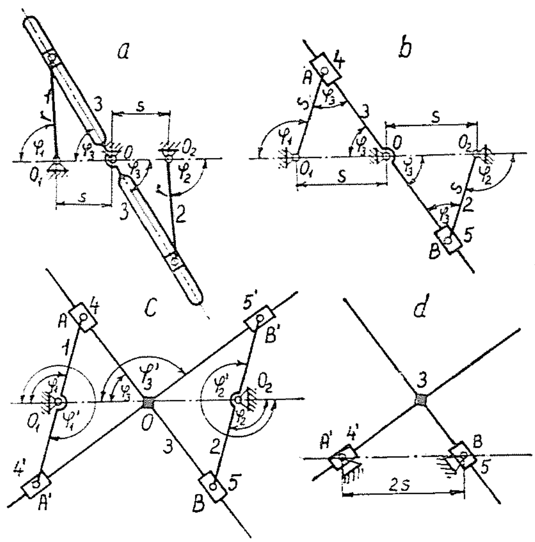

Fig. 10

b) By rotating the system 2—3″ in Fig. 8 about the axis z through 180°, a uniform transmission of rotation between intersecting shafts is secured (Fig. 11). Such a layout is symmetrical in respect to a plane (x, y) *i. e.* it is symmetrical as an image seen in a mirror.

c) Finally, rotate system 2—3″ through 180° about an optional vector $\bar{v} = \cos\beta\,\bar{j} + \sin\beta\,\bar{k}$ lying in the plane y, z. As a result the skew shafts shown in Fig. 12 are obtained. Between the shaft angle $2\,\alpha'$ of 1 and 2, the angle α and the angle of inclination γ of the two planes determined by the shafts 1—3′ and 2—3″, respectively, the following relationship holds true :

$$\cos\gamma = \frac{\cos^2\alpha - \cos^2\alpha'}{\sin^2\alpha} \tag{4.5}$$

Hence follows that a coupling of skew shafts gives a true uniform transmission of rotation if the two coupling elements mounted on the two ends of the shaft 3 are rotated relatively to the opposite-hand layout through an angle γ about the axis x.

In the cases described in *b)* and *c)* the couplings $K_{13'}$ and $K_{13''}$ are usually cardan joints.

An objectionable feature of the double cardan joint devices is that though the angular velocities of the shafts 1 and 2 are equally uniform, the

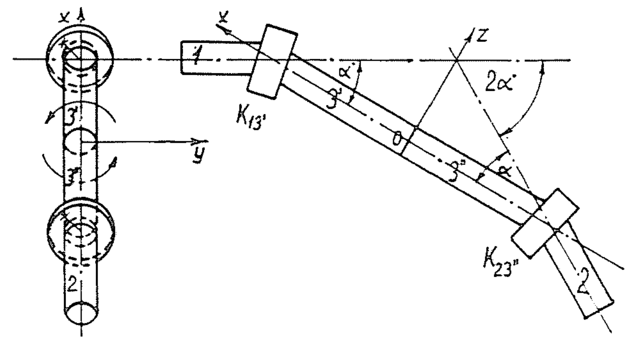

Fig. 11

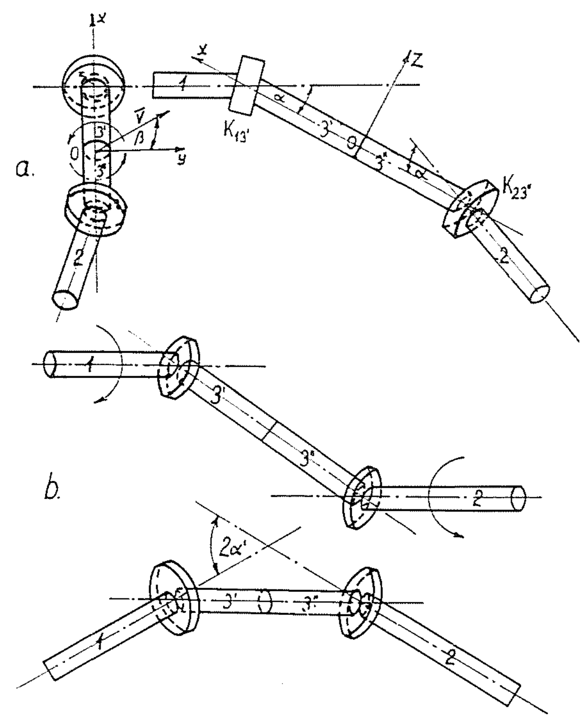

Fig. 12

angular velocity of 3 will keep varying, which may give rise to considerable inertia forces. Variability of the angular velocity of shafts 3 can be eliminated by application of synchronous drives.

