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AN ANALYSIS OF RANK PRESERVATION AND REVERSAL IN THE ANALYTIC HIERARCHY PROCESS[1]

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Abstract

The Analytic Hierarchy Process (AHP) is an extensively used tool for setting up the rank order of the alternatives in multiple criteria decision making problems. AHP has been widely applied in practice with inconsistent paired comparison matrices. In this case, however, the rank order of the alternatives does not remain stable, a rank reversal may occur. In this paper it is demonstrated that this phenomenon is inherent in AHP even if the matrix is slightly inconsistent only. Conditions and regions of such a rank reversal are also given.

Keywords: decision analysis, multiple criteria, ratio estimation.

1. Introduction

Indigenous to many real world systems is the problem of choosing the best alternative from a set of competing alternatives under conflicting criteria. The Analytic Hierarchy Process (AHP) is a multicriteria decision making method that represents the decision problem in a hierarchical network structure. AHP develops priorities for the alternatives based on the decision maker's judgment throughout the system by utilizing pairwise ratio estimates as entries of its paired comparison matrix and then determining the relative dominance (rank order) of the alternatives on a ratio scale.

Ever since the development of the AHP in the late 1970's (SAATY, 1977), a great number of criticisms of this approach have appeared in the

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literature. One of the more controversial aspects of AHP is the phenomenon of rank reversal. It has been shown that rank reversal in AHP may occur due to (i) if the pairwise comparison matrix is inconsistent, and (ii) when synthesizing the local ratio-scale priority weights (the normalized principal eigenvector components) into global weights by means of an additive function even if the matrix is perfectly consistent. This paper presents the analysis of rank preservation and reversal in AHP in relation to case (i). Case (ii) is *not* examined in this study. The interested reader may find a discussion of this issue e.g., in DYER and WENDELL (1985), SALO and HÄMÄLÄINEN (1992) and SCHONER, WEDLEY and CHOO (1993).

The occurrence of such a rank reversal might be serious in practice when a wrong alternative is chosen by the decision maker as the best. Some simple examples have made it clear that the introduction of a new alternative may reverse the rank order of the old alternatives if it is a replica (copy) of any of the old alternatives (BELTON and GEAR, 1983), or even if it is not a replica, but if it differs entirely from the old alternatives (DYER and WENDELL, 1985). Both proponents and opponents of AHP agree that these types of rank reversal may occur, but disagree on the legitimacy of them. This problem has been considered by numerous authors and a persistent debate has followed; see WATSON and FREELING (1983), SAATY and VARGAS (1984), BELTON and GEAR (1985), VARGAS (1985), HARKER and VARGAS (1987), SAATY (1987), SCHONER and WEDLEY (1989), DYER (1990), SAATY (1990) and HARKER and VARGAS (1990).

The major goal of this paper is to demonstrate that AHP is not an adequate tool to handle multicriteria decision making problems as long as the expert judgments contain even the slightest degree of inconsistency. We show that the eigenvalue approach used in AHP produces a perfect ranking if all the ratio estimates are consistent, but that the method cannot give the true ranking of the alternatives if these estimates are inconsistent. Conclusions are drawn on the simplest case where the paired comparison matrix of the inconsistent ratio estimates differs from that of the *consistent* ratio estimates in only a single pair of elements. Thus, a slight deviation from perfect consistency is introduced and is characterized by one perturbed pair of elements. If rank reversal occurs in this case, then, obviously this result also holds for matrices with an arbitrary number of perturbed pairs of elements. We consider a single criterion only.

Since decision makers almost always supply inconsistent ratio estimates the resulting components of the principal eigenvector may produce a biased ranking of the alternatives. By performing a comprehensive analysis we present conditions for the preservation and reversal of the rank order of the alternatives for different cases. To provide these conditions of a possible rank reversal it is required that the components of the principal eigenvector be given in explicit form. They have been developed in FARKAS and RÓZSA (1996) where a formal study of the solution to the algebraic eigenvalue problem of the paired comparison matrices is presented.

