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RESEARCH ARTICLE

# Conditional Lagrange Derivative with Gibbs Function

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***Abstract***
*In forming constitutive relations a method of Mindlin was used. By introducing the conditional Lagrange derivative and by using the laws of thermodynamics a formula is obtained for ε. In the first law Gibbs function is used. This formula should be satisfied in case of constitutive relation.*

***Keywords***
*Mindlin's method, conditional Lagrange derivative, Gibbs function*

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## 1 Introduction

In solid mechanics equations containing material properties are called the constitutive relations. Constitutive relations play an essential role in most problems of the mechanics of continua. On the one hand such equations are used to take into account all observations and tests on the physical behavior of materials. On the other hand they are needed to complete the set of basic equations to perform the necessary computations. There are equations which are successfully used for several cases in applications. For example Hooke's law is suitable to metallic materials in most problems and we have accepted testing methodology to measure its material constants. However there are problems, when dissipation plays important role and such model cannot be used.

While contitutive relation is necessary and there is no generally accepted way to find it, the aim of this paper is to find a method which forms a basis for tests and experimental studies.

## 2 Variational principles and Lagrange derivative

The method we use goes back to Mindlin. In [3] stress and strain and the higher order derivatives of them are used to form a constitutive relation. Mindlin uses the variation of strain, his study is based on the virtual work and on the variation of integrals of some function $W$ with respect to the strain tensor. Variational principles of solids contain expressions

$$\int_t \int_V \sigma : \delta\varepsilon dV dt,$$

where stress, strain and the variation of strain are denoted by $\sigma$, $\varepsilon$ and $\delta\varepsilon$ , respectively. This expression can also be written in a generalized form

$$\int_t \int_V \sigma : \delta\varepsilon dV dt = \delta \int_t \int_V W dV dt + \int_t \int_V D : \delta\varepsilon dV dt. \tag{1}$$

Symbol double dots denotes the scalar product of matrices as usual. The first term in the right hand side contains the potential part of the stress while the second term refers the remaining (for example dissipation). Function $D$ is not known generally, it is used to consider all virtual work effects.

Mindlin has investigated equation [3]

$$\int_V \sigma : \delta\varepsilon dV = \delta\int_V U\left(\varepsilon, \varepsilon\nabla, \varepsilon\nabla\nabla\right) dV.$$

By using elastic potential $U$ he determined stress $\sigma$ as a function of the strain $\varepsilon$, the first and the higher gradients of $\varepsilon$ . Then he derived a new constitutive relation in such a way, which can be used for high gradient dependent elastic materials. The first term on the right hand side of (1) can be written in form

$$\int_t \int_V L_\varepsilon\left(W\right) : \delta\varepsilon dV dt,$$

where Lagrange derivative [1] is denoted by $L_\varepsilon(\;)$ . For the uniaxial case

$$L_\varepsilon(\;) = \left(\frac{\partial}{\partial\varepsilon} - \frac{d}{dt}\frac{\partial}{\partial\dot\varepsilon} - \frac{d}{dx}\frac{\partial}{\partial\varepsilon_x}\right)(\;),$$

where dot and index denote derivatives, for example

$$\dot\varepsilon = \frac{\partial\varepsilon}{\partial t} \quad \text{and} \quad \varepsilon_x = \frac{\partial\varepsilon}{\partial x}.$$

Then strain tensor can be expressed as a Lagrange derivative $L_\varepsilon$ with respect to $\varepsilon$,

$$\sigma = L_\varepsilon\left(W\right). \tag{2}$$

In mechanics we may use several types of variational principles. These contain virtual work

$$\int_t \int_V \sigma \quad \delta\varepsilon dV dt \tag{3}$$

and complementary virtual work

$$\int_t \int_V \varepsilon : \delta\sigma dV dt. \tag{4}$$

From (2) we have for (3)

$$\sigma : \delta\varepsilon = L_\varepsilon\left(u\right) : \delta\varepsilon$$

and for (4)

$$\varepsilon : \delta\sigma = L_\sigma\left(w\right) : \delta\sigma$$

When there are additional equations describing further material properties instead of Lagrange derivatives we should use conditional Lagrange derivaties. Assume that the additional equations have forms

$$u_1 = 0, \quad u_2 = 0, \ldots \qquad w_1 = 0, \quad w_2 = 0, \ldots$$

for (3) and for (4).

