ELASTICITY THEORY OF PLANE PLATES OF UNIFORM THICKNESS

By

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(Received March 27, 1957)

1. Introduction

This paper deals with the theory of elasticity of structures bounded by two parallel planes (slabs, discs), hereafter referred to as plates. The theory of elasticity generally deals with such structures on the basis of the following simplifying assumptions [1], [2]:

a) Points located, prior to deformation, on lines perpendicular to the middle plane of the plate, will lie even after deformation on lines at right angle to the deformed middle plane ;

b) Normal stresses generated on planes parallel to the middle plane may be disconsidered in relation to stresses arising on cross-sectional planes ;

c) The middle plane of the plate deforms under load to a developable surface, or to a surface differing but little from such one.

In the following the problem of plates of uniform thickness, with plane middle surface, will be treated without the above-mentioned conventional simplificating suppositions of applied elasticity. The basis of treatment will be a system of particular solutions of the basic equations of the theory of elasticity, elaborated by the author [3].

2. The problem to be solved

For the investigations a rectangular co-ordinate system $O\ (x, y, z)$ is used whose plane xy coincides with the middle surface of the plate. Boundary planes of the plate parallel to plane xy are called *faces*, boundary surfaces perpendicular to the former are called *edge surfaces* of the plate. Thickness of the plate is denoted by symbol $2t$.

It is supposed that only distributed loads are acting on the plate. Unit values of the loading forces referred to faces of the plate are described by load components of direction x, y, z. Load components of direction x, y, z acting on face $z = +t$ are denoted by symbols p_x, p_y, p_z, and these are taken as positive if their directions coincide with the positive directions of the axes. Axially directed components of loads distributed on face $z = -t$ are denoted with symbols q_x, q_y, q_z. The latter are regarded as positive if their sense is opposite to the positive directions of the axes.

The aim is to determine the system of stresses generated by the load system acting on the plate. In the course of this procedure the following conditions are to be observed on the *faces* of the plate

$$\sigma_z\,(x, y, t) = p_z\,(x, y)\,, \qquad \sigma_z\,(x, y, -t) = q_z\,(x, y)\,, \tag{1}$$

$$\tau_{zx}\,(x, y, t) = p_x\,(x, y)\,, \qquad \tau_{zx}\,(x, y, -t) = q_x\,(x, y)\,, \tag{2}$$

$$\tau_{zy}\,(x, y, t) = p_y\,(x, y)\,, \qquad \tau_{zy}\,(x, y, -t) = q_y\,(x, y)\,. \tag{3}$$

On the edge surface of the plate very variable initial conditions are possible and therefore no closer stipulations are made here in relation to the latter. The method to be presented, however, yields no possibility of satisfying any initial conditions relative to the edge face.

3. A system of particular solution of the basic equations of the theory of elasticity

For the solution of the problem, a system of particular solutions of the basic equations of the theory of elasticity, elaborated by the author [3], is used, according to which

$$\begin{aligned} \xi &= m\,\frac{\partial S_a}{\partial x} - \frac{\partial S_b}{\partial x}\,, \\ \eta &= m\,\frac{\partial S_a}{\partial y} - \frac{\partial S}{\partial y}\,, \\ \zeta &= m\,\frac{\partial S_a}{\partial z} - (2\,m - 1)\,\frac{\partial S_b}{\partial z}\,, \end{aligned} \tag{4}$$

if namely

$$\begin{aligned} S_a &= \sum_{-\infty}^{+\infty} (-1)^j\, j\, F_{2j}\,(x, y) \cdot H_{2j}\,(z)\,, \\ S_b &= \sum_{-\infty}^{+\infty} (-1)^j\, F_{2j}\,(x, y) \cdot H_{2j}\,(z)\,. \end{aligned} \tag{5}$$

In the above formulae ξ, η, ζ, are displacements of direction x, y, z of points of the plate, m is Poisson's ratio, F and H are functions, between which the following relations subsist :

$$\frac{\partial^2}{\partial x^2} F_{2j} + \frac{\partial^2}{\partial y^2} F_{2j} = F_{2j+2}, \tag{6}$$

$$\frac{d}{dz} H_{j+1} = H_j. \tag{7}$$

The series figuring in the formulae must be convergent, so as to meet the following condition : either of the first two partial derivatives of the series can be produced by differentiating member by member.

The above system of displacements may be written with some simplification thus :

$$\begin{aligned} \xi &= \sum_{-\infty}^{+\infty} (-1)^j (jm-1) \frac{\partial F_{2j}}{\partial x} H_{2j}, \\ \eta &= \sum_{-\infty}^{+\infty} (-1)^j (jm-1) \frac{\partial F_{2j}}{\partial y} H_{2j}, \\ \zeta &= \sum_{-\infty}^{+\infty} (-1)^j (jm-2m+1) F_{2j} H_{2j-1}. \end{aligned} \tag{8}$$

The stress system corresponding to this system of displacements is

$$\begin{aligned} \sigma_x &= 2\,Gm \sum_{-\infty}^{+\infty} (-1)^j \frac{\partial^2 F_{2j}}{\partial x^2} H_{2j} + \\ &\quad + 2\,G \sum_{-\infty}^{+\infty} (-1)^j \frac{\partial^2 F_{2j}}{\partial y^2} H_{2j}, \\ \sigma_z &= 2\,Gm \sum_{-\infty}^{+\infty} (-1)^j (j-2) F_{2j} H_{2j-2}, \\ \tau_{xy} &= 2\,G \sum_{-\infty}^{+\infty} (-1)^j (jm-1) \frac{\partial^2 F_{2j}}{\partial x \cdot \partial y} H_{2j}, \\ \tau_{zx} &= 2\,Gm \sum_{-\infty}^{+\infty} (-1)^j (j-1) \frac{\partial F_{2j}}{\partial x} H_{2j-1}. \end{aligned} \tag{9}$$

Formulae of stress components σ_y and τ_{zy} are not written here and will not be written later either, because these can be produced immediately from formulae σ_x and τ_{zx} by interchanging the role of symbols x and y.