2. Overview of Analytic Hierarchy Process

This section reviews the basic definitions of AHP that are relevant to our subject. Basic notions and axiomatic foundation of AHP have been developed by SAATY (1986). He defined the positive square matrix $\mathbb{A} \in R_{M(n)}$ representing the paired comparisons of the *alternatives* with respect to a *criterion* C. The elements of the *paired comparison matrix* $\mathbb{A}=(a_{ij})$, correspond to the *strength of preference* values of alternative A_i over alternative A_j with respect to criterion C for each possible pair of the alternatives. The strength of preference values of the alternatives are inversely related (reciprocal condition).

DEFINITION 1: The positive matrix $\mathbb{A} \in R_{M(n)}$ is called a *reciprocal matrix* if its elements a_{ij} satisfy the relation

$$a_{ij} = \frac{1}{a_{ji}}. \tag{1}$$

DEFINITION 2: The positive matrix $\mathbb{A} \in R_{M(n)}$ is called a *consistent matrix* if its elements satisfy the relation

$$a_{ij} a_{jk} = a_{ik} \, \forall \, i, j, k. \tag{2}$$

Hereafter we recall (2) as the general consistency condition. If it does not hold for any triad that may be composed from elements of $\mathbb{A}$, then the positive matrix $\mathbb{A} \in R_{M(n)}$ is called an *inconsistent matrix*.

The ultimate goal of AHP is to derive the relative dominance and the rank order of the given set of alternatives. The relative dominance of the alternatives gives the overall priority of an alternative over the other alternatives with respect to a given set of criteria. We define this term in the following way.

DEFINITION 3: A positive real number representing a proportion of the total priority of the decision maker that is allocated to the ith alternative is called *relative dominance (relative standing)* of the ith alternative over the other alternatives.

DEFINITION 4: The preference order of the alternatives given by the decision maker is called *rank order* of the alternatives.

A rank order of the alternatives is transitive if

$$\mathrm{A}_i > \mathrm{A}_j \ \text{ and } \mathrm{A}_j > \mathrm{A}_k, \text{ imply } \mathrm{A}_i > \mathrm{A}_k \ \forall i, j, k. \tag{3}$$

The relative dominance of the alternatives is interpreted on a ratio scale, whereas the rank order of the alternatives is usually given on an ordinal scale. Thus, the rank order of the alternatives is automatically given if their relative dominance can be determined.

SAATY (1986, p. 848) proved that the relative dominance of the ith alternative is the ith component of the normalized principal right eigenvector of $\mathbf{A}$, if $\mathbf{A}$ is a consistent matrix. In addition, he presented a proof that this result also holds for a reciprocal matrix which is not necessarily consistent (SAATY, 1986, p. 853).

The question we raise at this point is whether the components of the principal eigenvector produce the true relative dominance, and hence the true rank order of the alternatives when, in fact, $\mathbf{A}$ is *not* a consistent matrix. We will investigate whether the rank order of the alternatives obtained for the consistent case is invariant to a slight perturbation in a_{ij} of a consistent matrix or not. This perturbation is assumed to be a continuous function of one parameter.

In the next sections we study the behaviour of the components of the principal eigenvector of the paired comparison matrix. We first define the specific case where the paired comparison matrix is perfectly consistent. Then, we discuss the simple perturbed case where one pair of elements is 'spoiled' and therefore, the matrix becomes inconsistent. Finally, we introduce the extended perturbed case where the matrix is augmented by a supplementary new column and row. The characteristic equations of these matrices will be given. The analysis of the rank reversal problem will be performed by comparing the corresponding elements of the principal eigenvectors for the three cases under consideration.

3. Paired Comparison Matrices of Specific Form

DEFINITION 5: If the positive matrix $\mathbf{A}_S \in R_{M(n)}$ is reciprocal and consistent, it will be called a *specific paired comparison matrix*.

