

# Two Theorems for Computation of Projections of Virtual Displacements and Its Application in Structural Analysis

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RESEARCH ARTICLE

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## Abstract

*A theorem for planar case and its generalization for spatial case are proposed to determine the projection of a virtual displacement to the orientation under the case of knowing the projections of a virtual displacement to the given two or three orientations for object systems subject to holonomic and scleronomic constraints. Some lemmas corresponding to the two theorems for special cases are given. Applications to structural static analysis are investigated using the two theorems in this paper. Result reveals that the two theorems and corresponding lemmas are easy to be used, shorten the distance between the principle of virtual displacement and its application, and the relating problems can be solved quickly with them.*

## Keywords

*projection of virtual displacement, principle of virtual displacement, engineering mechanics, structural analysis*

## 1 Introduction

It is well known that the principle of virtual displacement (or virtual work) is a main part in analytical statics, and the important basis for analytical dynamics and structural analysis. This principle provides an excellent tool for people to investigate the equilibrium laws of object systems, and plays a supporting role in classical mechanics.

Generally speaking, some current college textbooks [1-7] for engineering mechanics or structural analysis course often propose two main methods for building relations among virtual displacements of different points for the system with holonomic and scleronomic constraints, ie, analytical method and geometrical method. Using analytical method, one can determine the virtual displacements of interest points (usually points of forces) by taking differentials of their position coordinates, and geometrical methods by introducing the concept of the instant center of rotation of virtual displacements. These methods are very useful in application of the principle of virtual displacement. However, in actual experience, we realize that these methods are sometimes not sufficient for analyzing complex structures. For some difficult problems, such as the examples given in this paper, it is impossible or not sufficient to solve them out if only directly using the results given in these current textbooks. Then, how to shorten the distance between the principle of virtual displacement and its application, and give some feasible approaches to realize, is very important and great significance.

In this study, we will gain an insight into the application of the principle of virtual displacement and present two new theorems on how to build up the relations among the projections to different orientations of the virtual displacement of a point. Several examples are given to illustrate the application of two theorems and corresponding lemmas. The article is organized as follows. In Section 2, a theorem is given to find a projection under knowing the two projections to different orientations of a point for planar systems, and two lemmas for special cases are proposed. In Section 3, a theorem for spatial systems as the planar extending situation is given to find a projection under knowing the three projections to different orientations of a point, and three lemmas for special cases are proposed.

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## 2 The first theorem for plane cases

For a planar object system, if the projections of a virtual displacement  $\delta \mathbf{r}$  of a certain dot to orientations  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are  $\delta r_1$  and  $\delta r_2$  separately, shown in Fig. 1, then the projection of the virtual displacement to orientation  $\mathbf{n}_3$  is

$$\delta r_3 = \frac{\delta r_1 \sin(\theta - \phi) + \delta r_2 \sin \phi}{\sin \theta}, \quad (1)$$

where, angles  $\theta$  and  $\phi$  are angles between  $\mathbf{n}_1$  and  $\mathbf{n}_2$ ,  $\mathbf{n}_1$  and  $\mathbf{n}_3$  separately.

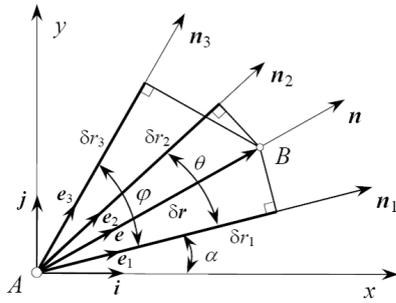


Fig. 1 A virtual displacement and its projections

**Proof.** Assume  $\mathbf{e}$ ,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$  are the unit vectors of the orientations of  $\mathbf{n}$ ,  $\mathbf{n}_1$ ,  $\mathbf{n}_2$  and  $\mathbf{n}_3$ , separately. Take the start point  $A$  of the virtual displacement  $\delta \mathbf{r}$  as the origin of the coordinate system  $xAy$ . Let  $\alpha$  is the angle between  $\mathbf{n}_1$  and  $x$  axis. According to the given conditions, one reads

$$\delta \mathbf{r} \cdot \mathbf{e}_1 = \delta r_1,$$

$$\delta \mathbf{r} \cdot \mathbf{e}_2 = \delta r_2.$$

Hence, one can get

$$\delta \mathbf{r} \cdot (\cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}) = \delta r_1 \quad (2)$$

$$\delta \mathbf{r} \cdot [\cos(\alpha + \theta) \mathbf{i} + \sin(\alpha + \theta) \mathbf{j}] = \delta r_2 \quad (3)$$

where  $\mathbf{i}$ ,  $\mathbf{j}$  are the standard unit vectors of  $x$  and  $y$  axis.