4. Four characteristic groups of functions H

Before embarking upon the solution of the problem proper, four characteristic groups of functions have to be learned. Characteristic data of these four cases are :

Case I.

$$\sigma_z(x, y, +t) = -\sigma_z(x, y, -t),$$
$$\tau_{zx}(x, y, \pm t) = 0,$$
$$\tau_{yz}(x, y, \pm t) = 0;$$

Case II.

$$\sigma_z(x, y, +t) = \sigma_z(x, y, -t),$$
$$\tau_{zx}(x, y, \pm t) = 0,$$
$$\tau_{yz}(x, y, \pm t) = 0;$$

Case III. (10)

$$\sigma_z(x, y, \pm t) = 0,$$
$$\tau_{zx}(x, y, +t) = -\tau_{zx}(x, y, -t),$$
$$\tau_{yz}(x, y, +t) = -\tau_{yz}(x, y, -t);$$

Case IV.

$$\sigma_z(x, y, \pm t) = 0,$$
$$\tau_{zx}(x, y, +t) = \tau_{zx}(x, y, -t),$$
$$\tau_{yz}(x, y, +t) = \tau_{yz}(x, y, -t).$$

With the proper designation of the H functions figuring in formulae (8) it can be easily attained that the ξ, η, ζ system of displacements results on faces $z = \pm t$ in the stress system corresponding to the cases stress I, II, III and IV. For this the H functions are to be assumed as follows :

Case I.

$$
\begin{aligned}
H_0 &= 0\,, \\
H_1 &= 1\,, \\
H_2 &= \frac{z}{1!}\,, \\
H_3 &= \frac{z}{2!} - \frac{t^2}{2}\,, \\
H_4 &= \frac{z^3}{3!} - \frac{t^2}{2}\cdot\frac{z}{1!}\,, \\
H_5 &= \frac{z^4}{4!} - \frac{t^2}{2}\cdot\frac{z^2}{2!} + \frac{5\,t^4}{24}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\,;
\end{aligned}
\tag{11}
$$

Case II.

$$
\begin{aligned}
H_0 &= 1\,, \\
H_1 &= \frac{z}{1!}\,, \\
H_2 &= \frac{z^2}{2!} - \frac{t^2}{6}\,, \\
H_3 &= \frac{z^3}{3!} - \frac{t^2}{6}\cdot\frac{z}{1!}\,, \\
H_4 &= \frac{z^4}{4!} - \frac{t^2}{6}\cdot\frac{z^2}{2!} + \frac{7\,t^4}{360}\,, \\
H_5 &= \frac{z^5}{5!} - \frac{t^2}{6}\cdot\frac{z^3}{3!} + \frac{7\,t^4}{360}\cdot\frac{z}{1!}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\,;
\end{aligned}
\tag{12}
$$

Case III.

$$
\begin{aligned}
H_0 &= 0\,, \\
H_1 &= 0\,, \\
H_2 &= 1\,, \\
H_3 &= \frac{z}{1!}\,, \\
H_4 &= \frac{z^2}{2!} - \frac{t^2}{2}\,, \\
H_5 &= \frac{z^3}{3!} - \frac{t^2}{2}\cdot\frac{z}{1!}\,, \\
H_6 &= \frac{z^4}{4!} - \frac{t^2}{2}\cdot\frac{z^2}{2!} + \frac{5\,t^4}{24}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\,;
\end{aligned}
\tag{13}
$$

Case IV.

$$H_0 = 0\,,$$

$$H_1 = 1\,,$$

$$H_2 = \frac{z}{1\,!}\,,$$

$$H_3 = \frac{z^2}{2\,!} - \frac{t^2}{6}\,,$$

$$H^4 = \frac{z^3}{3\,!} - \frac{t^2}{6} \cdot \frac{z}{1\,!}\,, \tag{14}$$

$$H_5 = \frac{z^4}{4\,!} - \frac{t^2}{6} \cdot \frac{z^2}{2\,!} + \frac{7\,t^4}{360}\,,$$

$$H_6 = \frac{z^5}{5\,!} - \frac{t^2}{6} \cdot \frac{z^3}{3\,!} + \frac{7\,t^4}{360} \cdot \frac{z}{1\,!}\,,$$

. . . .