According to *Definition 2*, any consistent paired comparison matrix can be expressed as the product of a (column) vector $\mathbf{u}$ and a (row) vector $\mathbf{v}^{\mathrm{T}}$:

$$\mathbf{A}_S = \mathbf{u}\mathbf{v}^{\mathrm{T}}, \tag{4}$$

where

$$\mathbf{v}^{\mathrm{T}} = [1, x_1, x_2, \ldots, x_{n-1}], \quad x_i > 0 \quad (i = 1, 2, \ldots, n-1)$$

and

$$\mathbf{u} = \left[1, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{n-1}}\right]^{\mathrm{T}}. \tag{5}$$

Introducing the diagonal matrix

$$\mathbb{D}^{-1} = \mathrm{diag} < 1, x_1, x_2, \ldots, x_{n-1} > \tag{6}$$

and the vector $\mathrm{e}^{\mathrm{T}} = [1, 1, \ldots, 1]$, obviously

$$\mathbb{D}^{-1}\mathbb{A}_S\mathbb{D} = \mathrm{ee}^{\mathrm{T}}, \tag{7}$$

and the characteristic polynomial of $\mathbb{A}_S$ is given by

$$\det[\lambda\mathbb{I} - \mathbb{A}_S] = \det[\lambda\mathbb{I} - \mathrm{ee}^{\mathrm{T}}] = \lambda^{n-1}(\lambda - n). \tag{8}$$

That means $\mathbb{A}_S$ has zero eigenvalue with multiplicity $n-1$ and a single positive eigenvalue, $\lambda = n$. The corresponding right and left eigenvectors of $\mathbb{A}_S$ are $\mathbf{u}$ and $\mathbf{v}^{\mathrm{T}}$, respectively.

In this context we examine matrices with positive elements only. These matrices have a positive real eigenvalue of maximal modulus which is a simple root of the characteristic equation and all elements of the corresponding right and left eigenvectors are positive. Hereafter we will refer to them as *maximal eigenvalue* and *principal eigenvectors* (right and left) of the positive matrix.

4. Paired Comparison Matrices of General Form

DEFINITION 6: If the positive matrix $\mathbb{A} \in R_{M(n)}$ is reciprocal, but it is not consistent, it will be called a *general paired comparison matrix.*

Two subcases will be distinguished: the *simple perturbed case* and the *extended perturbed case.*

4.1 Simple Perturbed Case

In this paper the simplest case of general paired comparison matrices will be considered only when a single pair of elements of a specific paired comparison matrix is 'spoiled'.

DEFINITION 7: If one pair of elements, say a_{12} and a_{21} of a 'spoiled' specific paired comparison matrix has the form $a_{12} = x_1 + \epsilon, \quad a_{21} = \frac{1}{x_1+\epsilon}$, then it will be called a *simple perturbed paired comparison matrix* depending on one parameter, ϵ.

If the components of the principal eigenvector of a simple perturbed paired comparison matrix in the function of ϵ are developed, then the rank order of the alternatives of the *specific* paired comparison matrix can be compared with that of the *simple perturbed* paired comparison matrix.

In FARKAS and RÓZSA (1996) it is proven that the characteristic polynomial $p_n(\lambda)$ of $\mathbb{A}$ is

$$p_n(\lambda) \equiv \det[\lambda\mathbb{I} - \mathbb{A}] = \lambda^{n-3}[\lambda^3 - n\lambda^2 - (n-2)Q], \tag{9}$$

where

$$Q = \frac{(\frac{\epsilon}{x_1})^2}{1 + \frac{\epsilon}{x_1}}. \tag{10}$$

It should be noted that

$$\frac{\epsilon}{x_1} > -1 \tag{11}$$

since the elements of $\mathbb{A}$ are (finite) positive numbers. If r is the maximal eigenvalue of $\mathbb{A}$, it can be obtained from the equation [cf.(9)]:

$$r^3 - nr^2 - (n-2)Q = 0, \tag{12}$$

where $r > n$ [see FARKAS and RÓZSA (1996)]. The components of the principal eigenvector can be obtained from the one-rank matrix, $\mathrm{adj}(r\mathbb{I} - \mathbb{A}) = [u_{ij}^S(r)]$, since its columns are proportional to the principal eigenvector.

4.2 Extended Perturbed Case

In this subsection, bordered paired comparison matrices will be considered.

DEFINITION 8: If a simple perturbed paired comparison matrix is bordered by one of its columns and by the corresponding row, it will be called an *extended perturbed paired comparison matrix*.

