LINEAR MULTISTAGE OPTIMIZATION PROBLEM[1]

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Abstract

A multiperiod decision supporting system has been developed. The model is based on linear programming formulation. Markov transition probability matrices are introduced to model the deterioration process. The model is used for helping road pavement maintenance policy. The road segments are divided into smaller groups. The characterisation of these groups is given by the pavement type, the traffic volume and the type of the maintenance policy. A Markov matrix belongs to every type of road surfaces, class of traffic volume and type of maintenance policy and it determines the deterioration process. The presented model and methodology are used to determine the optimal rehabilitation and maintenance policy of the Hungarian national road network for ten years.

The objective function of the model is a weighted sum of the cost of rehabilitation and the user cost. A computer program on workstation and on microcomputer has been developed which solves the large scale linear programming problem. The model has been used by the Ministry of Transport who is responsible for the 30.000 km road network of Hungary.

Keywords: economic analysis, reliability engineering, Laplace transform.

1. Introduction

To maintain the condition of a road network is one of the most important problems of the national economy. The current budget condition in Hungary needs effective economical politics in every possible field. The limited budget needs a much more effective decision supporting system to maintain the road network as good as possible.

The work to build up such a model began some years ago. First a large scale Road Data Bank was developed (see BAKÓ et al. [1]). The second step was to build up an optimization model. This model and the related programming system was developed 3 years ago (see BAKÓ et al.

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[2], GÁSPÁR et al. [5]). The developed system was intensively used by the Hungarian Ministry of Transport to solve the following problems:

- ensure a prescribed improvement in the state of the road system with minimal agency cost;
- distribute a certain amount of money between the road sections with different states in a way that the achieved improvements should be the best in some sense (e. g. to minimise the user cost on the total transportation network).

The first version of the model solves the problems mentioned above in one time period. This period could be one year or several years but only one period. This fact limits the application possibilities of the model. That is why a multiperiod model was suggested where the user cost and the cost/benefit analysis could be more effectively taken into consideration.

Several types of solution algorithms can be used depending on the given problem, the available data, the budget constraints etc. (see FEIGHAN et al. [4]). Two main types are the heuristics and the optimization. The heuristic techniques generally use a ranking algorithm and have limited applications. The optimization models depending on the problem to be solved integer (CHESTER–HARRISON [3]), linear (WANG et al. [8]) or dynamic programming algorithm (MARKOW [6]).

The linear programming model presented herein has some stochastic elements. Namely the road deterioration process is described by Markov transition probability matrices (see PREKOPA [7]).

2. Elements of the Optimization Model

The Hungarian road network has been divided into different groups according to their pavement type and traffic class. In our model the following denotation will be used

- i pavement type index
- s number of the pavement types
- j traffic class index
- f number of the traffic classes
- k maintenance policy index
- t number of the maintenance policies
- l year index
- T the number of the years

The condition of a road segment is described by different types of deterioration parameters. The developed model uses 3 parameters : bearing capacity note (5 classes), longitudinal unevenness note (5 classes) and pavement surface quality note (5 classes). Note 1 denotes the best, and note 5 denotes the worst condition. The segments using these notes could have 125 different conditions of state. This number determines the size of the unknown variable vector X_{ijkl}.

One co-ordinate of X_{ijkl} is the fraction of the road segments which belongs to a pavement condition state in the case of pavement type i, to the traffic class j, to the maintenance policy k, in the year l.

Let us denote the Markov transition probability matrix by Q_{ijk} which belongs to the pavement type i, traffic class j and maintenance policy k.

The matrix Q_{ijk} is quadratic, and the number of rows is equal to the number of the road condition states. The element q_{ijkmn} of Q_{ijk} means the probability that the road segment being in state m at the beginning of the planning period will be in state n at the end of the planning period.

Let us denote the unknown vector by Y_{ijl} which is the fraction of the road segments belonging to the pavement type i, to the traffic class j after the planning period l.

The initial fraction of road segment is denoted by b_{ij} which belongs to the pavement type i, to the traffic class j.

There are several conditions to fulfil. The first condition is related to the fraction of the road segment at the initial year:

$$\sum_{k=1}^{t} UX_{ijkl} = b_{ij}, \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, f, \tag{1}$$

where $\mathbf{U}$ is a 125×125 unit matrix.

