THE FUNDAMENTAL BENDING FREQUENCY OF AXIAL COMPRESSOR BLADES IN CASE OF ELASTIC FIXING

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After the end of World War II — as a result of the general adoption of gas turbine aircraft power plants — cases of compressor or turbine blade failure were repeatedly reported. These failures were not due to the direct aerodynamic and centrifugal loads acting on the blades but occurred as a result of blade fatigue caused by vibration phenomena. All reports published in this question agree that fatigue is the result of blades vibrating in their fundamental bending mode. In order to investigate vibration conditions on compressor blades it is indispensable to know the exact value of the fundamental bending frequency.

The compressor blade may be regarded as a beam fixed at its extremity. Different methods have been evolved in the literature for determining the natural frequencies of a similar beam. Vibration frequency of a blade fixed on the rotating rotor is increased by the stiffening effect of centrifugal loads acting upon the blade. This effect is taken into account by some of these methods. In the following we propose to consider the fact that the fixing of the blade, as a rule, cannot be regarded as a strictly rigid one. As the literature furnishes no easily applicable relations for this case, we suggest a formula deduced in the following paragraphs — by using the Rayleigh method — to determine the fundamental bending frequency of an elastically fixed beam (blade) mounted into a rotating rotor.

For the sake of simplicity let us suppose that the section of the blade along its full length remains constant and that its twisting has a small value negligible in vibration analysis. These conditions are generally valid for the case of axial compressor blading.

The strain of a section — at a distance x from the fixing — of an elastically fixed blade is the sum of two components (Fig. 1):

1. the strain component y_1 due to elastic blade strain and
2. the strain component y_2 due to a twist by an angle φ of the fixture.

Thus we have

$$y = y_1 + y_2 . \tag{1}$$

For y_2 we can write

$$y_2 = x\varphi = x\frac{M}{K}$$

where M is the bending moment applied at the fixing section of the blade and caused by the elastic strain y_1, while K denotes the elastic constant for the fixing, which is taken as invariant for the following discussions. The value of K shall be determined by experiment, taking into account centrifugal and lateral loads acting upon the blade root, because the latter factors will considerably alter it.

M may be computed by taking into account the elastic form of the beam :

$$M = J\,E\,[y_1'']_{x=0} \tag{2}$$

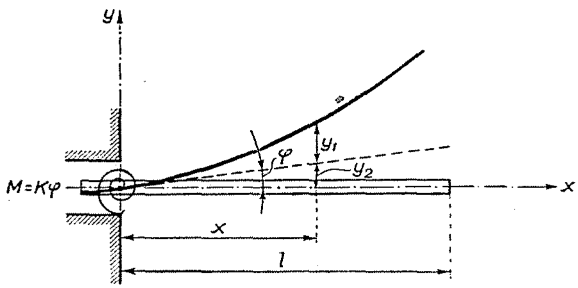

Fig. 1

Thus we have

$$y_2 = \frac{x}{K}\,J\,E\,[y_1'']_0$$

and

$$y = y_1 + \frac{x}{K}\,J\,E\,[y_1'']_0 \tag{3}$$

The mass of a blade element of a length dx and at a distance x is given by $\mu\,dx$, its velocity being ya when passing through the rest position. Here μ will denote the unit mass (mass per unit length) of the beam, while a denotes the circular frequency of the vibration. The kinetic energy of the beam as a whole will be given therefore by

$$E_l = \int_0^l \mu\,\frac{y^2\,a^2}{2}\,dx\,.$$

For a beam of constant section $\mu =$ const. and since in this case we are dealing with natural frequency vibrations, a must have the same value for every mass element of the beam. We may therefore write :

$$E_l = \frac{1}{2}\,\mu a^2 \int_0^l y^2\,dx$$

or, by taking into account Eq. (3) :

$$E_l = \frac{1}{2}\mu a^2 \int_0^l \left(y_1 + \frac{x}{K} JE\,[y_1'']_0\right)^2 dx\,. \tag{4}$$