5. Synchronous drives

True uniform transmission of rotation between shaft 1 and 2 has been obtained by application of two couplings and a third shaft 3, the latter rotating with variable angular velocity.

In order to eliminate the variability in speed also at the shaft 3, a direct coupling suitable to secure true uniform transmission of rotation between two intersecting shafts has to be designed.

As has been seen in paragraph 4b for intersecting shafts, a true uniform transmission of rotation can be obtained through a device symmetrical to a plane, *i. e.* representing an opposite-hand view of the counter-part. The same requirement can also be fulfilled by an only, symmetrically designed joint.

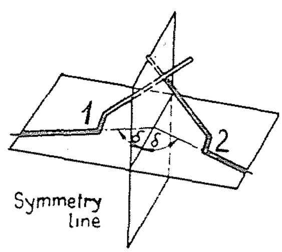

Fig. 13

Such a solution is shown in Fig. 13, where two shaft ends developed symmetrically are connected directly to each other. The contact of the shaft ends is at a single spot. While the shafts are rotating, the contact spot as well as the point of intersection of the center lines are located, always in a plane bisecting the shaft angle and perpendicular to the common plane of the two shafts. This device ensures keeping up a perfect symmetry by a complete revolution, *i. e.* the transmission of rotation is true uniform.

With this solution the contact of the shaft ends is at a point of the circumference. As a consequence, a large surface pressure and considerable bending forces arise. Bending forces can be taken care of through a distribution of the transmission of moments over several points of the circumference, and the contact at a point can be avoided by means of halfcylinders making contact on plane surfaces and rotating in their seats (Fig. 14).

By another solution moments are transmitted through balls placed along the circumference (Fig. 15). The balls are located at the points of intersection of semi-circular races made in the driving and driven shafts. A ball-and-socket joint located at the point of intersection of the shaft center lines prevents axial displacements. This is important as any axial displacement of the shafts upsets the symmetry and brakes off the uniformity of the transmission of rotation.

The races must be made so that their point of intersection always lies in the plane of symmetry and their angle of intersection is fairly large. Should the angle of intersection of the races be too small, they, might come

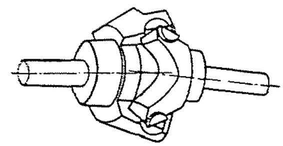

Fig. 14

into covering at a certain angular displacement φ_1, and the ball would lose its definite location in the plane of symmetry. Therefore it is most advisable to design the races in such a way that they intersect each other at some constant angle, for any angle of rotation φ_1. This requirement may be met by

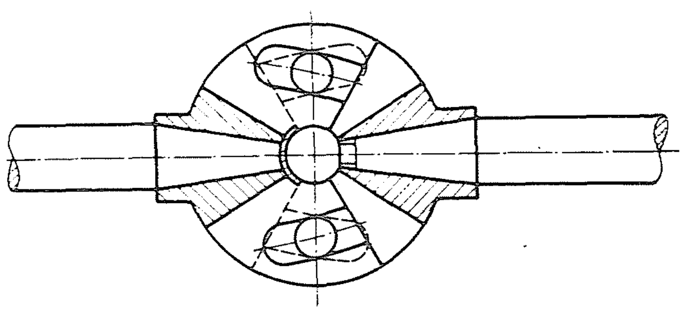

Fig. 15

races designed as logarithmic spirals. In order to facilitate processing logarithmic spirals can succesfully be substituted by osculatory circles.

Such couplings securing a true uniform transmission of rotation through direct connection of intersecting shafts are termed synchronous drives.

By using synchronous drives instead of the couplings shown in Fig. 9, 11 and 12, shafts parallel, intersecting or skew can be connected in such a way that not only shaft 2, but also 3 rotates with the same angular velocity as driving shaft 1.

Summary

Kinematic and dynamic analyses of several possibilities for connecting shafts are described in the present paper. Construction methods for the determination of moments and forces acting on shafts are expounded.

General principles for couplings suitable to ensure uniform transmission of rotation for shafts of any chosen location are examined and examples of their applications are shown.

Finally, several examples of direct coupling devices suitable for ensuring a true uniform transmission of rotation for intersecting shafts, viz. of synchronous drives are given.

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