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APPLICATION OF CONTINUOUS CASH FLOW STREAMS IN THE FIELD OF RELIABILITY ENGINEERING USING LAPLACE TRANSFORM

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Abstract

It is suggested that economic problems with stochastic character can be easily solved if the cost variables, which are stochastic in nature, have as their source the stochastic character of the failure. The economical aspects of reliability and the knowledge of reliability engineering in applied economic analysis are of much more importance than it is devoted in literature. This paper presents some examples of this issue.

In the first part, the present value computation technique using transform technique is outlined, which, is rarely applied in economics analysis. The second part shortly summarises the most important theorems of reliability engineering. Finally, in the main part, a cost model is presented which can be used to address stochastic economic problems, where sources of stochastism are failure processes.

Keywords: economic analysis, reliability engineering, Laplace transform.

1. Introduction

Economic analysis (engineering economics) and reliability engineering usually appear as independent disciplines to be dealt with and taught separately. Although most comprehensive textbooks of both fields (such as PARK and SHARP-BETTE, 1990; GOSSEN, 1991; IRESON and COOMBS, 1988; GROSH, 1989) make some direct or indirect hints at their relations to the other discipline, yet, these references – often only in some words – suggest a very loose and superficial connection. In my opinion, this is not the case. Many stochastic economic problems can only be managed with thorough knowledge of both economic analysis and reliability engineering.

In this paper I summarise some traditional basic concepts and new approaches of economic analysis and reliability engineering with the help of which I will find a solution for problems involving both disciplines. I think that all this will also demonstrate the close connection of both scientific fields.

2. Some Parts of Economic Analysis

2.1 Present Value of Cash Flow Series

The theoretical and practical importance of present value and net present value is well known. Only the most important relations will be summarised below.

2.1.1 Present Value of Discrete Cash Flow Series

$$P(i) = \sum_{n=0}^{\infty} F_n(1+i)^n, \tag{1}$$

where

i: the discount rate
$P(i)$: the present value under i
n: the number of compounding periods (usually years)
F_n: cash flow in period n

2.1.2 Continuous Compounding

In (engineering) economic analysis, year is usually used as the interest period because investments in engineering projects are of long duration and a calendar year is a convenient period for accounting and tax computation. (PARK and SHARP-BETTE, 1990).

However, we can also use a compounding period that is more frequent than the annual one. In this case, we must introduce the terms nominal interest rate and effective interest rate. The relation of both terms is given by the following equation:

$$i_{eff} = (1 + \frac{r}{M})^M - 1, \tag{2}$$

where

i: the effective annual interest rate
r: the nominal interest rate per year
M: the number of interest (compounding) periods per year
r/M: the interest rate per interest period

Assuming a compounding of infinite frequency, we obtain:

$$i_{eff} = \lim_{M\to\infty} (1 + \frac{r}{M})^M - 1 = e^r - 1. \tag{3}$$

In this case, we speak of continuous compounding. (It is worth mentioning that approximation of an in reality annual compounding by a continuous compounding does not evolve a serious error. For example, if r =10%, then $i \approx 10.5\%$. Therefore it is usually allowable to use this approximation, in such cases where the use of continuous compounding seems to be much more practical.)

2.1.3 Present Value of Continuous Cash Flow

It is often appropriate to treat cash flows as though they were continuous rather than discrete. An advantage of the continuous flow representation is its flexibility for dealing with patterns other than the uniform and gradient ones. (PARK and SHARP-BETTE, 1990).

Formula of present value of continuous cash flow differs from the discrete case shown in *Eq.* (1) in as much as F_n becomes continuous $f(t)$ and the effective annual interest rate i for continuous compounding is $e^r - 1$, integration of the argument instead of summation yields

$$P(r) = \int_0^\infty f(t)e^{-rt}dt, \tag{4}$$

where:

$f(t)$: the continuous cash flow function of the project
r : the nominal interest rate [$r=\ln(1+i)$]
t : the time expressed in years

2.2 Application of Transform Techniques in Computation of Present Values

Papers on application of various transform techniques in economic analyses have been appearing in bibliography of engineering economics in the early seventies (BUCK and HILL, 1971, 1974, 1975 and MUTH, 1977). I will shortly demonstrate the application of the Laplace transformation.