In the extreme position of the blade its potential energy L is the sum

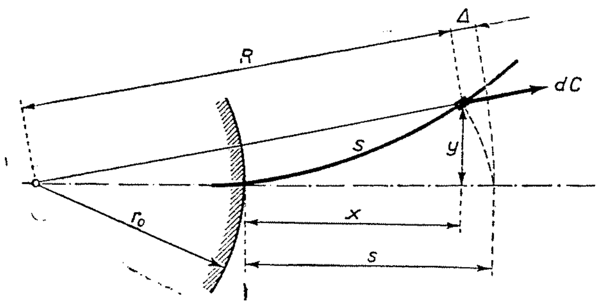

Fig. 2

obtained from the strain work of the beam flexured as a consequence of vibration and from the work performed by the elastic fixture :

$$L = \frac{1}{2} JE \int_0^l y_1''^2\,dx + \frac{1}{2} M\varphi = \frac{1}{2} JE \int_0^l y_1''^2\,dx + \frac{1}{2}\,\frac{M^2}{K}$$

or, by considering Eq. (2), we have :

$$L = \frac{1}{2} JE \left(\int_0^l y_1''^2\,dx + \frac{JE}{K}\left\{[y_1'']_0\right\}^2\right). \tag{5}$$

In order to take into account the effect of rotor rotation we must define the work performed by the centrifugal force while the blade returns from its strained extreme position to its neutral position. With the notations given in Fig. 2 the value of elementary centrifugal force may be defined by[1]

$$dC = R\,\Omega^2 \mu\,dx \approx (r_0 + x)\,\Omega^2 \mu\,dx$$

(where Ω denotes the angular velocity of rotation). This force performs work along a displacement Δ.

[1] G. Mesmer: Freie Schwingungen stabförmiger Körper. Ing. Arch. 1937.

We may write the following formula for Δ, using again the notations of Fig. 2

$$\Delta = r_0 + s - R \tag{6}$$

On the other hand

$$R^2 = (r_0 + x)^2 + y^2$$

so that

$$R - (r_0 + x) = \frac{y^2}{R + r_0 + x} \approx \frac{y^2}{2\,(r_0 + x)}$$

i. e.

$$R = \frac{y^2}{2\,(r_0 + x)} + r_0 + x\,.$$

By using the known formula for arc length we have

$$s = \int_0^x \sqrt{1 + y'^2}\, dx \approx \int_0^x \left(1 + \frac{y'^2}{2}\right) dx = x + \frac{1}{2}\int_0^x y'^2\, dx\,.$$

By substituting the expressions for R and s into Eq. (6), we have

$$\Delta = \frac{1}{2}\left(\int_0^x y'^2\, dx - \frac{y^2}{r_0 + x}\right). \tag{7}$$

The work performed by the elementary centrifugal force will be

$$dL_c = dC \cdot \Delta = \frac{1}{2}\,\mu\,\Omega^2 \left[(r_0 + x) \int_0^x y'^2\, dx - y^2\right] dx$$

while total work as performed by the centrifugal force acting upon the blade as a whole will be

$$L_c = \frac{1}{2}\,\mu\,\Omega^2 \int_0^l \left[(r_0 + x) \int_0^x y'^2\, dx - y^2\right] dx$$

or, by taking into account Eq. (3),

$$L_c = \frac{1}{2}\,\mu\,\Omega^2 \int_0^l \left[(r_0 + x) \int_0^x \left(y_1' + \frac{JE}{K}\,[y_1'']_0\right)^2 dx - \left(y_1 + \frac{x}{K}\,JE\,[y_1'']_0\right)^2\right] dx. \tag{8}$$

The work performed by the centrifugal force will therefore increase the kinetic energy of the blade. Thus by using the Rayleigh method we may write that total kinetic energy is equal to the sum of potential energy and the work due to centrifugal force, or :

$$E_l = L + L_c .$$

By substituting the values expressed by relations (4), (5) and (8), we have :

$$\frac{1}{2}\mu\alpha^2\int_0^l\left(y_1+\frac{x}{K}JE\,[y_1'']_0\right)^2 dx = \frac{1}{2}JE\left(\int_0^l y_1''^2\,dx+\frac{JE}{K}\{[y_1'']_0\}^2\right)+$$

$$+\frac{1}{2}\mu\Omega^2\int_0^l\left[(r_0+x)\int_0^x\left(y_1'+\frac{JE}{K}[y_1'']_0\right)^2 dx-\left(y_1+\frac{x}{K}JE[y_1'']_0\right)^2\right]dx.$$

Thus vibration frequency may be determined by the following formula :

$$(2\,\pi\nu)^2 = \alpha^2 =$$

$$=\frac{\frac{JE}{\mu}\left(\int_0^l y_1''^2\,dx+\frac{JE}{K}\{[y_1'']_0\}^2\right)+\Omega^2\int_0^l(r_0+x)\int_0^x\left(y_1'+\frac{JE}{K}[y_1'']_0\right)^2 dx\,dx}{\int_0^l\left(y_1+\frac{x}{K}JE\,[y_1'']_0\right)^2 dx}-\Omega^2 \quad (9)$$