The second condition defines the vector $\mathbf{Y}_{ijl}$ at the initial year:

$$\sum_{k=1}^{t} Q_{ijk}X_{ijkl} = Y_{ijl}, \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, f, \tag{2}$$

For the consecutive years the following conditions have to be fulfilled:

$$\sum_{i=1}^{s} UX_{ijk(l+1)} - Y_{ijl} = 0, \;\; j = 1, 2, \ldots, f, \;\; k = 1, 2, \ldots, t, \quad l = 1, 2, \ldots, T-1 \tag{3}$$

The conditions (3) define the unknown vectors $\mathbf{Y}_{ijl}$ which contain the fractions at the beginning of the planning period l by the sum of $X_{ijk(l+1)}$ which contains the fractions at the end of the planning period $(l-1)$.

One of the maintenance policies has to be applied on every road segment in each year:

$$\sum_{v=1}^{125}\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{v=1}^{t}(X_{ijkl})_v = 1, \quad l = 1, 2, \ldots, T. \tag{4}$$

The segments are divided into 3 groups: acceptable (good), unacceptable (bad) and the rest. Let us denote the three sets by J (good), R (bad) and by E (rest of the segments) and by H the whole set of segments. The relations for these sets are given by:

$$J \cap R = \emptyset \qquad J \cap E = \emptyset$$
$$R \cap E = \emptyset \qquad J \cup R \cup E = H \tag{5}$$

The following conditions are related to these sets at the initial year:

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \geq \alpha_1 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{J},$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \leq \alpha_2 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{R}, \tag{6}$$

$$(b_E)_v \leq \sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \leq (\bar{b}_E)_v, \quad v \in \mathbb{E},$$

where $\mathbb{J}$,$\mathbb{R}$,$\mathbb{E}$ are given above, and

- $\sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{J}$ the share of the good road segments before the planning period,
- $\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v, \ v \in \mathbb{J}$ the actual share of the good road segments after the first year,
- $\sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{R}$ the share of the bad road segments before the planning period,
- $\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \ v \in \mathbb{R}$ the share of the bad road segments after the first year,
- $\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \ v \in \mathbb{E}$ the share of the other road segment group after first year,
- $b_E \in$ the lower bound vector of the other road segment group,
- $b_E \in$ the upper bound vector of the other road segment group,
- α_1 and α_2 given constants.

The meaning of the first condition is that the amount of 'good' road segment after the first year has to be greater than or equal to a given value, in this case the actual share of the good road segments before the first year. The second relation does not allow a higher share of 'bad' roads after the first year than a specified value, the actual share of the bad road segments before the planning period. The third relation gives upper and lower limits to the amount of the rest of the road after the first year.

For the further years similar inequalities could be used

$$\sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ijl} \mathbb{R} \sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ij(l+1)} \quad l = 1, 2, \ldots, T-1 \tag{7}$$

where $\mathbb{R}$ could be one of the relations $<, >, =, <=, >=$ and these relations could be given in connection with each condition states (e.g each row could have different relations).

Instead of (6) and (7) a condition state could be applied for the end of the planning period (e.g for $l = T$):

$$\begin{aligned}
&\sum_{i=1}^{s}\sum_{j=1}^{f}(Y_{ijT})_v \geq \alpha_1 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, \quad v \in \mathbb{J}, \\
&\sum_{i=1}^{s}\sum_{j=1}^{f}(Y_{ijT})_v \leq \alpha_2 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, \quad v \in \mathbb{R}, \\
&(\underline{b}_E)_v \leq \sum_{i=1}^{s}\sum_{j=1}^{f}(Y_{ijT})_v \leq (\bar{b}_E)_v, \quad v \in E
\end{aligned} \tag{8}$$

Let us denote by C_{ijk} the unit cost vector of the maintenance policy k on the pavement type i and traffic volume j.

One more condition is in connection with the yearly budget bound of each maintenance action:

$$\sum_{i=1}^{s}\sum_{j=1}^{f} r^{(l-1)} C_{ijk} X_{ijkl} = r^{(l-1)} M_k, \quad l = 1, 2, \ldots, T, \quad k = 1, 2, \ldots, t, \tag{9}$$

where r is the interest rate, C_{ijk} is the unit cost vector of the maintenance policy k on the pavement type i and traffic volume j and M_k is the budget bound available for maintenance policy k in the initial year.

Now the objective of the problem is formalized. The objective is to minimise the total cost of maintenance

$$C = \sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}\sum_{l=1}^{T} X_{ijkl} C_{ijk} \rightarrow \min! \tag{10}$$

If the available budget B is known two further budget limitation conditions are added to the constraints.