2.2.1 Application of the Laplace Transformation in Determination of Continuous Cash Flow Present Value

The general formula of present value of continuous cash flows has been demonstrated in *Eq.* (4). As BUCK and HILL (1971) recognized, the general form of this integral bears a close resemblance to the definition of the

Laplace transforms. That is, if the function $f(t)$ is considered to be continuous, then the Laplace transform of $f(t)$, written $L\{f(t)\}$, is defined as a function $F(s)$ of the variable s by the integral

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{5}$$

over the range of values of s for which the integral exists. Replacing s in *Eq.* (5) with the continuous compound interest rate r simply generates *Eq.* (4); thus, taking a Laplace transform on the cash flow streams over an infinite horizon time (PARK and SHARP-BETTE, 1990):

$$P(r) = L\{f(t)\}. \tag{6}$$

As Laplace transform of the most important functions can be found in tables (e. g. FODOR, 1966), and the operational rules of Laplace transform are also known (e.g. FODOR 1966), the almost optional present value $f(t)$ can be relatively simply determined. On the basis of the above, we can also determine the present values of the general form of the most important continuous cash flows. By tabulating these present values, we obtain a well applicable tool for quick determination of present values of cash flows with different characteristics, represented in *Table 1*.

3. Some Parts of Reliability Engineering

3.1 Basic Formulations

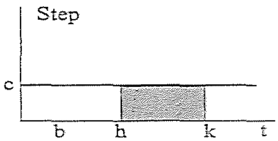
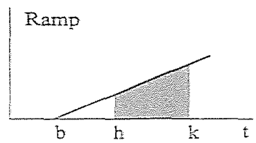
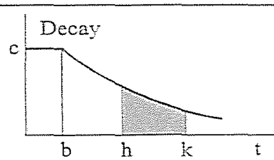
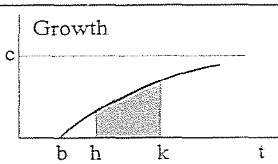
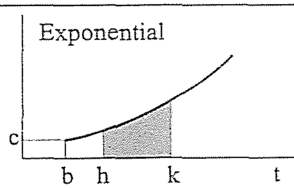
Let T be a non-negative continuous random variable that represents the useful life (or length of life, or time to failure) of a component (or unit or piece of equipment). The failure law for the component can be described in several ways. Perhaps the most fundamental formulation is in terms of $F(t)$, the cumulative distribution function defined as the probability that the unit 'lives' for at most time t, and which we write as:

$$F(t) = P\{T \leq t\}. \tag{7}$$

This is also referred to as the unreliability function. An equivalent and sometimes more useful formulation is the reliability function $R(t)$, the probability that the component lives longer than time t, which is designated as:

$$\begin{aligned} R(t) &= P\{T > t\} \\ &= 1 - P\{T \leq t\} \\ &= 1 - F(t). \end{aligned} \tag{8}$$

Table 1

Time Form	$f(t)$	$PV(r)$
	c	$\frac{c}{r}\left(e^{-hr} - e^{-kr}\right)$
	ct	$\frac{c}{r^2}\left(e^{-hr} - e^{-kr}\right) + \frac{c}{r}\left[(h-b)e^{-hr} - (k-b)e^{-kr}\right]$
	ce^{-jt}	$\frac{ce^{+bj}}{r+j}\left(e^{-h(j+r)} - e^{-k(j+r)}\right)$
	$c(1-e^{-jt})$	$\frac{c}{r}\left(e^{-hr} - e^{-kr}\right) - \frac{ce^{+bj}}{r+j}\left(e^{-h(j+r)} - e^{-k(j+r)}\right)$
	ce^{-jt}	$\frac{ce^{-bj}}{r-j}\left(e^{h(j-r)} - e^{k(j-r)}\right)$ $j \neq r$