In order to compute the actual fundamental bending frequency of the blade, we must assume a proper equation for the curve y_1, *i. e.* for the strained shape of the blade. It is known that for a beam with rigidly fixed end a remarkable agreement can be arrived at (within 0,41%) if the shape of the beam is identical with the strained shape of a fixed beam loaded by uniformly distributed load. This is given for the co-ordinate system in Fig. 1 by the equation

$$y_1 = y_{01}\left(2\frac{x^2}{l^2}-\frac{4}{3}\frac{x^3}{l^3}+\frac{1}{3}\frac{x^4}{l^4}\right) \quad (10)$$

where $y_{01}=\dfrac{p\,l^4}{8\,JE}$ gives the end deflection of a fixed beam having a uniformly distributed loading of p unit intensity. In the latter case

$$[y_1'']_0=\left[4\,y_{01}\left(\frac{1}{l^2}-2\frac{x}{l^3}+\frac{x^2}{l^4}\right)\right]_{x=0}=4\,\frac{y_{01}}{l^2}\,. \quad (11)$$

In case of axial compressors, the effect exerted by the centrifugal force upon the elastically strained shape of the vibrating blade is rather small so that Eq. (10) may be accepted as defining the elastically strained shape of a rotating blade. The application of Eq. (10) will promise considerable agreement, as relation (3) — after substitution of Eq. (10) — will satisfy both geometrical and dynamic boundary conditions. Geometrical boundary conditions require the deflection in the fixing section to be zero, while the maximum angle of twist of the section must be adequate to the fixing moment of $\frac{p\,l^2}{2}$ value, *i.e.* for $x = 0$ we have

$$y = 0 \quad \text{and} \quad [y']_0 = \frac{p\,l^2}{2\,K}\,.$$

Dynamic boundary conditions require zero value at the extremity of the beam for both bending moment and shear force, *i. e.* for $x = l$ we have

$$y'' = 0 \quad \text{and} \quad y''' = 0\,.$$

By taking into account Eqs. (10) and (11) we may easily prove, that Eq. (3) will satisfy the above conditions.

Considering Eqs. (10) and (11), the values of the integrals in Eq. (9) will assume the following values :

$$I_1 = \int_0^l y_1''^2\,dx = \frac{16}{5}\,\frac{y_{01}^2}{l^3}$$

$$I_2 = r_0 \int_0^l \int_0^x y_1'^2\,dx\,dx = \frac{2}{5}\,y_{01}^2 r_0$$

$$I_3 = 2\,r_0 \int_0^l \int_0^x y_1'\,\frac{J\,E}{K}\,[y_1'']_0\,dx\,dx = \frac{16}{5}\,y_{01}^2\,\frac{J\,E}{K l}\,r_0$$

$$I_4 = r_0 \int_0^l \int_0^x \left(\frac{J\,E}{K}\,[y_1'']_0\right)^2 dx\,dx = 8\,y_{01}^2\,\frac{J^2\,E^2}{K^2\,l^2}\,r_0$$

$$I_5 = \int_0^l x \int_0^x y_1'^2\,dx\,dx = \frac{122}{405}\,y_{01}^2\,l$$

$$I_6 = 2\int_0^l x \int_0^x y' \frac{JE}{K} [y_1'']_0 \, dx\, dx = \frac{272}{135} y_{01}^2 \frac{JE}{K} \approx 2\, y_{01}^2 \frac{JE}{K}$$

$$I_7 = \int_0^l x \int_0^x \left(\frac{JE}{K} [y_1'']_0\right)^2 dx\, dx = \frac{16}{3} y_{01}^2 \frac{J^2 E^2}{K^2 l^2}$$

$$I_8 = \int_0^l y_1^2 \, dx = \frac{104}{405} y_{01}^2 \, l$$

$$I_9 = 2\int_0^l y_1 \frac{x}{K} J E [y_1'']_0 \, dx = \frac{104}{45} y_{01}^2 \frac{JE}{K}$$

$$I_{10} = \int_0^l \left(\frac{x}{K} J E [y_1'']_0\right)^2 dx = \frac{16}{3} y_{01}^2 \frac{J^2 E^2}{K^2 l^2} \,.$$