The budget limitation condition for the initial year is:

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} X_{ijkl}C_{ijk} \leq B. \tag{11}$$

For the years $l = 2, 3, \dots, T$ this condition is the following:

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} r^{(l-1)} X_{ijkl}C_{ijk} \leq r^{(l-1)}B. \tag{12}$$

The cost of travelling depends on the pavement type i and the traffic class of j. Let us denote the travelling cost vector by K_{ij}. The v-th co-ordinate of this vector belongs to the condition state $(l \leq v \leq 75)$.

The objective in this case is to minimise the total user (travelling) cost :

$$C = \sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}\sum_{l=1}^{T} X_{ijkl}K_{ij} \rightarrow \text{min!} \tag{13}$$

3. Variations of the Model

The model can be used to solve different problems depending on the given aim. Two main types usually formalise the necessary fund model and the budget bound one.

The first model is used to determine a solution which fulfills the conditions given above and the total rehabilitation cost is minimised.

In this case we have to find vectors $\mathbf{X}_{ijkl}$ and $\mathbf{Y}_{ijk}$ which fulfil the following conditions:

$$\sum_{k=1}^{t} UX_{ijkl} = b_{ij}, \quad i = 1, 2, \dots, s, \quad j = 1, 2, \dots, f$$

$$\sum_{k=1}^{t} (Q_{ijk}X_{ijkl}) = Y_{ijl}, \quad i = 1, 2, \dots, s, \quad j = 1, 2, \dots, f$$

$$\sum_{i=1}^{s} UX_{ijk(l+1)} - Y_{ijl} = 0, \quad j = 1, 2, \dots, f, \quad k = 1, 2, \dots, t, \quad l = 1, 2, \dots, T$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} X_{ijkl} = 1, \quad l = 1, 2, \dots, T \tag{14}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijkl})_v \geq \alpha_1 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{J},$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijk1})_v \leq \alpha_2 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ijv})_v, v \in \mathbb{R},$$

$$(b_E)_v \leq \sum_{i=1}^{s}\sum_{j=1}^{f}(Y_{ijT})_v \leq (b_E)_v, \quad v \in \mathbb{E},$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ijl} \mathbb{R} \sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ij(l+1)} \quad l = 1, 2, \dots, T-1,$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f} r^{(l-1)} C_{ijk} X_{ijkl} = r^{(l-1)} M_k, \quad l = 1, 2, \dots, T, \quad k = 1, 2, \dots, t$$

and minimizes the objective

$$C = \sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}\sum_{l=1}^{T} X_{ijkl} C_{ijk} \rightarrow \min! \tag{15}$$

The budget bound model contains some further conditions ((11),(12)) and the objective in this case is to minimise the total travelling cost.

The budget bound model is

$$\sum_{k=1}^{t} U X_{ijkl} = b_{ij}, \quad i = 1, 2, \dots, s, \quad j = 1, 2, \dots, f,$$

$$\sum_{k=1}^{t} (Q_{ijk} X_{ijkl}) = Y_{ijl}, \quad i = 1, 2, \dots, s, \quad j = 1, 2, \dots, f,$$

$$\sum_{i=1}^{s} U X_{ijk(l+1)} - Y_{ijl} = 0, \quad j = 1, 2, \dots, f, \quad k = 1, 2, \dots, t, \quad l = 1, 2, \dots, T-1,$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} X_{ijkl} = 1, \quad l = 1, 2, \dots, T, \tag{16}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}(Q_{ijk}X_{ijkl})_v \geq \alpha_1 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbb{J},$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} Q_{ijk}X_{ijk1})_v \leq \alpha_2 \sum_{i=1}^{s}\sum_{j=1}^{f}(b_{ij})_v, v \in \mathbf{R},$$

$$(\underline{b}_E)_v \leq \sum_{i=1}^{s}\sum_{j=1}^{f}(Y_{ijT})_v \leq (b_E)_v, \quad v \in \mathbf{E},$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ijl}\mathbf{R}\sum_{i=1}^{s}\sum_{j=1}^{f} Y_{ij(l+1)} \quad l = 1, 2, \ldots, T-1.$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f} r^{(l-1)}C_{ijk}X_{ijkl} = r^{(l-1)}M_k, \quad l = 1, 2, \ldots, T, \quad k = 1, 2, \ldots, t$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} X_{ijkl}C_{ijk} \leq B$$

$$\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t} r^{(l-1)}X_{ijkl}C_{ijk} \leq r^{(l-1)}B$$

and minimizes the objective:

$$C = \sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}\sum_{l=1}^{T} X_{ijkl}K_{ijk} \rightarrow \min! \tag{17}$$

Another variation of the model contains both. The conditions given in (16). and the objective function of the problem is the weighted sum of the total rehabilitation and the total travelling cost:

$$\delta\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{k=1}^{t}\sum_{l=1}^{T} X_{ijkl}C_{ijk} + \lambda\sum_{i=1}^{s}\sum_{j=1}^{f}\sum_{l=1}^{T} Y_{ijl}K_{ij} \rightarrow \min! \tag{18}$$

The above mentioned model could be used for solving different tasks depending on the aim. If $\delta = 0$, then the model serves the necessary funds which are needed for ensuring the a given condition level of roads with minimal rehabilitation cost. If $\lambda = 0$ this model is the budget bound model.