In the H functions (11), (13) and (12), (14), respectively, the coefficients are in easily recognizable relation to coefficients figuring in the power series

$$\frac{1}{\operatorname{ch} t} = 1 - \frac{t^2}{2} + \frac{5\,t^4}{24} - \ldots\,,$$

and

$$\frac{t}{\operatorname{sh}\, t} = 1 - \frac{t^2}{6} + \frac{7\,t^4}{360} - \ldots\,.$$

5. Plates loaded on their faces

Let plates loaded on their faces be dealt with first. In this case one eigenfunction of the differential equation

$$\frac{\partial^2 \Phi\,(x, y)}{\partial\, x^2} + \frac{\partial^2 \Phi\,(x, y)}{\partial\, y^2} + \lambda^2 \Phi\,(x, y) = 0\,; \qquad \lambda^2 = \text{const} \tag{15}$$

vanishing at the edge line is chosen for function F_0. For the sake of brevity, the eigenfunction in question is denoted by symbol Φ, the eigenvalue pertaining to it by symbol λ^2. In this case, considering (6),

$$\begin{aligned} F_0 &= \Phi , \\ F_2 &= -\lambda^2 \Phi , \\ F_4 &= +\lambda^4 \Phi , \\ &\dots\dots\dots\dots , \\ F_{2j} &= (-1)^j \lambda^2 \Phi , \\ &\dots\dots\dots\dots\dots \end{aligned} \tag{16}$$

For functions F_{-2}, F_{-4}, F_{-6}, ... no nearer data are given.

The next step is to investigate what shape the formulae of displacement functions (4) will assume if for functions F values (16) and for functions H, in the same order, values (11) — (14) are substituted therein. In the course of these investigations some endless series will be regrouped with omission of the examination of the necessary convergence, and also other transformations will be made on the series. Therefore it will have to be ascertained whether or not the displacement functions obtained as the results of calculation satisfy the basic equations of the theory of elasticity.

Calculations are executed seperately for the four special cases mentioned in the previous chapter.

Case I. The values of S_a figuring in formulae (4) is determined in the first place. For this the values of (11) and (16) are to be substituted into formula (5). Thus with notations

$$Z \equiv \lambda z , \qquad T \equiv \lambda t , \tag{17}$$

the following formula is obtained :

$$\begin{aligned} S_a = \lambda \Phi Z \Big[1 &+ 2 \left(\frac{Z^2}{3!} - \frac{T^2}{2} \right) + \\ &+ 3 \left(\frac{Z^4}{5!} - \frac{T^2}{2} \cdot \frac{Z^2}{3!} + \frac{5\,T^4}{24} \right) + \\ &+ 4 \left(\frac{Z^6}{7!} - \frac{T^2}{2} \cdot \frac{Z^4}{5!} + \frac{5\,T^4}{24} \cdot \frac{Z^2}{3!} - \frac{61\,T^6}{720} \right) + \dots \Big] . \end{aligned}$$

By re-grouping members of this series the following series may be produced:

$$S_a = \lambda \Phi Z \left[\frac{1}{1!} \left(1 - 2\frac{T^2}{2} + 3\frac{5T^4}{24} - 4\frac{61T^6}{720} + \ldots \right) + \right.$$
$$+ \frac{Z^2}{3!} \left(2 - 3\frac{T^2}{2} + 4\frac{5T^4}{24} - \ldots\ldots\ldots\ldots \right) +$$
$$+ \frac{Z^4}{5!} \left(3 - 4\frac{T^2}{2} + \ldots\ldots\ldots\ldots\ldots\ldots \right) +$$
$$\left. + \frac{Z^6}{7!} \left(4 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) + \ldots \right] \cdot$$

By substituting the known formulae of development into series:

$$S_a = \frac{\lambda \Phi Z}{2} \left[\frac{1}{1!} \left(\frac{2}{\operatorname{ch} T} - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \right) + \right.$$
$$+ \frac{Z^2}{3!} \left(\frac{4}{\operatorname{ch} T} - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \right) +$$
$$+ \frac{Z^4}{5!} \left(\frac{6}{\operatorname{ch} T} - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \right) +$$
$$\left. + \frac{Z^6}{7!} \left(\frac{8}{\operatorname{ch} T} - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \right) + \ldots \right],$$

or by re-grouping of members, the formula

$$S_a = \frac{\lambda \Phi Z}{2} \left[\frac{1}{\operatorname{ch} T} \cdot \frac{1}{Z} \left(\frac{2Z}{1!} + \frac{4Z^3}{3!} + \frac{6Z^5}{5!} + \frac{8Z^7}{7!} + \ldots \right) - \right.$$
$$\left. - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \cdot \frac{1}{Z} \left(\frac{Z}{1!} + \frac{Z^3}{3!} + \frac{Z^5}{5!} + \frac{Z^7}{7!} + \ldots \right) \right]$$

is obtained. Instead of this formula with reference to the known formulae of development into series the formula

$$S_a = \frac{\lambda \Phi}{2} \left[\frac{1}{\operatorname{ch} T} \cdot \frac{d}{dZ} (Z \operatorname{sh} Z) - \frac{T \operatorname{sh} T}{\operatorname{ch}^2 T} \operatorname{sh} Z \right]$$

and by further transformation of the formula, we obtain

$$S_a = \frac{\lambda \Phi}{2 \operatorname{ch}^2 T} (\operatorname{ch} \cdot T \cdot \operatorname{sh} Z + Z \operatorname{ch} T \cdot \operatorname{ch} Z - T \operatorname{sh} T \cdot \operatorname{sh} Z) \cdot \tag{18}$$