It is traditional also to describe the failure law in terms of the density function

$$f(t) = F'(t) \tag{9}$$

which must have the following properties that

$$f(t) =\geq 0, \tag{10a}$$

$$\int_0^\infty f(t)dt = 1. \tag{10b}$$

3.2 Renewal Processes

We have frequent occasion in reliability engineering to work with renewal processes, as typified by the following situation. Let an equipment consist of renewable units. Let this equipment start to work at time $t = 0$. The m-th unit of the equipment works until it fails at time t_{m1}, when it is instantaneously replaced by a new one. This new unit in turn functions until its failure at time t_{m2}, whereupon it is immediately replaced, and so on. The failures occur at random times depending on the same probability law about which various assumptions can be made. Let $N_m(t)$ be the integer random variable that designates the number of failures of m-th unit by time t. It is desired to formulate an expression for

$$P_{mn}(t) = P\{N_m(t) = n\} \tag{11}$$

the probability of n failures of m-th unit by time t. Let (τ_{mi} be the failures-free working time of m-th unit between i-th and $(i-1)$-th failures. If all τ_{mi} are independent random variables with the same distribution, then we can define the expected value and the variance of $N_m(t)$ discrete random variable, in knowledge of the commutative distribution function $F_m(t)$ of the continuous random variable τ_m. The expected value of $N_m(t)$, i. e. the number of failures of m-th unit by time t, is the so-called renewal function $D_m(t)$. (GNEDENKO et al., 1965)

$$D_m(t) = E[N_m(t)] = f(\tau, F_m(t)), \tag{12}$$

$$D_m(t) = \sum_{n+1}^{\infty} F_{mn}(t). \tag{13}$$

Instead of function $D_m(t)$ its derivative is often examined:

$$d_m(t) = D'_m(t). \tag{14}$$

This is the so-called renewal density function that gives the mean of the failures occurring during the next unit of time for all moments t (if the unit of time is small) (GNEDENKO et al., 1965). From *Eq.* (9) and *Eq.* (14), it can be deduced that the renewal density function can be expressed in the following form of the infinite series:

$$d_m(t) = \sum_{n=1}^{\infty} f_{mn}(t). \tag{15}$$

As the examined equipment consists of M independent renewable units, thus the renewal function and the renewal density function of the equipment are:

$$D(t) = \sum_{m=1}^{M} D_m(t), \tag{16}$$

$$d(t) = \sum_{m=1}^{M} d_m(t). \tag{17}$$

(GNEDENKO et al., 1965).

3.3 Special Renewal Processes

3.3.1 Poisson Process

When exponential distribution is assumed for $\tau_m, F_m(t)$ is as follows:

$$F_m(t) = 1 - e^{-\lambda_m t}, \tag{18}$$

where λ_t, the parameter of the exponential distribution, is the so-called Poisson (renewal) process. In this case $P_{mn}(t)$, $D_m(t)$ and $d_m(t)$ can be written in the following simple forms (GNEDENKO et al., 1965):

$$P_{mn}(t) = P\{N(t) = n\} = \frac{(\lambda_m t)^n)}{n!} e^{-\lambda_m t}, \tag{19}$$

$$D_m(t) = \lambda_m t, \tag{20}$$

$$d_m(t) = \lambda_m. \tag{21}$$

3.3.2 Renewal Process with Normal Distribution

If the distribution of τ_m is a normal distribution and we assume that $\sigma << \mu$, where σ is the standard deviation and μ is the expected value of τ_m, then $D_m(t)$ and $d_m(t)$ are given as follows (GNEDENKO – BELJAEV – SOLOVJEV, 1965):

$$D_m(t) = \sum_{n=1}^{\infty} \Phi\Big(\frac{t - n\mu}{\sigma\sqrt{n}}\Big), \tag{22}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

which values can be easily determined by the help of tables (e. g. GROSH, 1989), and

$$d_m(t) = \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{2\pi n}} e^{-\frac{(t-n\mu)^2}{2n\sigma^2}}. \tag{23}$$