Using the integral notations introduced by the above formulae, expression (9) will assume the following form :

$$\alpha^2 = \frac{\dfrac{JE}{\mu}\left(I_1 + 16\, y_{01}^2 \dfrac{JE}{K l^4}\right)}{I_8 + I_9 + I_{10}} + \Omega^2 \left(\frac{I_2 + I_3 + I_4 + I_5 + I_6 + I_7}{I_8 + I_9 + I_{10}} - 1\right) . \tag{12}$$

Let us characterize the elasticity of the fixture by the nondimensional ratio of the two strain components of the blade end section, *i. e.* let us introduce

$$\zeta = \frac{y_{02}}{y_{01}} \,. \tag{13}$$

It is known, that $y_{01} = \dfrac{p\, l^4}{8\, JE}$ and $y_{02} = l\, \varphi = l \dfrac{M}{K} = \dfrac{p\, l^3}{2\, K}$, so that we have

$$\zeta = 4 \frac{JE}{K l}$$

and finally

$$\frac{JE}{K l} = \frac{\zeta}{4} \,. \tag{14}$$

Thus our computations will involve the nondimensional factor ζ instead of the elastic constant K.

It should be noted that the experimental determination of ζ is easier than that of K. The point is that if we measure the total strain of the end section of the fixed blade loaded subsequently, by uniformly distributed loading, in two opposite directions we shall determine Y.

Then obviously we may compute ζ from

$$\zeta = \frac{\frac{Y}{2} - \frac{pl^4}{8\,JE}}{\frac{pl^4}{8\,JE}} = 4\,\frac{JE}{pl^4}\,Y - 1. \tag{15}$$

Taking into account the value yielded given by (14), the new expressions for the above integrals will assume the following forms :

$$\left.\begin{aligned}
I_1 &= \frac{16}{5}\,\frac{y_{01}^2}{l^3}\\
I_2 &= \frac{2}{5}\,y_{01}^2\,r_0\\
I_3 &= \frac{4}{5}\,y_{01}^2\,\zeta\,r_0\\
I_4 &= \frac{1}{2}\,y_{01}^2\,\zeta^2\,r_0\\
I_5 &= \frac{122}{405}\,y_{01}^2\,l\\
I_6 &= \frac{1}{2}\,y_{01}^2\,\zeta\,l\\
I_7 &= \frac{1}{3}\,y_{01}^2\,\zeta^2\,l\\
I_8 &= \frac{104}{405}\,y_{01}^2\,l\\
I_9 &= \frac{26}{45}\,y_{01}^2\,\zeta\,l\\
I_{10} &= \frac{1}{3}\,y_{01}^2\,\zeta^2\,l
\end{aligned}\right\} \tag{16}$$

If we finally substitute these integral expressions, as well as Eq. (14) into Eq. (12), the circular frequency of the vibration will be determined by the following formula (after dividing by y_{01}^2 both numerator and nominator) :

$$\alpha^2 = \frac{\frac{JE}{\mu}\left(\frac{16}{5}\frac{1}{l^3} + 4\frac{\zeta}{l^3}\right)}{l\left(\frac{104}{405} + \frac{26}{45}\zeta + \frac{1}{3}\zeta^2\right)} +$$

$$+ \Omega^2 \left[\frac{r_0\left(\frac{2}{5} + \frac{4}{5}\zeta + \frac{1}{2}\zeta^2\right) + l\left(\frac{122}{405} + \frac{1}{2}\zeta + \frac{1}{3}\zeta^2\right)}{l\left(\frac{104}{405} + \frac{26}{45}\zeta + \frac{1}{3}\zeta^2\right)} - 1 \right]$$

or after the suitable rearrangement of the terms :

$$\alpha^2 = \frac{162}{13}\;\frac{JE}{\mu l^4}\;\frac{104 + 130\zeta}{104 + 234\zeta + 135\zeta^2} +$$

$$+ \Omega^2\left(\frac{r_0}{l}\;\frac{162 + 324\zeta + 203\zeta^2}{104 + 234\zeta + 135\zeta^2} + \frac{122 + 203\zeta + 135\zeta^2}{104 + 234\zeta + 135\zeta^2} - 1\right). \quad (17)$$

The latter formula may be rewritten in the following form :

$$\alpha^2 = 12{,}46\;\frac{JE}{\mu l^4}\;\psi_1 + \Omega^2\left(\frac{r_0}{l}\psi_2 + \psi_3 - 1\right) \quad (18)$$

where the coefficients are given by

$$\left.\begin{aligned} \psi_1 &= \frac{104 + 130\zeta}{104 + 234\zeta + 135\zeta^2} \\ \psi_2 &= \frac{162 + 324\zeta + 203\zeta^2}{104 + 234\zeta + 135\zeta^2} \\ \psi_3 &= \frac{122 + 203\zeta + 135\zeta^2}{104 + 234\zeta + 135\zeta^2}\,. \end{aligned}\right\} \quad (19)$$

Relation (18) formulates the square of the fundamental bending circular frequency of a constant-section fixed beam for the most general case, thus covering all simpler cases too.