A similar method may be applied in the case of S_b figuring in formulae (4). On this occasion by substituting values of (11), (16) and (17) into formula (5) of expression S_b it is found that

$$S_b = \lambda \Phi Z \left[1 + \left(\frac{Z^2}{3\,!} - \frac{T^2}{2} \right) + \right.$$
$$+ \left(\frac{Z^4}{5\,!} - \frac{T^2}{2} \cdot \frac{Z^2}{3\,!} + \frac{5\,T^4}{24} \right) +$$
$$\left. + \left(\frac{Z^6}{7\,!} - \frac{T^2}{2} \cdot \frac{Z^4}{5\,!} + \frac{5\,T^4}{24} \cdot \frac{Z^2}{3\,!} - \frac{61\,T^6}{720} \right) + \ldots \right],$$

or

$$S = \lambda \Phi \left[\frac{Z}{1\,!} \left(1 - \frac{T^2}{2} + \frac{5\,T^4}{24} - \frac{61\,T^6}{720} + \ldots \right) + \right.$$
$$+ \frac{Z^3}{3\,!} \left(1 - \frac{T^2}{2} + \frac{5\,T^4}{24} - \ldots\ldots\ldots\ldots \right) +$$
$$+ \frac{Z^5}{5\,!} \left(1 - \frac{T^2}{2} + \ldots\ldots\ldots\ldots\ldots\ldots \right) +$$
$$\left. + \frac{Z^7}{7\,!} \left(1 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) + \ldots \right].$$

Hence

$$S_b = \frac{\lambda \Phi}{2\,\mathrm{ch}^2 T} \cdot 2\,\mathrm{ch}\,T \cdot \mathrm{sh}\,Z. \tag{19}$$

With knowledge of (18) and (19), instead of formulae (4) of the displacement functions, the following formulae may be written

$$\xi = \alpha^{\mathrm{I}} \frac{\partial \Phi}{\partial x} \left(\frac{m-2}{m} \mathrm{ch}\,T \cdot \mathrm{sh}\,Z + Z\,\mathrm{ch}\,T \cdot \mathrm{ch}\,Z - T\,\mathrm{sh}\,T \cdot \mathrm{sh}\,Z \right),$$
$$\eta = \alpha^{\mathrm{I}} \frac{\partial \Phi}{\partial y} \left(\frac{m-2}{m} \mathrm{ch}\,T \cdot \mathrm{sh}\,Z + Z\,\mathrm{ch}\,T \cdot \mathrm{ch}\,Z - T\,\mathrm{sh}\,T \cdot \mathrm{sh}\,Z \right), \tag{20}$$
$$\zeta = \alpha^{\mathrm{I}}\ \lambda \Phi \left(\frac{2-2m}{m} \mathrm{ch}\,T \cdot \mathrm{ch}\,Z + Z\,\mathrm{ch}\,T\,\mathrm{sh}\,Z - T\,\mathrm{sh}\,T \cdot \mathrm{ch}\,Z \right).$$

In these formulae

$$\alpha^{\mathrm{I}} = \frac{m\,\lambda}{2\,\mathrm{ch}^2 T}.$$

If the above values found for the displacement functions are substituted into the basic equations of the theory of elasticity, it becomes evident that the functions satisfy those equations.

With the displacement functions known, the stress formulae can also be easily determined:

$$\begin{aligned}
\sigma_x &= 2\,G\,\alpha^{\mathrm{I}}\frac{\partial^2 \Phi}{\partial x^2}(\operatorname{ch} T\cdot\operatorname{sh} Z + Z\operatorname{ch} T\cdot\operatorname{ch} Z - T\operatorname{sh} T\cdot\operatorname{sh} Z) + \\
&\quad + 2\,G\,\alpha^{\mathrm{I}}\frac{\partial^2 \Phi}{\partial y^2}\cdot\frac{2}{m}\operatorname{ch} T\cdot\operatorname{sh} Z,\\
\sigma_z &= 2\,G\,\alpha^{\mathrm{I}}\;\lambda^2\Phi\,(-\operatorname{ch} T\cdot\operatorname{sh} Z + Z\operatorname{ch} T\cdot\operatorname{ch} Z - T\operatorname{sh} T\cdot\operatorname{sh} Z),\\
\tau_{xy} &= 2\,G\,\alpha^{\mathrm{I}}\frac{\partial^2 \Phi}{\partial x\cdot\partial y}\left(\frac{m-2}{m}\operatorname{ch} T\cdot\operatorname{sh} Z + \right.\\
&\qquad\qquad \left. + Z\operatorname{ch} T\cdot\operatorname{ch} Z - T\operatorname{sh} T\cdot\operatorname{sh} Z\right),\\
\tau_{zx} &= 2\,G\,\alpha^{\mathrm{I}}\frac{\partial \Phi}{\partial x}\qquad(Z\operatorname{ch} T\cdot\operatorname{sh} Z - T\operatorname{sh} T\cdot\operatorname{ch} Z)\cdot
\end{aligned}\tag{21}$$

Formulae of stresses σ_y and τ_{yz}, here not presented, can be produced from formulae of σ_x and τ_{zx} by substituting y for x, and x for y.