This case occurs when any of the alternatives (say the kth one) contained in the given decision problem is repeated in the course of the decision process. The repeated alternative is called a *replica* or a *copy*.

Introducing the characteristic polynomial

$$P_{n+1}(\lambda, k) \equiv \det \begin{bmatrix} \lambda\mathbb{I}_n - \mathbb{A} & -\lambda \mathrm{e}_k^{(n)} \\ -\lambda \mathrm{e}_k^{(n)\mathrm{T}} & 2\lambda \end{bmatrix}, \tag{13}$$

where $\mathbf{e}_k^{(n)\mathrm{T}} = [\overset{1}{\breve{0}}, \ldots, 0, \overset{k}{\breve{1}}, 0, \ldots, \overset{n}{\breve{0}}]$, the characteristic equation for the extended perturbed paired comparison matrix, $\mathbf{A}_E$, can be written as

$$P_{n+1}(\lambda, k) = 0. \tag{14}$$

Let the maximal eigenvalue of $\mathbf{A}_E$ be denoted by $\lambda_{\max}(k)$. In FARKAS and RÓZSA (1996) it is shown that (14) leads to the equations

$$\lambda_{\max}^3(j) - (n+1)\lambda_{\max}^2(j) - 2(n-2)Q = 0, \quad \text{if} \quad j = 1, 2, \tag{15}$$

and

$$\lambda_{\max}^3(k) - (n+1)\lambda_{\max}^2(k) - (n-1)Q = 0, \quad \text{if} \quad k = 3, 4, \ldots, n, \tag{16}$$

where Q is given in (10), and $\lambda_{\max}(q) > n+1$ and $\lambda_{\max}(q) > r$ [see in FARKAS and RÓZSA (1996)].

The elements of the principal eigenvectors are denoted by

$$\mathbf{u}_q^E(\lambda_{\max}(q)) = [\mathbf{u}_{iq}^E\Big(\lambda_{\max}(q)\Big)], \qquad q = 1, \ldots, n.$$

In the next section we will consider the case when the first column of the simple perturbed paired comparison matrix is repeated.

5. The Issue of Rank Reversal

The concept of rank reversal is introduced in the following way. Suppose that for two consecutive elements, u_i and u_{i+1} of the principal eigenvector of a *specific* paired comparison matrix

$$u_i < u_{i+1} \tag{17}$$

holds. Further, suppose that for the corresponding two elements, $u_{ij}^S(r)$ and $u_{i+1,j}^S(r)$ of the principal eigenvector of a *simple perturbed* paired comparison matrix,

$$u_{ij}^S(r) > u_{i+1,j}^S(r) \tag{18}$$

holds (for any j). In this case we say that the rank order of the alternatives i and $(i+1)$ has been reversed due to the perturbation. The reversal of the rank order between the ith and $(i+1)$th alternatives can be defined in a similar way for the other cases.

In this section we will investigate whether such a rank reversal might occur at all

i) for the specific versus the simple perturbed case, and
ii) for the specific versus the extended perturbed case, respectively.

By comparing the principal eigenvectors for the specific, for the simple perturbed, and for the extended perturbed paired comparison matrices $\mathbb{A}_S, \mathbb{A}$ and $\mathbb{A}_E$, respectively, it turns out that their elements are proportional to each other for the elements with indices $3 \leq j \leq n$. Consequently, no rank reversal can occur between any pair of these alternatives regardless of whether the elements a_{12} and a_{21} of the specific paired comparison matrix are perturbed or not, nor which of the columns k is repeated. For all other alternatives, however, the occurrence of a rank reversal can not be precluded. In subsections *5.1* and *5.2* the conditions of rank reversal between the first two alternatives will be determined. The location of any rank reversal will be given in the function of $\frac{\epsilon}{x_1}$ and r. The analysis of the rank reversal issue will be presented in subsection 5.3.

According to the theory developed by Saaty the rank order (order of magnitude) of the components of the principal eigenvectors determine the rank order of the corresponding alternatives. Therefore, rank reversal *must not occur* either in the specific versus the simple perturbed case or in the specific versus the extended perturbed case. Based on the results shown in subsections *5.1* and *5.2*, we present a detailed analysis of the rank reversal issue in *5.3* where it becomes clear that the theory of relative dominance given by Saaty contradicts the facts emerging in the decision making processes.