This renewal density function has a very characteristic wave shape, which is depicted in *Fig. 1.*

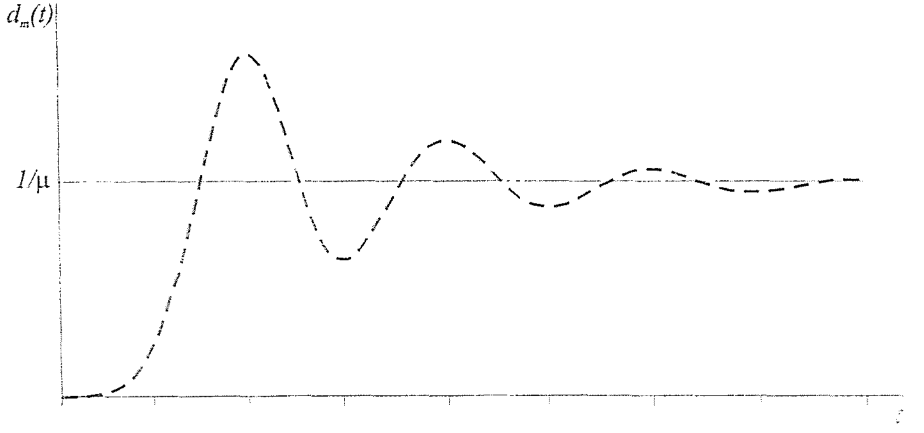


Fig. 1. Density function of a renewal process with normal distribution

4. Some Problems Treatable by Laplace Transform Technique and Reliability Engineering

As shown in *Table 1* at present there are much more advanced methods for present value computations permitting treatment of more complicated cash flow series and cash flow streams. However, these opportunities cannot be utilised if models cannot be filled in with data of appropriate accuracy, and analyses in engineering economics lose their meaning due to lack of appropriate data. An essential part of these necessary data represent the cost data, thus, their estimation with the highest possible accuracy is a task of key importance.

In the opinion of IRESON and COOMBS (1966), the majority of cost estimating methods is based on the premise that the system cost is in a quantifiable way logically related to some of the system's physical or performance characteristics. This is generally derived from historical cost data by regression analysis. The most common form of estimating algorithm is:

$$C = f(x_1, \ldots, x_n), \tag{24}$$

where

C: the amount of cost

f: a mathematical function

$x_1, \ldots, x_n$: cost variables correlated with some physical or performance characteristics of the system (equipment).

The undoubted advantage of this approach is its easy application. Its disadvantage, on the other hand, is that it gives a deterministic model for the changes of the cost – which very often means an exaggerated simplification. There is, however, a different approach as well. We can derive from historical cost data the functions of cost variables. In this case, naturally, the C cost becomes a random variable. This method also gives a stochastic model, however, the overcomplicated structure of the model makes its application impossible.

In my opinion a mixture of these two models can give in many cases a better model than any. I suggest that, in the case of such cost variables, which have significant weight on the cost changes and which have very strong stochastic character, to keep the stochastic character and, concerning the rest, to use deterministic functions.

At first sight, this mixture model may certainly seem to be too complicated and hardly applicable in practice, but in many cases it is not so. However, very often the cost variables, which have to be managed in a stochastic way, mean some renewal process, so they are easily manageable with the help of some reliability engineering knowledge.

On the basis of the above mentioned facts, I want to present another model. But before this I summarise some important conditions of the applicability of the model:

1. The cost should be considered as the sum of the functions of cost variables.
2. These variables should be independent from each other.
3. The stochastic character of the change of the amount of cost has its source in the stochastic character of the failure (Of course in the presented model we can manage stochastism of other sources as well, but this paper focuses on problems which can be approached with the help of reliability engineering).
4. We should possess functions of cost variables to be managed in a deterministic way (i. e. with sufficient data to determine these functions).
5. We should possess sufficient data to determine the probability functions of the renewal processes.
6. Following each failure the renewal (replacement) takes place in very short time.