Case II. In this case after calculations, similar to the above, the following formulae, similar to those of case I, are obtained for the displacement functions:

$$\begin{aligned}
\xi &= \alpha^{\mathrm{II}}\frac{\partial \Phi}{\partial x}\left(\frac{m-2}{m}\operatorname{sh} T\cdot\operatorname{ch} Z + Z\operatorname{sh} T\cdot\operatorname{sh} Z - T\operatorname{ch} T\cdot\operatorname{ch} Z\right),\\
\eta &= \alpha^{\mathrm{II}}\frac{\partial \Phi}{\partial y}\left(\frac{m-2}{m}\operatorname{sh} T\cdot\operatorname{ch} Z + Z\operatorname{sh} T\cdot\operatorname{sh} Z - T\operatorname{ch} T\cdot\operatorname{ch} Z\right),\\
\zeta &= \alpha^{\mathrm{II}}\;\lambda\Phi\left(\frac{2-2m}{m}\operatorname{sh} T\cdot\operatorname{sh} Z + Z\operatorname{sh} T\cdot\operatorname{ch} Z - T\operatorname{ch} T\cdot\operatorname{sh} Z\right)\cdot
\end{aligned}\tag{22}$$

In these formulae

$$\alpha^{\mathrm{II}} = \frac{m\lambda}{2\operatorname{sh}^2 T}\;.$$

The above displacement functions satisfy the basic equations of the theory of elasticity in every respect.

In possession of the displacement functions the stress formulae can also be given:

$$\sigma_x = 2\,G\,a^{\mathrm{II}} \frac{\partial^2 \Phi}{\partial x^2} (\operatorname{sh} T \cdot \operatorname{ch} Z + Z \operatorname{sh} T \cdot \operatorname{sh} Z - T \operatorname{ch} T \cdot \operatorname{ch} Z) +$$

$$+ 2\,G\,a^{\mathrm{II}} \frac{\partial^2 \Phi}{\partial y^2} \cdot \frac{2}{m} \operatorname{sh} T \cdot \operatorname{ch} Z\,,$$

$$\sigma_z = 2\,G\,a^{\mathrm{II}}\, \lambda^2 \Phi\, (- \operatorname{sh} T \cdot \operatorname{ch} Z + Z \operatorname{sh} T \cdot \operatorname{sh} Z - T \operatorname{ch} T \cdot \operatorname{ch} Z)\,,$$

$$\tau_{xy} = 2\,G\,a^{\mathrm{II}} \frac{\partial^2 \Phi}{\partial x \cdot \partial y} \left(\frac{m-2}{m} \operatorname{sh} T \cdot \operatorname{ch} Z + \right. \tag{23}$$

$$\left. + Z \operatorname{sh} T \cdot \operatorname{sh} Z - T \operatorname{ch} T \cdot \operatorname{ch} Z \right),$$

$$\tau_{zx} = 2\,G\,a^{\mathrm{II}} \frac{\partial \Phi}{\partial x} (Z \operatorname{sh} T \cdot \operatorname{ch} Z - T \operatorname{ch} T \cdot \operatorname{sh} Z)\,.$$

Formulae of σ_y and τ_{yz} can now again be produced from those of σ_x and τ_{zx} by interchanging the symbols of x and y.

Case III. The formulae now valid can be produced, by using functions H under (13) in the same way as the formulae of case I, from functions H with symbol (11). However, since functions H with symbol (13) are equal to the differential quotients of those under (11) according to z, the formulae valid for the present case can be more simply deduced from the formulae of case I. by differentiating with respect to z. The displacement functions thus produced correspond in every respect to the basic equations of the theory of elasticity.

Case IV. The displacement functions sought for are derivatives of those under II. with respect to z. These displacement functions also satisfy the basic equations of the theory of elasticity in every respect.

6. Plates not loaded on their faces

It will now be investigated how the functions F figuring in formulae (8) of the displacement functions are to be designated, if the following is stipulated:

$$\sigma_z\,(x, y, \pm t) = 0\,,$$

$$\tau_{zx}\,(x, y, \pm t) = 0\,,$$

$$\tau_{zy}\,(x, y, \pm t) = 0\,.$$

This investigation leads to different results according to which one of the groups of functions (11)—(14) is chosen for functions F. Accordingly, four cases are distinguished:

Case I.: If the group of formulae (11) is chosen for functions H, then, according to (9), $\sigma_{zx}(x, y, \pm t)$ and $\sigma_{zy}(x, y, \pm t)$ are equal beforehand to zero, and the formula of stress σ_z takes the following form

$$\sigma_z(x, y, z) = 2\,G\,m\,(-\,F_0 H_4 + 2\,F_8 H_6 - 3\,F_{10} H_8 + \ldots)\,.$$

To make also σ_z vanish on the plate faces, a triharmonic function has to be chosen for function F_0. Then

$$F_6 = F_8 = F_{10} = \ldots = 0\,,$$

and the displacement functions (8) are

$$\begin{aligned} \xi &= -\frac{\partial F_2}{\partial x} z + (2\,m - 1)\frac{\partial F_4}{\partial x}\left(\frac{z^3}{6} - \frac{t^2 z}{2}\right), \\ \eta &= -\frac{\partial F_2}{\partial y} z + (2\,m - 1)\frac{\partial F_4}{\partial y}\left(\frac{z^3}{6} - \frac{t^2 z}{2}\right), \\ \zeta &= F_2 + F_4\left(\frac{z^2}{2} - \frac{t^2}{2}\right). \end{aligned} \tag{24}$$