5.1 Rank Reversal: Specific Case versus Simple Perturbed Case

Consider the case when $x_1 > 1$. We know that $r > n$ holds. The first two elements of the principal eigenvectors for the specific and the simple perturbed cases, denoted by A and B are given [see FARKAS and RÓZSA (1996)]:

	Specific case:	Simple perturbed case:
A:	1	$A^S : r-(n-1)$
B:	$\frac{1}{x_1}$	$B^S : \frac{1}{x_1}\left\{\frac{1}{1+\frac{\epsilon}{x_1}} + \frac{n-2}{r}\frac{\frac{\epsilon}{x_1}}{1+\frac{\epsilon}{x_1}}\right\}.$

It is apparent that the rank order is A>B. Rank reversal occurs if $A^S < B^S$, i.e., if

$$r[r-(n-1)] < \frac{1}{x_1(1+\frac{\epsilon}{x_1})}[r+(n-2)\frac{\epsilon}{x_1}]. \tag{19}$$

After rearranging (19) the inequality

$$y(x_1) \equiv [r-(n-1)]x_1^2 - \frac{2(n-2)+r(r-n)[r-(n-2)]}{(n-2)}x_1 + [r-(n-1)] < 0 \tag{20}$$

is obtained. The two zeroes of the polynomial $y(x_1)$, denoted by $x_1^{(L)}$ and $x_1^{(U)}$, are functions of r and n:

$$x_1^{(L)} = f_L(r,n), \qquad x_1^{(U)} = f_U(r,n),$$

and they provide the boundaries of the interval over which (20) is satisfied:

$$f_L(r,n) < x_1^S < f_U(r,n). \tag{21}$$

From (20) we obtain

$$\left.\begin{aligned} f_U(r,n) &= \frac{\alpha+\beta}{\gamma} \\ f_L(r,n) &= \frac{\alpha-\beta}{\gamma} \end{aligned}\right\}, \tag{22}$$

where

$$\begin{aligned} \alpha &= 2(n-2)+r(r-n)[r-(n-2)], \\ \beta &= [r-(n-2)]\sqrt{(r-n)\{r^2(r-n)+4(n-2)\}}, \\ \gamma &= 2(n-2)[r-(n-1)]. \end{aligned}$$

5.2 Rank Reversal: Specific Case versus Extended Perturbed Case

Assume that $x_1 > 1$. We know that $\lambda_{\max}(q) > n+1$ and $\lambda_{\max}(q) > r$ hold for $q = 1,2,\ldots,n$. In order to simplify notation, let $\lambda_{\max}(j) = \lambda$ $(j = 1,2)$. The first two elements of the corresponding principal eigenvectors for the specific and the extended perturbed cases are given [see FARKAS and RÓZSA (1996)]:

	Specific case:	Extended perturbed case:
A:	1	$A^E : \lambda - (n-1)$
B:	$\frac{1}{x_1}$	$B^E : 2\frac{1}{x_1}\left\{\frac{1}{1+\frac{\epsilon}{x_1}} + \frac{n-2}{\lambda}\frac{\frac{\epsilon}{x_1}}{1+\frac{\epsilon}{x_1}}\right\}.$

Rank reversal occurs if $A^E < B^E$, i.e., if

$$\lambda - [\lambda - (n-1)] < \frac{2}{x_1(1+\frac{\epsilon}{x_1})}[\lambda + (n-2)\frac{\epsilon}{x_1}]. \tag{23}$$