Then the conditional Lagrange derivatives in cases (3) and (4) are

$$L_\varepsilon\left(u_0 + \lambda_1 u_1 + \lambda_2 u_2 + \ldots\right), \tag{5}$$

and

$$L_\sigma\left(w_0 + \lambda_1 w_1 + \lambda_2 w_2 + \ldots\right), \tag{6}$$

where $\lambda_1$, $\lambda_2$,... are scalar Lagrange multipliers.

## 3 Conditional Lagrange derivative with the first law of thermodynamics

Take the first law of thermodynamics in form

$$u_1\left(\varepsilon, \vartheta\right) \equiv \rho\dot e - \sigma\dot\varepsilon + h_x - \rho\dot r = 0$$

as the "additional" condition. The notations are: $\vartheta$ – temperature, $e$ – internal energy, $\rho$ – mass density, $h$ – heat flux, $r$ – heat source intensity, $h_x = \partial h / \partial x$. Let us study the uniaxial case. Then from (5)

$$\sigma = L_\varepsilon\left(u_0 + \lambda u_1\right),$$

where

$$u_0 = u_0\left(\varepsilon\right) = a\left(\varepsilon\right).$$

In this case we derive [2]

$$\sigma = \rho\left(\frac{\partial a}{\partial\varepsilon} + \vartheta\frac{\partial S}{\partial\varepsilon}\right) + \frac{1}{\dot\varepsilon}\left(\rho\vartheta\frac{\partial S}{\partial\vartheta}\dot\vartheta + h_x\right) - \rho\frac{r}{\dot\varepsilon}. \tag{7}$$

Here and later on small strains are assumed. Further notation are: $a$ – strain energy, $S$ – entropy

Then we will use of complementary conditional Lagrange derivative. Now similarly to the previous case take from (6)

$$\varepsilon = L_\sigma\left(w_0\left(\sigma\right) + \lambda w_1\right) \tag{8}$$

and assume that the condition is the first law of thermodynamics

$$w_1 = 0$$

by using Gibbs function it has the form

$$w_1\left(\sigma, \vartheta\right) \equiv \rho\left(G + \vartheta S\right)^{\cdot} - \varepsilon\dot\sigma + h_x - \rho\dot r = 0.$$

Now from (9)

$$\lambda L_\sigma\left(w_1\right) = 0 \tag{9}$$

we substitute $w_1$ into (8)

$$\varepsilon = L_\sigma\left(w_0\left(\sigma\right)\right) + \lambda L_\sigma\left(w_1\left(\sigma, \vartheta\right)\right)$$

is obtained. Then we derive

$$\varepsilon = \frac{1}{\dot\sigma}\left(\rho\left(G_\sigma\dot\sigma + G_\vartheta\dot\vartheta\right) + h_\sigma\sigma_x + h_\vartheta\vartheta_x - \rho r\right).$$

We have two equations, multiply (10) by

$$E \equiv \frac{\dot\sigma}{\dot\varepsilon}$$

and add to (7). The result is

$$\sigma + E\varepsilon = \rho\left(\frac{\partial a}{\partial \varepsilon} + \vartheta\frac{\partial S}{\partial \varepsilon} - E\frac{\partial G}{\partial \sigma}\right) + \rho\left(\vartheta\frac{\partial S}{\partial \vartheta} - \frac{\partial G}{al\vartheta}\right)\frac{\dot{\vartheta}}{\dot{\varepsilon}} + \frac{1}{\dot{\varepsilon}}\left(h_{\varepsilon}\varepsilon_{x} - h_{\sigma}\sigma_{x}\right) \tag{10}$$

For constant temperature

$$\sigma + \varepsilon\frac{d\sigma}{d\varepsilon} = \rho\left(\frac{\partial a}{\partial \varepsilon} + \vartheta\frac{\partial S}{\partial \varepsilon} - \frac{\partial G}{\partial \sigma}\frac{d\sigma}{d\varepsilon}\right) \tag{11}$$

is obtained because then $\dot{\vartheta} = h = 0$ and

$$E \equiv \frac{d\sigma}{d\varepsilon}.$$

In the widely used standard tensile test one end of the specimen of length $\ell_0$ is fixed while the other is pulled with a small constant speed $v_0$. Then strain rate is apporimately

$$\dot{\varepsilon} \approx \frac{v_0}{\ell_0} \neq 0,$$

that is, strain rate is cosidered to be small and constant. While in tensile test constant temperature is also assumed, Eq. (11) should be satisfied.

## 4 Summary

Based on Mindlin's paper a method is suggested to calculate the possible forms of constitutive relations. By introducing conditional Lagrange derivatives stress and strain was calculated. Stress comes from our previous result [2] and strain was determined using Gibbs function. At the end a differential equation was obtained, which is available to analyze tensile test.


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