The following stress system corresponds to these displacement functions

$$\begin{aligned} \sigma_z &= 2\,G\left[-\,m\frac{\partial^2 F_2}{\partial x^2} z - \frac{\partial^2 F_2}{\partial y^2} z + \right. \\ &\qquad\qquad \left. + (2\,m - 1)\frac{\partial^2 F_4}{\partial x^2}\left(\frac{z^3}{6} - \frac{t^2 z}{2}\right)\right], \\ \sigma_z &= 0\,, \\ \tau_{xy} &= 2\,G\left[-\,(m - 1)\frac{\partial^2 F}{\partial x \cdot \partial y} z + \right. \\ &\qquad\qquad \left. + (2\,m - 1)\frac{\partial^2 F_4}{\partial x \cdot \partial y}\left(\frac{z^3}{6} - \frac{t^2 z}{2}\right)\right]; \\ \tau_{zx} &= 2\,G\,m\frac{\partial F_4}{\partial x}\left(\frac{z^2}{2} - \frac{t^2}{2}\right). \end{aligned} \tag{25}$$

Case II.: Let now the values included in group of formulae (12) be chosen for functions H. In this case, according to (9), $\sigma_{zx}(x, y, \pm t)$ and $\sigma_{zy}(x, y, \pm t)$ are beforehand of zero value and the value of stress

$$\sigma_z(x, y, z) = 2\,G\,m\,(F_2 H_0 - F_6 H_4 + 2\,F_8 H_6 - 3\,F_{10} H_8 + \ldots)\,.$$

To make also σ_z vanish on the face of the plate, it is necessary to choose a tri-harmonic function for F_0. Then

$$F_2 = F_4 = F_6 = \ldots = 0 ,$$

and, accordingly, formulae (8) of the displacement function take the form

$$\begin{aligned} \xi &= -\frac{\partial F_0}{\partial x} , \\ \eta &= -\frac{\partial F_0}{\partial y} , \\ \zeta &= 0 . \end{aligned} \tag{26}$$

The formulae of stresses, on the other hand, are the following:

$$\begin{aligned} \sigma_x &= -2G \frac{\partial^2 F_0}{\partial x^2} = 2G \frac{\partial^2 F_0}{\partial y^2} , \\ \sigma_z &= 0 , \\ \tau_{xy} &= -2G \frac{\partial^2 F_0}{\partial x \cdot \partial y} , \\ \tau_{zx} &= 0 . \end{aligned} \tag{27}$$

From the above formulae it appears that in the present case the stress and deformation states are two-dimensional. The function F_0 figuring in the formulae is $2\,G$-times the Airy stress function.

Case III.: Now functions of formula group (13) are chosen for functions H. In this case $\sigma_z\,(x, y, \pm t)$ is equal beforehand to zero and the values of stresses τ_{zx} and τ_{zy} are

$$\tau_{zx}\,(x, y, z) = 2\,G\,m \left(\frac{\partial F_4}{\partial y} H_3 - 2 \frac{\partial F_6}{\partial y} H_5 + 3 \frac{\partial F_8}{\partial y} H_7 - \ldots \right),$$

$$\tau_{zy}\,(x, y, z) = 2\,G\,m \left(\frac{\partial F_4}{\partial x} H_3 - 2 \frac{\partial F_6}{\partial x} H_5 + 3 \frac{\partial F_8}{\partial x} H_7 - \ldots \right).$$

To assure that τ_{zx} and τ_{zy} vanish on faces of the plate it is necessary that $F_4\,(x, y) = c = \text{const.}$ In this case

$$\frac{\partial F_4}{\partial x} = \frac{\partial F_4}{\partial y} = 0 ; \qquad F_6 = F_8 = F_{10} = \ldots = 0 ,$$

and the displacement functions are of the following simple form :

$$\xi = -(m-1)\frac{\partial F_2}{\partial x},$$

$$\eta = -(m-1)\frac{\partial F_2}{\partial y}, \tag{28}$$

$$\zeta = c\,z,$$

and the stress formulae take the shape

$$\sigma_x = 2\,G\left[(m-1)\frac{\partial^2 F_2}{\partial x^2} - c\right] = 2\,G\left[(m-1)\frac{\partial^2 F_2}{\partial y^2} - m\,c\right],$$

$$\sigma_z = 0, \tag{29}$$

$$\tau_{xy} = -\,2\,G\,(m-1)\,\frac{\partial^2 F_2}{\partial x \cdot \partial y},$$

$$\tau_{zx} = 0.$$

It is evident that, in the case on hand, too, a plane stress state is dealt with.

Case IV.: If functions figuring in group of formulae (14) are chosen for H functions stress $\sigma_z\,(x, y, \pm t)$ beforehand becomes of zero value, and the formulae of stresses τ_{zx} and τ_{zy} take the following form :

$$\tau_{zx}\,(x, y, z) = 2\,G\,m\left(\frac{\partial F_4}{\partial x} H_3 - 2\frac{\partial F_6}{\partial x} H_5 + 3\frac{\partial F_8}{\partial x} H_7 - \ldots\right),$$

$$\tau_{zy}\,(x, y, z) = 2\,G\,m\left(\frac{\partial F_4}{\partial y} H_3 - 2\frac{\partial F_6}{\partial y} H_5 + 3\frac{\partial F_8}{\partial y} H_7 - \ldots\right).$$