After rearranging (23) the inequality

$$y^E(x_1) \equiv \frac{1}{2}(n-2)[\lambda-(n-1)]x_1^2 - \{2(n-2)+\frac{\lambda}{2}[\lambda-(n-2)][\lambda-(n+1)]\}x_1$$
$$+(\lambda-n)(n-2) < 0 \tag{24}$$

is obtained. The zeroes of the polynomial $y^E(x_1)$, denoted by $\tilde{x}_1^{(L)}$ and $\tilde{x}_1^{(U)}$, are functions of λ and n:

$$\tilde{x}_1^{(L)} = f_L^E(\lambda, n), \qquad \tilde{x}_1^{(U)} = f_U^E(\lambda, n),$$

and they give the boundaries of the interval over which (24) is satisfied:

$$f_L^E(\lambda, n) < x_1^E < f_U^E(\lambda, n). \tag{25}$$

From (24) we obtain

$$\left.\begin{aligned} f_L^E(\lambda, n) &= \frac{\alpha^E + \beta^E}{\gamma^E} \\ f_U^E(\lambda, n) &= \frac{\alpha^E - \beta^E}{\gamma^E} \end{aligned}\right\}, \tag{26}$$

where

$$\alpha^E = 2(n-2) + \frac{\lambda}{2}[\lambda-(n-2)][\lambda-(n+1)],$$
$$\beta^E = [\lambda-(n-2)]\sqrt{[\lambda-(n+1)]\{(\frac{\lambda}{2})^2[\lambda-(n+1)]+2(n-2)\}},$$
$$\gamma^E = (n-2)[\lambda-(n-1)].$$

5.3 Analysis of Rank Reversals

The basic idea in analysing a possible rank reversal is based on the derivation of a direct functional relationship among the maximal eigenvalues of the specific, the simple perturbed, and the extended perturbed matrices, n, r and $\lambda_{\max}(j)$, respectively. For a given ratio $(\frac{\epsilon}{x_1})$, the maximal eigenvalue r of the simple perturbed matrix $\mathbb{A} \in R_{M(n)}$ can be found by solving *Eq.* (12) where Q is given in (10):

$$r^3 - nr^2 - (n-2)\frac{(\frac{\epsilon}{x_1})^2}{1+\frac{\epsilon}{x_1}} = 0. \tag{27}$$

Similarly, for a given ratio $(\dfrac{\epsilon}{x_1})$, the maximal eigenvalue $\lambda_{\max}(j)$, $(j = 1, 2)$ of the extended perturbed matrix $\mathbb{A}_E \in R_{M(n)}$ can be found by solving *Eq.* (15):

$$\lambda^3_{\max}(j) - (n+1)\lambda^2_{\max}(j) - 2(n-2)\frac{\left(\dfrac{\epsilon}{x_1}\right)^2}{1+\frac{\epsilon}{x_1}} = 0. \tag{28}$$

Since a given ratio for $(\frac{\epsilon}{x_1})$ applied to the case of a simple perturbed matrix is identical to that applied to an extended perturbed matrix, this term can be eliminated from *Eqs.* (27) and (28). Thus, we obtain

$$\lambda^3_{\max}(j) - (n+1)\lambda^2_{\max}(j) = 2(r^3 - nr^2). \tag{29}$$

Fig. 1 shows the functional relationship among the maximal eigenvalues n, r and $\lambda_{\max}(j)$, $(j = 1, 2)$ for a paired comparison matrix of order 3. This graphical representation of the problem demonstrates that for any given ratio $(\frac{\epsilon}{x_1})$, the maximal eigenvalues of the simple perturbed and the extended perturbed matrices, r and $\lambda_{\max}(j)$, $(j = 1, 2)$ can easily be found.

Furthermore, using (29), $\lambda_{\max}(j)$ can be considered to be a function of r:

$$\lambda_{\max}(j) = \lambda(r). \tag{30}$$

Substituting (30) into (26) and introducing the functions

$$\left.\begin{aligned} f_L^E(\lambda(r), n) &= g_L(r, n) \\ f_U^E(\lambda(r), n) &= g_U(r, n) \end{aligned}\right\}, \tag{31}$$