7. The performance of the equipment should be uniform in time.

The conditions No. 2, 6 and 7 can be broken up, but it makes the model more complicated, thus I will not refer to them in this paper.

With all these conditions the model can be described as

$$c(t) = \sum_{l=1}^{L} k_l(t) + \sum_{m=1}^{M} a_m d_m(t), \tag{25}$$

where:

$c(t)$: the cost density function showing the change of cost during a time unit (of course, if this time unit is small).

$k_l(t)$: the time function of the cost component induced by the l-th cost variables during the time unit.

$d_m(t)$: the renewal density function of the m-th unit of the equipment (see *Eq.* (14)).

a_m: the costs incurred at the failure (replacement) of the m-th unit of the equipment.

As in the economic analysis in general the task is to determine the present value, so in the following the determination of the present value will be presented.

For analysis the time interval should last from 0 to T.

As in the case where the amount of the cost is a random variable, the aim is to determine the expected value of the amount of cost until T.

The present value of the deterministic part of *Eq.* (25)

$$\sum_{l=1}^{L} k_l(t)$$

gives the sum of present value of the components. The present value of the components can be easily determined with the help of *Table 1*. The results can also be considered as expected values, where the variances are zeros.

The determination of the present value of the stochastic part of *Eq.* (25)

$$\sum_{m=1}^{M} a_m d_m(t)$$

seems to be more complicated. For the sake of better understanding, let us examine a simple stochastic case. Let the t timing of F cash flow be a random variable (as can be seen in *Fig. 2*).

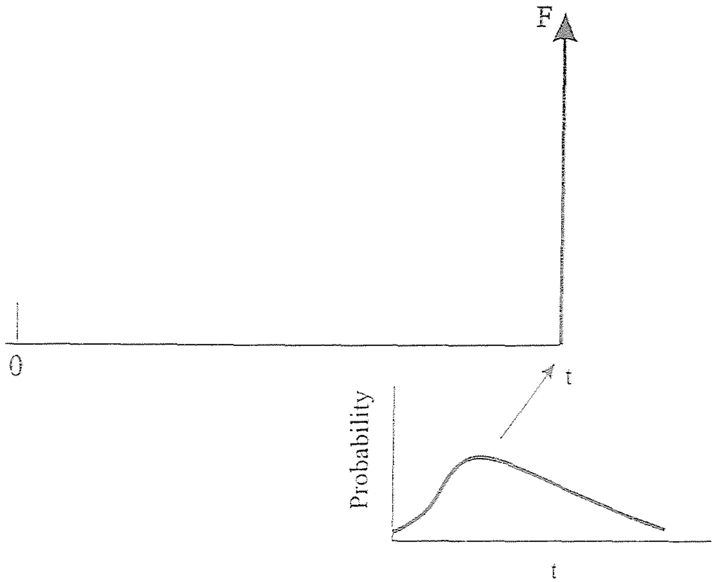


Fig. 2. Cash flow with uncertain timing (source: PARK and SHARP-BETTE, 1990)

Following this, the expected value and the variance of the present value of F at a nominal rate of r (on the basis of PARK and SHARP-BETTE, (1990) are given as follows:

$$E[P(r)] = \int_0^\infty F e^{-rt} f(t) dt \tag{26}$$

$$= \int_0^\infty f(t) e^{-rt} dt$$

$$= FE(e^{-rt}), \tag{27}$$

where $f(t)$ denotes the probability density function about the timing of F. From Chapter 2, the expression for

$$\int_0^\infty f(t) e^{-rt} dt$$

is known as the Laplace transform of the function $f(t)$ and is denoted by $L(r)$. Since the Laplace transform of most standard forms of probability functions is known (e.g. in Park and SHARPE-BETTE, 1990), we may easily calculate $E(e^{-rt})$. That is,

$$E(e^{-rt}) = L[f(t)] = L(r). \tag{28}$$

The variance computation is

$$Var(e^{-rt}) = L(2r) - L(r)^2. \tag{29}$$

The task is to determine $L_m(r)$, the Laplace transform pair of $d_m(t)$. The expected value and the variance of present value of the stochastic part of *Eq.* (25) are:

$$E[P(r)] = \sum_{m=1}^{M} E[P_m(r)] = \sum_{m=1}^{M} a_m L_m(r) \tag{30}$$

and

$$Var[P(r)] = \sum_{m=1}^{M} Var[P_m(r)] = \sum_{m=1}^{M} a_m^2 \Big(L_m(2r) - L_m(r)^2 \Big) \tag{31}$$

or if time horizon lasts from 0 to T:

$$E[P^T(r)] = \sum_{m=1}^{M} E[P_m^T(r)] = \sum_{m=1}^{M} a_m L_m^T(r) \tag{32}$$

and

$$Var[P^T(r)] = \sum_{m=1}^{M} Var[P_m^T(r)] = \sum_{m=1}^{M} a_m^2 \Big(L_m^T(2r) - L_m^T(r)^2 \Big), \tag{33}$$

where P^T and L^T denoted the present value and the Laplace transform from 0 to T.

If d_m is the renewal density function of a Poisson renewal process (see *Eq.* (21)), we have a very simple matter, because

$$E[P_m(r)] = L\Big\{a_m \frac{1}{\lambda_m}\Big\} = \frac{a_m}{r\lambda_m} \tag{34}$$

and

$$E[P_m^T(r)] = L^T\Big\{a_m \frac{1}{\lambda_m}\Big\} = \frac{a_m}{r\lambda_m}\Big(1 - e^{-Tr}\Big). \tag{35}$$

In the case of renewal process with normal distribution, we do not have to face such difficult problems either, if $\mu << T$. Then *Eq.* (23) is well approachable with the function:

$$d_m(t) \cong \frac{1}{\mu} \tag{36}$$

(see *Fig. 1*) (GNEDENKO et al., 1965). Then the case is equal to the earlier one presented for the Poisson renewal process (see *Eq.* (34) and (35).

If μ is not $<< T$, then the task is fairly difficult. We have to determine the Laplace transform pair of *Eq.* (23). As we know the Laplace transform of a sum is the sum of the Laplace transforms of the elements, so the Laplace transform pair of *Eq.* (23) can be written as:

$$L_m^T(r) = \sum_{n=1}^{\infty} L_n^T(r). \tag{37}$$

As ($\sigma << \mu$, thus $L_{mT}(r)$ can be closely approximated as

$$L_m^T(r) = \sum_{n=1}^{>\frac{T}{\mu}} L_n^T(r), \tag{38}$$

where $>\frac{T}{\mu}$ denotes the rounding-off to the next whole number.

The last task is to determine $L_n^T(r)$, the Laplace transform pair of

$$\frac{1}{\sigma\sqrt{2n\pi}} \int_0^T e^{-\frac{(t-n\mu)^2}{2n\sigma^2}} dt,$$

which can be given as follows:

$$\begin{aligned} L_n^T(r) &= \frac{1}{\sigma\sqrt{2n\pi}} \int_0^T e^{-\frac{(t-n\mu)^2}{2n\sigma^2}} e^{-rt} dt \\ &= e^{-n(r\mu - \frac{\sigma^2 r^2}{2})} \Big[\Phi\Big(\sqrt{n}(\frac{T}{n} + r\sigma - \frac{\mu}{\sigma})\Big) - \Phi\big(\sqrt{n}(r\sigma - \frac{\mu}{\sigma})\big)\Big] \end{aligned} \tag{39}$$

If

$$p = r\mu - \frac{\sigma^2 r^2}{2}$$

and

$$q = r\sigma - \frac{\mu}{\sigma},$$

then

$$L_n^T(r) = e^{-np}\Big[\Phi\Big(\sqrt{n}(\frac{T}{n}+q)\Big) - \Phi\Big(q\sqrt{n}\Big)\Big]. \tag{40}$$

On the basis of the above, the expected present value of this case can be also easily determined as:

$$E[P_m^T(r)] = \sum_{n=1}^{>\frac{T}{\mu}} e^{-np}\Big[\Phi\Big(\sqrt{n}(\frac{T}{n}+q)\Big) - \Phi\Big(q\sqrt{n}\Big)\Big]. \tag{41}$$

And finally, I present *Fig. 3*, which represents a $c(t)$ cost density function, and consists of three different cost components with different characters.