By taking the inner products  $\delta \mathbf{r} \cdot \mathbf{i}$  and  $\delta \mathbf{r} \cdot \mathbf{j}$  as unknowns, and solving the equations of (2) and (3), one can get

$$\delta \mathbf{r} \cdot \mathbf{i} = \frac{\delta r_1 \sin(\alpha + \theta) - \delta r_2 \sin \alpha}{\sin \theta} \quad (4)$$

$$\delta \mathbf{r} \cdot \mathbf{j} = \frac{\delta r_2 \cos \alpha - \delta r_1 \cos(\alpha + \theta)}{\sin \theta} \quad (5)$$

Then, the projection of the virtual displacement to orientation  $\mathbf{n}_3$  is

$$\begin{aligned} \delta r_3 &= \delta \mathbf{r} \cdot \mathbf{e}_3 = \delta \mathbf{r} \cdot [\cos(\alpha + \phi) \mathbf{i} + \sin(\alpha + \phi) \mathbf{j}] \\ &= \cos(\alpha + \phi) \cdot \frac{\delta r_1 \sin(\alpha + \theta) - \delta r_2 \sin \alpha}{\sin \theta} \\ &\quad + \sin(\alpha + \phi) \cdot \frac{\delta r_2 \cos \alpha - \delta r_1 \cos(\alpha + \theta)}{\sin \theta} \\ &= \frac{\delta r_1 \sin(\alpha + \phi) + \delta r_2 \sin \phi}{\sin \theta}. \end{aligned}$$

**Lemma 1.** When  $\theta \neq 0(180^\circ)$ ,  $\delta r_1 = \delta r_2 = 0$ , then  $\delta r_3 = 0$ .

**Lemma 2.** When  $\theta = 90^\circ$ , then  $\delta r_3 = \delta r_1 \cos \phi + \delta r_2 \sin \phi$ .

For demonstrating the effectiveness of this theorem in solving the balancing problems of planar object systems, two examples are given below.

**Example 1.** Determine the force in member  $CD$  of the truss, shown in Fig. 2. Assume all members are pin connected, and  $AD = AE = ED = EC = CG = DG = DB = BG = a$ .

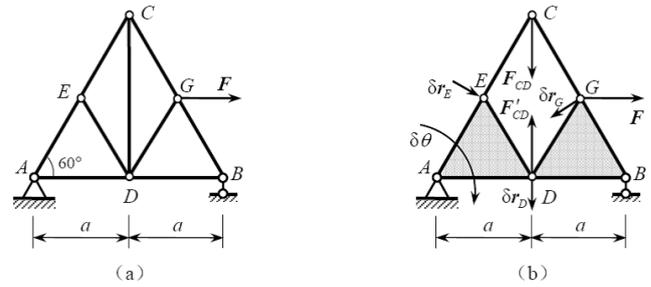


Fig. 2 Example 1

**Solution.** To calculate the force in rod  $CD$ , it is isolated from the system in Fig. 2 (b). Then, the system in Fig. 2 (b) is a mechanism. Obviously, equilateral triangles  $AED$  and  $DBG$  can be regarded as rigid plates, and  $\Delta AED$  can be assumed to rotate about  $A$  point with  $\delta \theta$ . Then,  $\delta r_E = \delta r_D = a \delta \theta$ . Considering the orientation of  $\delta r_D$  and character of support  $B$ , point  $B$  is the virtual displacement center of  $\Delta DBG$ , therefore  $\delta r_G = \delta r_D = a \delta \theta$ . Based on the principle of virtual displacement, one reads

$$F \cdot \delta r_G + F'_{CD} \cdot \delta r_D + F_{CD} \cdot \delta r_C = 0 \quad (6)$$