the interval (25) can be expressed in function of r as

$$g_L(r, n) < x_1^E < g_U(r, n). \tag{32}$$

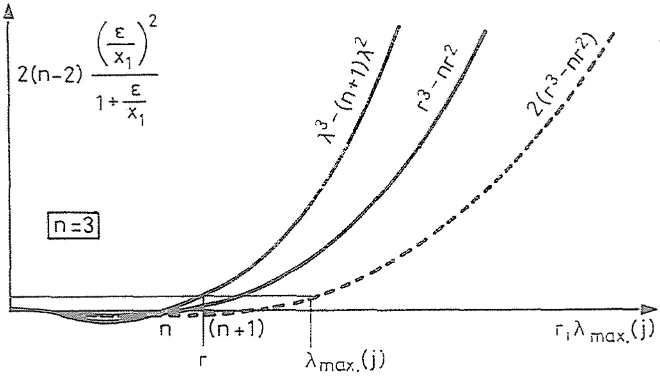

Fig. 1. Relationship among the maximal eigenvalues n, r and $\lambda_{\max}(j)$, $(j = 1, 2)$, for $n = 3$

For feasible values of x_1, the functions $f_U(r,n)$, $f_L(r,n)$ and $g_U(r,n)$, $g_L(r,n)$ are plotted for $n = 3$ in *Fig. 2* [see (21) and (32)]. In order to study the phenomenon of rank reversal, the behaviour of these functions is investigated in the cases

i) if $x_1 > 1$, and
ii) if $x_1 < 1$.

In case i), since $g_U(r,n) > f_U(r,n)$ for $n < r < \infty$, the graph of $g_L(r,n)$ depicted by a dashed line proceeds above the graph of $f_U(r,n)$ depicted by a solid line.

Case ii) is more complicated. It can be shown that there exists a certain value $r_0 > n$ for which $f_L(r_0,n) = g_L(r_0,n)$.

For the other values of r we get

$$g_L(r,n) < f_L(r,n), \quad \text{if} \quad n < r < r_0,$$

and

$$g_L(r,n) > f_L(r,n), \quad \text{if} \quad r_0 < r < \infty,$$

$$r_0 = \frac{K(n-1) + \sqrt{K^2(n-1)^2 + 4K(n-2)}}{2K}, \tag{33}$$

where

$$K = \frac{(n-2)}{\frac{\lambda}{2}[\lambda - (n-1)]}, \tag{34}$$

and λ is defined by (29) as a function of r.