Thus, if rigid fixing is employed, obviously $\zeta = 0$ and the square of the circular frequency will be given by

$$a^2 = 12{,}46 \frac{JE}{\mu l^4} + \Omega^2 \left(1{,}558 \frac{r_0}{l} + 0{,}173\right). \tag{20}$$

If, on the other hand, rotational speed (Ω) is low, so that the effects of centrifugal force may be neglected, we have for the case of elastic fixing

$$a^2 = 12{,}46 \frac{JE}{\mu l^4} \psi_1 . \tag{21}$$

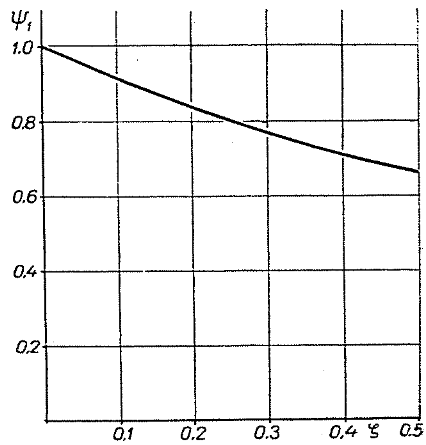

Fig. 3

For a rigidly fixed beam and without any centrifugal force effect we arrive at the known formula :

$$a_0^2 = 12{,}46 \frac{JE}{\mu l^4} \tag{22}$$

or

$$a_0 = \frac{3{,}53}{l^2} \sqrt{\frac{JE}{\mu}} \tag{23}$$

It can be seen, that the function ψ_1 determines the squared ratio of the frequencies of elastically fixed and rigidly fixed beams in the absence of rotation, *i. e.* without centrifugal force effects. Thus we have

$$\psi_1 = \left(\frac{a}{a_0}\right)^2 = \left(\frac{\nu}{\nu_0}\right)^2 . \tag{24}$$

Formula (18) may be rewritten in the following form :

$$a^2 = a_0^2 \psi_1 + B\Omega^2 \tag{25}$$

where

$$B = \frac{r_0}{l}\psi_2 + \psi_3 - 1\,. \tag{26}$$

Variations of ψ_1 as a function of ζ are shown in Fig. 3, while the variations of B against $\frac{r_0}{l}$ — using ζ as a parameter — may be seen in Fig. 4. The use of

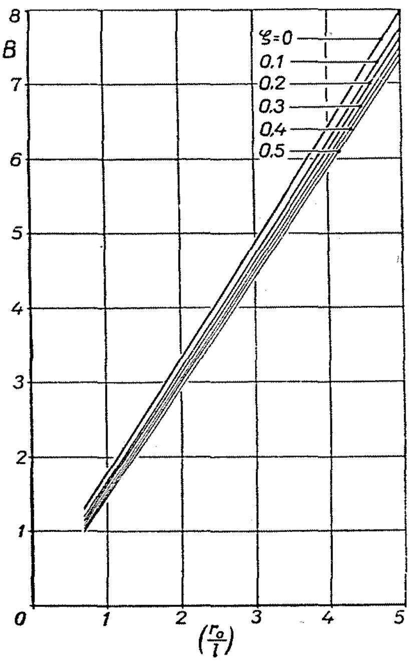

Fig. 4

these curves ensures satisfactory accuracy for computations, whereas increased accuracy can be achieved by using the formulae (19).

If the blade section is not constant, its variations as well as those of J may, as a rule, be sufficiently approximated by a linear or square function. Computing integrals enumerated above will necessarily take more time, but will not involve difficulties of principle. If no analytical function is found to approximate F and J, integrating may be performed graphically, *i. e.* by using the Simpson rule.

Summary

Vibration failures of axial compressor blades are mostly due to vibrations in the fundamental bending mode. Fundamental bending frequency may be easily computed by using Rayleigh's method. The method proposed by the author takes into account the elasticity of blade fixing as well as centrifugal force field effects.

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