Based on the theorem of projection of virtual displacement (i.e. the projections of the virtual displacements of the points from a straight line belonging to a body, on that line, are equal.), the projections of  $\delta r_C$  to the orientations  $E \rightarrow C$  and  $G \rightarrow C$  are zeros. Then, by employing the lemma 1 of the above theorem,  $\delta r_C$  must be zero. Therefore, based on the above analysis and (6), one can get

$$-F \cdot a \delta \theta \cdot \cos 30^\circ - F'_{CD} \cdot a \delta \theta = 0.$$

Because of  $\theta \neq 0$ , then

$$F'_{CD} = F_{CD} = -F \cdot \cos 30^\circ = -\frac{\sqrt{3}}{2} F.$$

(bar  $CD$  in compression).

**Example 2.** Determine the force in member  $DG$  of the truss, shown in Fig. 3. Assume all members are pin connected.

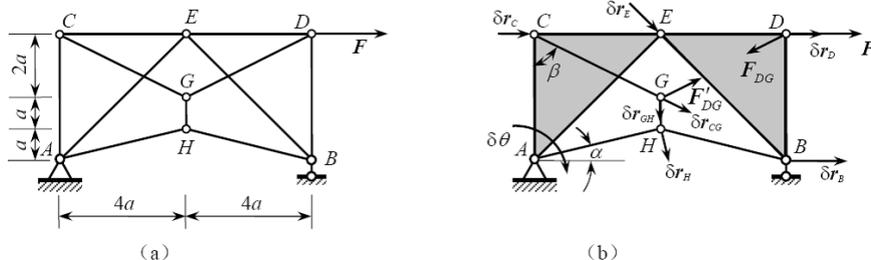


Fig. 3 Example 2

**Solution.** To calculate the force in rod  $DG$ , it is isolated from the system in Fig. 3 (b). Obviously, triangles  $ACE$  and  $DBE$  can be regarded as rigid plates, and  $\triangle ACE$  can be assumed to rotate about  $A$  point with  $\delta\theta$ . Then,  $\delta r_C = 4a\delta\theta$ ,  $\delta r_E = 4\sqrt{2}a\delta\theta$ . Considering that the orientation of  $\delta r_B$  is horizontal, the virtual displacement  $\delta r_D$  must be horizontal, and

$$\delta r_D = \delta r_E \cdot \cos 45^\circ = 4a\delta\theta,$$

$$\delta r_B \cdot \cos 45^\circ = \delta r_E, \delta r_B = 8a\delta\theta.$$

For the bar  $BH$ , it is easy to get that

$$\delta r_H \cdot \cos(90^\circ - 2\alpha) = \delta r_B \cdot \cos \alpha$$

$$\delta r_H = \frac{1}{2 \sin \alpha} \delta r_B = \frac{4a}{\sin \alpha} \delta\theta.$$

Then, for the bar  $GH$ , the projection of  $\delta r_G$  to the orientation  $G \rightarrow H$  is

$$\delta r_{GH} = \delta r_H \cdot \cos \alpha = \frac{4a \cos \alpha}{\sin \alpha} \delta\theta.$$

For the bar  $CG$ , the projection of  $\delta r_G$  to the orientation  $C \rightarrow G$  is

$$\delta r_{CG} = \delta r_C \cdot \cos(90^\circ - \beta) = 4a \sin \beta \delta\theta.$$

For the point  $G$ , the projections  $\delta r_{GH}$  and  $\delta r_{CG}$  are known. By employing the formula (1) of the above theorem, the projection of  $\delta r_G$  to the orientation  $G \rightarrow D$  is

$$\begin{aligned} \delta r_{GD} &= \frac{\delta r_{GH} \sin\{\beta - [\beta + 2(90^\circ - \beta)]\} + \delta r_{CG} \sin[\beta + 2(90^\circ - \beta)]}{\sin \beta} \\ &= \left[ \frac{-4a \cos \alpha \sin 2\beta}{\sin \alpha \sin \beta} + 4a \sin \beta \right] \delta\theta \\ &= (-8a \cot \alpha \cos \beta + 4a \sin \beta) \delta\theta \\ &= \left( -8a \cdot 4 \cdot \frac{2}{2\sqrt{5}} + 4a \cdot \frac{4}{2\sqrt{5}} \right) \delta\theta = -\frac{24}{\sqrt{5}} a \delta\theta. \end{aligned}$$