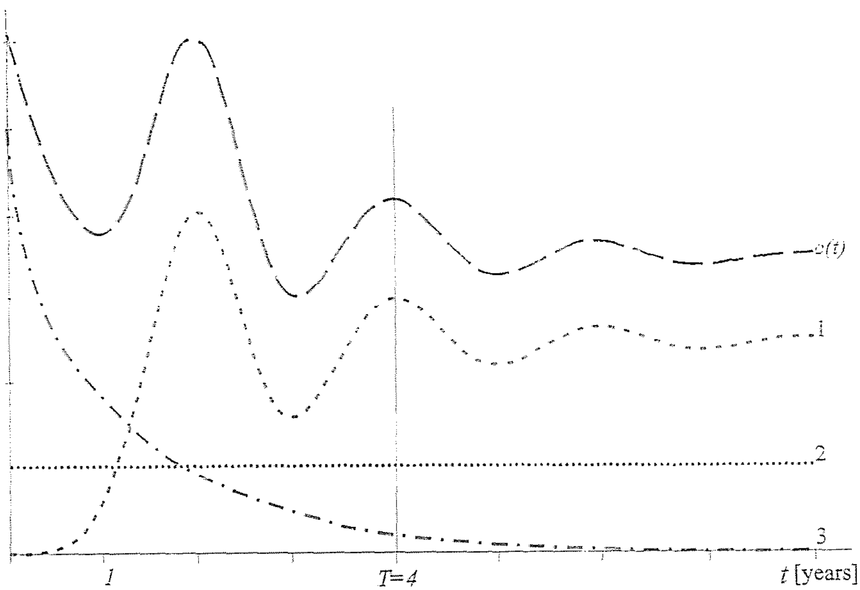


Fig. 3. Cost density function, which consists of three different cost components with different characters.

Line 1 represents the renewal density function of a renewal process with normal distribution ($\mu = 2, \sigma = 0.5$) (see *Eq.* (23)), where the cost of a failure is $a_1 = \$5{,}000$. We can calculate the expected present value of this cost component at nominal rate 10% and time from 0 to 4 with the help of *Eq.* (39) and *Eq.* (40) as:

$$E[P_1^4(10\%)] =$$

$$\$5,000 \sum_{n=1}^{2} e^{-n0.1988} \Big[\Phi\Big(\sqrt{n}(\frac{4}{n} - 3.95)\Big) - \Phi\Big(- 3.95\sqrt{n}\Big)\Big] = \$5,518. \quad (41)$$

Line 2 represents the renewal density function of a Poisson renewal process (λ=1 year), where the cost of a failure is a_2=\$1,000. The expected present value of this cost component at nominal rate 10% time from 0 to 4 can be calculated on the basis of *Table 1* and *Eq.* (21):

$$E[P_2^4(10\%)] = \frac{\$1,000}{0.1}(1 - e^{-0.4}) = \$3,297.$$

Line 3 represents the cash flow stream of a deterministic cost component ($f(t) = 5,000e^{-0.9t}$). The present value of this stream until T=4 can be easily determined with the help of *Table 1*

$$E[P_2^3(10\%)] = \frac{\$5,000}{0.1 + 0.9}\Big(1 - e^{-4(0.1+0.9)}\Big)$$

The expected present value of $c(t)$ time from 0 to 4 is the sum of the three components:

$$E[P^4(10\%)] = E[P_1^4(10\%)] + E[P_2^4(10\%)] + E[P_3^4(10\%)] = \$13,723.$$

5. Summary

The computation of the present value of many stochastic economic problems can be easily managed using a cost model based on reliability engineering approach. This is therefore achieved by using the Laplace transform pairs of the main renewal processes, as they give simple functional forms.

It is recommended that further literature reviews need to deal with these issues more thoroughly.

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