To make τ_{zx} and τ_{zy} equal to zero on faces of the plate it is necessary that

$$F_4 = c = \text{const.}$$

In that case

$$\frac{\partial F_4}{\partial x} = \frac{\partial F_4}{\partial y} = 0, \qquad F_6 = F_8 = F_{10} = \ldots = 0,$$

so that the formulae of displacement functions:

$$\xi = -(m-1)\frac{\partial F_2}{\partial x} z\,,$$

$$\eta = -(m-1)\frac{\partial F_2}{\partial y} z\,, \tag{30}$$

$$\zeta = (m-1) F_2 + c\left(\frac{z^2}{2} - \frac{t^2}{6}\right).$$

And the stress formulae in turn

$$\sigma_x = 2G\left[(m-1)\frac{\partial^2 F_2}{\partial y^2} - mc\right] z\,,$$

$$\sigma_z = 0\,, \tag{31}$$

$$\tau_{xy} = -2G(m-1)\frac{\partial^2 F_2}{\partial x \cdot \partial y} z\,,$$

$$\tau_{zx} = 0\,.$$

Evidently, in this case, too, there subsists a plane stress state.

7. Solution of the problem

In posession of formulae learned in Chapters 5 and 6 the problem can be solved in two steps. In the first step through use of formulae learned in Chapter 5, a system of displacements ξ^*, η^*, ζ^* is produced which satisfies only initial conditions (1)—(3). In the second step this displacement system is completed with the application of formulae presented in Chapter 6 by a displacement system ξ^{**}, η^{**}, ζ^{**} which enables the united solution

$$\xi = \xi^* + \xi^{**}\,,$$
$$\eta = \eta^* + \eta^{**}\,,$$
$$\zeta = \zeta^* + \zeta^{**}$$

to satisfy all other stipulations of the problem.

Since initial edge conditions of the plate may be very variable, the determination of the displacement system ξ^*, η^*, ζ^* alone will be dealt with here. This displacement system will also be composed of two parts.

First a displacement system $\xi^{(a)}$, $\eta^{(a)}$, $\zeta^{(a)}$ is sought to satisfy on plate faces the conditions

$$\begin{aligned} \sigma_z(x,y,t) &= p_z, & \sigma_z(x,y,-t) &= q_z,\\ \tau_{zx}(x,y,t) &= 0, & \tau_{zx}(x,y,-t) &= 0,\\ \tau_{zy}(x,y,t) &= 0, & \tau_{zy}(x,y,-t) &= 0. \end{aligned}$$

Then this displacement system is completed by a displacement system $\xi^{(b)}$, $\eta^{(b)}$, $\zeta^{(b)}$ which assures compliance with the following conditions :

$$\begin{aligned} \sigma_z(x,y,t) &= 0, & \sigma_z(x,y,-t) &= 0,\\ \tau_{zx}(x,y,t) &= p_x, & \tau_{zx}(x,y,-t) &= q_x,\\ \tau_{zy}(x,y,t) &= p_y, & \tau_{zy}(x,y,-t) &= q_y. \end{aligned}$$

With knowledge of the two displacement systems

$$\begin{aligned} \xi^* &= \xi^{(a)} + \xi^{(b)},\\ \eta^* &= \eta^{(a)} + \eta^{(b)},\\ \zeta^* &= \zeta^{(a)} + \zeta^{(b)}. \end{aligned}$$

The displaceeent systems $\xi^{(a)}, \eta^{(a)}, \zeta^{(a)}$ and $\xi^{(b)}$, $\eta^{(b)}$, $\zeta^{(b)}$ are determined according to the following instructions :

a) *Determination of displacement system* $\xi^{(a)}$, $\eta^{(a)}$, $\zeta^{(a)}$. This problem can be traced back to the problem treated in Chapter 5 if the loading systems $p_z(x,y)$ and $q_z(x,y)$ given are resolved into the two loading systems appropriate to cases I and II respectively, dealt with in Chapter 5. One loading system (with symbol I) is assumed to have at point (x, y, t) an intensity of $r^{\mathrm{I}}(x, y)$ and at point $(x, y, -t)$ an intensity of $-r^{\mathrm{I}}(x, y)$, where

$$r^{\mathrm{I}}(x,y) = \frac{1}{2}(p_z - q_z).$$

Thus, the second (with symbol II) loading system's intensity, at points (x, y, t) and $(x, y, -t)$ alike,

$$r^{\mathrm{II}}(x,y) = \frac{1}{2}(p_z + q_z).$$

Of these two loading systems the first corresponds to case I and the second to case II, both presented in Chapter 4.

To make possible the application of the formulae of Chapter 5 to the above loading systems, the afore-written functions $r^{\mathrm{I}}(x, y)$ and $r^{\mathrm{II}}(x, y)$

are developed in series according to the eigenfunctions of the partial differential equation (15) provided these developments in series are possible. Thereby the series

$$r^{\mathrm{I}}(x, y) = \sum_{k=1}^{\infty} r_k^{\mathrm{I}}(x, y) = \sum_{k=1}^{\infty} c_k^{\mathrm{I}} \varphi_k(x, y),$$

$$r^{\mathrm{II}}(x, y) = \sum_{k=1}^{\infty} r_k^{\mathrm{II}}(x, y) = \sum_{k=1}^{\infty} c_k^{\mathrm{II}} \varphi_k(x, y) \tag{32}$$