By employing the principle of virtual displacement, one can get

$$F \cdot \delta r_D + F_{DG} \cdot \delta r_D + F'_{DG} \cdot \delta r_G = 0 \quad (7)$$

Therefore, based on the above analysis and (7), one can get

$$F \cdot 4a\delta\theta - F_{DG} \cdot 4a\delta\theta \cdot \cos(90^\circ - \beta) - F'_{DG} \cdot \frac{24}{\sqrt{5}} a \delta\theta = 0.$$

Because  $\delta\theta \neq 0$ ,  $F_{DG} = F'_{DG}$ , then

$$F_{DG} = \frac{F}{\sin \beta + \frac{6}{\sqrt{5}}} = \frac{F}{\frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}}} = \frac{\sqrt{5}F}{8} = 0.279F.$$

### 3 The second theorem for spatial cases

If the projections of the virtual displacement  $\delta \mathbf{r}$  of a dot to three non-coplanar orientations  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are  $\delta r_1, \delta r_2, \delta r_3$ , then the projection  $\delta r_4$  of the virtual displacement to the orientation  $\mathbf{n}_4$  is

$$\delta r_4 = (e_{4x}, e_{4y}, e_{4z}) \begin{bmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{bmatrix}^{-1} \begin{pmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{pmatrix}. \quad (8)$$

where  $e_{ix}, e_{iy}, e_{iz}$  are projections of unit vector  $\mathbf{e}_i$  of spatial orientation  $\mathbf{n}_i$  to the Cartesian coordinate axes  $x, y, z$ , separately. ( $i = 1, 2, 3, 4$ )

**Proof.** Assuming that  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the standard unit orthogonal vectors of the spatial coordinate system, based on the given conditions, one reads

$$\mathbf{e}_i = e_{ix}\mathbf{i} + e_{iy}\mathbf{j} + e_{iz}\mathbf{k}, \quad i = 1, 2, 3, 4$$

and

$$\begin{cases} e_{1x}\delta \mathbf{r} \cdot \mathbf{i} + e_{1y}\delta \mathbf{r} \cdot \mathbf{j} + e_{1z}\delta \mathbf{r} \cdot \mathbf{k} = \delta r_1 \\ e_{2x}\delta \mathbf{r} \cdot \mathbf{i} + e_{2y}\delta \mathbf{r} \cdot \mathbf{j} + e_{2z}\delta \mathbf{r} \cdot \mathbf{k} = \delta r_2 \\ e_{3x}\delta \mathbf{r} \cdot \mathbf{i} + e_{3y}\delta \mathbf{r} \cdot \mathbf{j} + e_{3z}\delta \mathbf{r} \cdot \mathbf{k} = \delta r_3 \end{cases}$$

or

$$\begin{bmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{bmatrix} \begin{pmatrix} \delta \mathbf{r} \cdot \mathbf{i} \\ \delta \mathbf{r} \cdot \mathbf{j} \\ \delta \mathbf{r} \cdot \mathbf{k} \end{pmatrix} = \begin{pmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{pmatrix}.$$

Because the three orientations  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are non-coplanar, the coefficient matrix is reversible. Hence, one can get

$$\begin{pmatrix} \delta \mathbf{r} \cdot \mathbf{i} \\ \delta \mathbf{r} \cdot \mathbf{j} \\ \delta \mathbf{r} \cdot \mathbf{k} \end{pmatrix} = \begin{bmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{bmatrix}^{-1} \begin{pmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{pmatrix}.$$



## 4 Conclusions

This paper mainly faces to the difficulty of computation of virtual work, investigates on the projection of a dot's virtual displacement to a given orientation, propose two theorems and corresponding lemmas, and discuss their application for analyzing the forces of structural members. Computation procession reveals that the formula for planar situation is easy to use, and the formula for spatial situation is normalized and easy to be remembered. Combining with current methods, these results will be very helpful to analyze inner forces of complicit structures. It's necessary to say that the two theorems can be used in dynamics when dynamical problems convert into static problems in form by utilizing D'Alembert's principle.

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