4. Application

The models have been applied for solving the Hungarian pavement management system in the case of the multiperiod model. 4 pavement types were applied:

asphalt concrete (i=1)
asphalt macadam (i=2)
turnpike surface No1 (i=3)
turnpike surface No2 (i=4)

The traffic categories were the following:
In the case asphalt concrete and asphalt macadam:

$$0 \leq ADT \leq 3000$$
$$3001 \leq ADT \leq 8000$$
$$8001 \leq ADT$$

and in the case of turnpike

$$0 \leq ADT \leq 8000$$
$$8001 \leq ADt$$

where ADT is the average daily traffic. The pavement condition state was reduced to 41. In this case the size of the Markov matrices 42×41 and the unknown variables vector $\mathbb{X}_{ijkl}$ and $\mathbb{Y}_{ijl}$ is 41.

Some maintenance policy could not be used in some surface type and traffic categories.

After these reductions the number of unknown variables at one period is 1558, and in the case of our model 15580. The number of conditions depends on the applied model. The approximate number of the rows (conditions) at the above mentioned models is 8000.

Different types of inner point and simplex algorithms were tested to solve the 8000×15000 LP problem. Finally we found the best a simplex algorithm which was developed here in Hungary.

The Road Data Bank is handled in an IBM 80486 PC. The computer code to solve model runs both on a workstation and on an above mentioned PC. If no initial solution is known a workstation is used to solve the problem. In other case we could handle the problem on 486 type PC.

Notation

b_{ij}	share of road belongs to i, j
B	maintenance budget
C_{ijk}	maintenance cost
d_i	share of road surface type i
E	set of the 'other' road segments
f	traffic volume class number

H	set of road segments
i	pavement type index
j	traffic class index
J	set of 'good' road segments
k	maintenance policy index
K_{ij}	travelling cost
M_k	budget bound for maintenance policy k
Q_{ijk}	transition probability matrix belongs to i, j, k
r	interest rate
R	set of 'bad' road segments
s	pavement type number
t	maintenance policy number
U	unit matrix
X_{ijkl}	unknown variable belongs to i, j, k before the planning period l
Y_{ijl}	unknown variable belongs to i, j after the planning period l

References

1. BAKÓ, A. – GYULAI, L. – ERBEN, P.: Structure of the Road Data Bank, *Proceedings of the Pavement Management System*, Budapest, (1989) pp. 43–47.
2. BAKÓ, A. – GÁSPÁR L. – KLAFSZKY, E. – SZÁNTAY, T.: Optimisation Techniques for Planning Highway Pavement Improvements, *Annales of Operations Research*, Vol. 58 (1995) pp. 55–66.
3. CHESTER, A. – HARRISON, R.: Highway Design and Maintenance Standard Model. V.1, World Bank 1987, p. 180.
4. FEIGHAN, K. J. – SHAHIN, M. Y. – SINHA, K. C. – WHITE, T. D.: An Application of Dynamic Programming and Other Mathematical Techniques to Pavement Management Systems, Transportation Research Record 1200, 1988.
5. GÁSPÁR, L. – BAKÓ, A.: Compilation of the Hungarian Network-Level Pavement Management System, *Revue Generale des Routes et des Aerodromes* Vol. 170, (1993), pp. 34–37.
6. MARKOW, M. J. – BRADEMEYER, B. D. – SHERWOOD, G. M. - KENIS, W. J.: The Economic Optimization of Pavement and Maintenance and Rehabilitation Policy S.N.A.C. on Managing Pavement, 1988, pp. 2.169–2.182.
7. PREKOPA, A.: Valószínűségszámítás, (Probability Theory), Műszaki Könyvkiadó, (1972) p. 440 (in Hungarian).
8. WANG, K. C. P. – ZANIEWSKI, J. – WAY, G. – DELTON, J.: Revision for the Arizona Department of Transportation Pavement Management System, *Transportation Research Board 72th Annual Meeting*, Jan. 10–14, 1993 P. N. 1023, pp. 1–17.