are obtained. Here $\varphi_1(x, y), \varphi_2(x, y), \ldots \varphi_k(x, y), \ldots$ represent the normalized eigenfunctions while the meaning of factors c_k^{I} and c_k^{II} is

$$c_k^{\mathrm{I}} = \int_{(A)} \varphi_k(x, y) \cdot r^{\mathrm{I}}(x, y) \cdot dA,$$

$$c_k^{\mathrm{II}} = \int_{(A)} \varphi_k(x, y) \cdot r^{\mathrm{II}}(x, y) \cdot dA,$$

that is to say, if A is the area of the middle plane of the plate. With the use of the development into series (32) loading system I is resolved into loading systems composed only of forces of direction z, whose intensities at points $(x, y, +t)$ are $r_1^{\mathrm{I}}, r_2^{\mathrm{I}}, \ldots r_k^{\mathrm{I}}, \ldots$, and at points $(x, y, -t)$: $-r_1^{\mathrm{I}}, -r_2^{\mathrm{I}}, \ldots -r_k^{\mathrm{I}}, \ldots$, while loading system II is resolved into components again only of direction z, whose intensities at points $(x, y, \pm t)$ are $r_1^{\mathrm{II}}, r_2^{\mathrm{II}}, \ldots r_k^{\mathrm{II}}, \ldots$.

Having denoted the stresses generated by loading systems r_k^{I} and r_k^{II} by symbols $\sigma_{x\,k}^{\mathrm{I}}, \sigma_{y,k}^{\mathrm{I}}, \ldots$, and $\sigma_{x\,k}^{\mathrm{II}}, \sigma_{y\,k}^{\mathrm{II}}, \ldots$, respectively, and the corresponding displacements by $\xi_k^{\mathrm{I}}, \eta_k^{\mathrm{I}}, \zeta_k^{\mathrm{I}}$ and $\xi_k^{\mathrm{II}}, \eta_k^{\mathrm{II}}, \zeta_k^{\mathrm{II}}$, let us determine their values. We start from formulae (20)—(23). Functions Φ, figuring in these formulae, and expediently denoted by Φ_k^{I} in the first case, and by Φ_k^{II} in the second case are to be designated so that

$$\sigma_{z,k}^{\mathrm{I}}(x, y, \pm t) = \pm r_k^{\mathrm{I}},$$

and

$$\sigma_{z,k}^{\mathrm{II}}(x, y, \pm t) = + r_k^{\mathrm{II}}.$$

This stipulation is easily complied with since functions $\bar{\Phi}_k^{\mathrm{I}}$ and $\bar{\Phi}_k^{\mathrm{II}}$ are proportionate to stresses $\sigma_{z,k}^{\mathrm{I}}(x, y, +t)$, and $\sigma_z^{\mathrm{II}}(x, y, +t)$, respectively. With knowledge of this a simple calculation may confirm that the following are to be used for functions Φ^{I} and Φ^{II}. In this case:

$$\Phi_k^{\mathrm{I}}\,(x,y) = \frac{c_k^{\mathrm{I}}\,\varphi_k\,(x,y)}{2\,G\,a^{\mathrm{I}}\,\lambda^2\,(T - \operatorname{sh} T \cdot \operatorname{ch} T)}\,,$$

$$\Phi_k^{\mathrm{II}}\,(x,y) = \frac{c_k^{\mathrm{II}}\,\varphi_k\,(x,y)}{2\,G\,a^{\mathrm{II}}\,\lambda^2\,(-\,T - \operatorname{sh} T \cdot \operatorname{ch} T)}\,.$$

By substituting functions Φ_k^{I} and Φ_k^{II} for Φ figuring in formulae (20) and (22), we obtain the displacement systems ξ_k^{I}, η_k^{I}, ζ_k^{I} and ξ_k^{II}, η_k^{II}, ζ_k^{II}, respectively. Hereafter the sought displacements $\xi^{(a)}$, $\eta^{(a)}$, $\zeta^{(a)}$ are calculated by formulae

$$\xi^{(a)} = \sum_{1}^{\infty} (\xi_k^{\mathrm{I}} + \xi_k^{\mathrm{II}})\,,$$

$$\eta^{(a)} = \sum_{1}^{\infty} (\eta_k^{\mathrm{I}} + \eta_k^{\mathrm{II}})\,,$$

$$\zeta^{(a)} = \sum_{1}^{\infty} (\xi_k^{\mathrm{I}} + \xi_k^{\mathrm{II}})\,.$$

b) *Determination of displacement system* $\xi^{(b)}$, $\eta^{(b)}$, $\zeta^{(b)}$. This displacement system is determined by formulae of cases III and IV of Chapter 5 in the same way as the displacement system $\xi^{(a)}$, $\eta^{(a)}$, $\zeta^{(a)}$ was established above. Therefore it is superfluous to deal with the problem here.

Remark. The application to a specific problem of the afore-outlined principles will be demonstrated in a subsequent issue of this periodical. The problem to be treated there will give a comparison how far the results of the afore-outlined, more exact calculations diverge from results computed with the simplifying assumptions mentioned in the Introduction.

Summary

The Kirchhoff theory of plates of uniform thickness (slabs, discs) uses simplifying assumptions, besides of the hypotheses of the classical theory of elasticity. Based on a system of solution of the basic equations elaborated by the author [3], this paper indicates a way by which the problem in question can be handled without those simplifying assumptions.

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