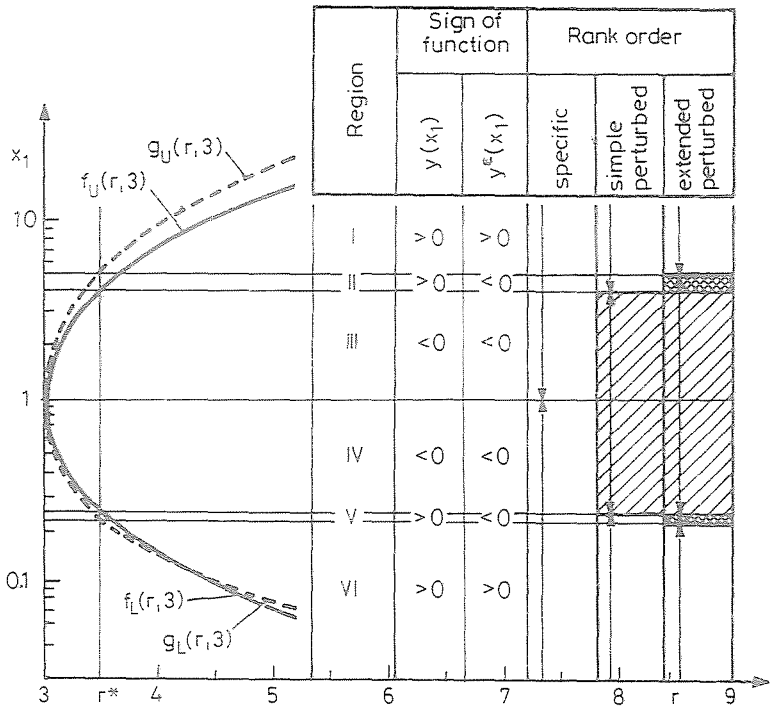

Fig. 2. Characteristic regions of rank reversals

Since the curves of the functions $g_U(r,n)$ and $f_U(r,n)$ plotted in *Fig. 2* have no intersection within the range $1 < x_1 < \infty$, for each value of r, intervals of different lengths can be found in a manner that the sign of the function $g_U(r,n)$ differs from the sign of $f_U(r,n)$. In an analogous manner, intervals of different lengths can also be found within the range $0 < x_1 < 1$ where the signs of the functions $f_L(r,n)$ and $g_L(r,n)$ differ. The only exception occurs at their intersection for $r = r_0$ given in (33) and (34) where the function $g_L(r,n)$ crosses $f_L(r,n)$. These ranges of x_1 contain the set of all possible values of x_1 which might appear in any simple perturbed matrix and in their corresponding extended perturbed matrix, therefore in these intervals rank reversals will always occur. Eventually, the spectral properties of the defined matrices cause these rank reversals.

Given any ratio of $(\frac{\epsilon}{x_1})$, the corresponding maximal eigenvalue of a simple perturbed matrix can be found from *Fig. 1.* Let it be denoted by r^*. By finding the intersections of the plotted functions and a vertical segment drawn over r^*, six characteristic regions (I. through VI.) can be distinguished. From (20) and (24), it is obvious that the quadratic functions $y(x_1)$ for $r = r^*$ and $y^E(x_1)$ for $\lambda_{\max} = \lambda(r^*)$ are negative between their two roots, whereas beyond their roots, they are positive as shown in *Fig. 2.* Rank orders are shown for each region by the arrows indicating their directions. Thus, a downward-sloping arrow represents a descending rank order, while an upward-sloping arrow indicates an ascending rank order. Note that *Fig. 2* refers to the case where a descending rank order is given for the specific matrix, since $x_1 > 1$.

In making use of these symbols, it is easy to find the regions of all possible occurrences of rank reversals between any pairwise combination of the defined cases. Rank reversals occur within a specified interval of the values of x_1 if the arrows are pointing into opposite directions in the associated two columns of *Fig. 2.* Single hatched areas indicate the regions for x_1 where rank reversals of A and B occur between the specific and the general cases, whereas double hatched symbols are used to indicate the rank reversals between the simple perturbed and the extended perturbed cases. Hence, rank reversals occur:

i) within regions III and IV, between the specific and the simple perturbed cases;

ii) within regions II, III, IV and V, between the specific and the extended perturbed cases;

iii) within regions II and V, between the simple perturbed and the extended perturbed cases.

Accordingly, no rank reversals occur between a pairwise combination of the defined cases if neither column is hatched for a particular value of x_1 actually chosen.

Whether the components of the principal eigenvector really correspond to the relative dominance of the alternatives is a matter of utmost importance. In contrast to the rank order of a set of alternatives, the concept of relative dominance has not been uniformly interpreted in the AHP literature. Therefore, everyone is entitled to define this term as they wish. We have chosen *Definition 3* in this paper which is generally acceptable. Nevertheless, whichever definition of the relative dominance is used, the resulting vector (set of the scores) must not conflict with the properties of the ordinal scale. Saaty's definition (see in SAATY, 1987, p. 161) is equivalent to the statement that the components of the principal eigenvector provide the relative dominance of the alternatives.

Through this paper we studied whether the eigenvalue method is a proper tool for finding the true ranking of a set of alternatives or not. We have shown that a continuous change in the perturbation parameter results in the alteration of the order of magnitude of the elements of the principal eigenvector. According to Saaty's definition, this fact would imply the change of the rank order of the alternatives. However, the rank order of the alternatives must be invariant. Thus, we arrived at a contradiction. Consequently, Saaty's definition of the relative dominance of the alternatives being equivalent to the elements of the principal eigenvector *conflicts with reality if the paired comparison matrix is inconsistent.*

6. Conclusions

An analysis of the eigenvalue approach used in AHP was presented. We showed that the key issue resulting in a reversal of the ranking of the alternatives is the inconsistency of the ratio estimates. If the paired comparison matrix is inconsistent, then the introduction of a new alternative preserves inconsistency if either it is a replica or if it differs entirely from the old alternatives. Even if the paired comparison matrix is consistent, the introduction of any type of a new alternative but one alters the consistency of the matrix. This feature of AHP limits its scope and application to real world problems, because in practice the consistency of the paired comparison matrix can never be guaranteed a priori.

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