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RESEARCH ARTICLE

Present value under uncertain asset life: an evaluation of relative error

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Abstract

We evaluate the relative errors attributable to using the expected economic life of an asset to calculate present value instead of the correct approach of calculating the expected present value when the cessation point of the asset is uncertain. We compare the continuous-time case of exponential cash flows and exponentially distributed asset life with its discrete-time analogue of geometric cash flows and geometrically distributed asset life. We find that if the discount rate and growth rate are equal, the error is always zero. If the growth rate exceeds the discount rate, the error can easily be severe and reach 100%. If the discount rate exceeds the growth rate, the error cannot exceed 30%, but such large errors may occur for any expected asset life. In the discrete case, this error limit depends on expected asset life and is somewhat lower for shorter lives. For realistic cases, even a small percentage point difference between the discount rate and the growth rate can lead to considerable errors.

Keywords

present value · uncertain life · relative error

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1 Introduction

In the discounted cash flow (DCF) framework, the worth of a capital asset (henceforth simply asset) is computed by discounting its expected future cash flows at an appropriate rate (discount rate) that expresses the cost of capital. As described in related textbooks (e.g., especially [20, 32, 33] and also [17, 18, 23, 31, 36]), the computation can typically take one of two forms: discounting discrete cash flows (i.e., cash flows occurring at equally spaced time periods, typically years) at a discrete discount rate (i.e., the effective rate for one period) or discounting a continuous cash flow stream (i.e., cash flow is a continuous function of time) at a continuous discount rate (i.e., the rate assuming an infinitesimal period of capitalization). Mathematically, the former approach is defined by Eq. (1), and the latter is defined by Eq. (2). We assume, as usual, an identical discount rate across periods.

$$P_d = \sum_{n=1}^{N} F_n \left(1 + i\right)^{-n} \tag{1}$$

$$P_c = \int_0^T F\left(t\right) \mathrm{e}^{-rt}\, \mathrm{d}t \tag{2}$$

where P is the present value; the indices d and c refer to the given approach, that is, “discrete” and “continuous”, respectively; n is the index of periods in the discrete case; N (an integer) is the life of the asset expressed in periods; F_n is cash flow at the end of period n; T (a real number) is the life of the asset expressed in periods; t is time; $F(t)$ is the continuous cash flow function; i is a discrete discount rate; r is a continuous discount rate; and e is the base of the natural logarithm.

The discrete and continuous discount rates are related to one another as

$$i = \mathrm{e}^r - 1\text{, or equivalently } r = \ln(1 + i) \tag{3}$$

We note that we use the term “asset”, generally defined as any series of cash flows; therefore, we consider only present value. In this general interpretation, cash flows of a capital budgeting project, for example, are not distinguished as usually done (e.g., initial investment, net return or revenues/disbursements, salvage value – see, e.g., in [37, 41]), and consequently net present value is – as it is conceptually – just a specific present value. For convenience, we start the indexing in Eq. (1) from 1, although our findings in this paper are not affected by starting from 0, as is done in some textbooks.

In Eqs. (1) and (2), essentially all variables, such as the cash flows, discount rates and asset lives may be uncertain; that is, they may be random variables. Therefore, valuation and investment decisions should be based on the *expected* present value (see, e.g., [32, 37] and references therein). Although textbooks usually recognize the stochastic nature of these variables, for computational simplicity, they substitute the expected values of the random variables into the present value formula instead of taking the expectation of the whole expression. This is an obvious source of computational error, leading to biased present value and, possibly, to incorrect investment decision.

In this paper, we are concerned with error attributable to the uncertainty of the economic life of an asset, N or T (henceforth usually simply "life") – that is, the moment in time when the series of cash flows terminates. For a more tractable analysis, we assume non-stochastic discount rates and that cash flows are stochastically independent of asset life (for a numerical method of dealing with correlation between cash flows and life in discrete time, see [38]). Based on this latter assumption, for notational parsimony, we omit the expectations operator for cash flows, so that F itself will henceforth denote expected cash flow. Under our assumptions, the usual but incorrect textbook computations (henceforth referred to as "conventional") are as follows (e.g., [32, 33]):

$$\hat{P}_d = \sum_{n=1}^{E(N)} F_n (1+i)^{-n} \tag{4}$$

$$\hat{P}_c = \int_0^{E(T)} F(t)\, \mathrm{e}^{-rt}\, \mathrm{d}t \tag{5}$$

where $E(.)$ is the expectations operator, and a hat indicates that a present value is approximate.

In contrast, the correct approach is to calculate $E(P)$, on which investment decisions should be based. The difference between $E(P)$ and $\hat{P}$ clearly depends on the cash flow pattern and the distribution of asset life. In this paper, we examine the geometric gradient series pattern and geometrically distributed asset life for the discrete case, and the exponential pattern and exponentially distributed life for the continuous case. In other words, these patterns are growing annuities, which are generally used in practice because their present values can be given in nice closed forms. We assume that growth rates are non-stochastic. The geometric and exponential distributions are also each other's discrete versus continuous counterparts and are perhaps also the most frequently used distributions in reliability engineering, in modeling failure rates (e.g., [21, 24]). These distributions are the only "memoryless" distributions; that is, the life of an item does not depend on how long the item has survived (e.g., [21, 24]). They describe completely random failures – the middle (and longest) section of the bathtub curve in reliability engineering (e.g., [27]). Thus, we can also say that incorporating life uncertainty into present value calculations is an explicit consideration of underlying technical characteristics, and it is desirable to reconcile technology and economics [1].

Our contributions in this paper are threefold. First, under the above-described pattern, distribution, and parameter assumptions, we evaluate the relative error of computing present value based on expected life instead of the expected present value. This error measure is defined as

$$\varepsilon = \frac{\hat{P}}{E(P)} - 1 \tag{6}$$

where ε denotes the relative error. In this formulation, a positive error means that the conventional approach overstates the correct present value, and a negative error implies understatement. (Note that these points apply to the magnitude of the present values; thus, in the case of negative present values, the reverse is true if we look at the actual values. To avoid ambiguity, we will be concerned with the magnitude, i.e., the absolute value, of present values in this paper.)

We are the first to look at the relative error, which we deem a better descriptor of computational error than the absolute error, i.e., $\hat{P} - E(P)$, which was examined by Chen and Manes [15]. Relative error is dimensionless and is expressed as a percentage deviation, whereas absolute error is expressed in dollars. As we demonstrate, in using relative error, we are able to establish limiting conditions on errors that are unobtainable in working with absolute error. For example, we establish that the relative error cannot exceed approximately 30%, provided that the discount rate exceeds the growth rate. Such limits are not obtainable using absolute error. Additionally, we present convenient graphical illustrations of the behavior of relative error, which, again, cannot be done with absolute error, which depends on actual dollar amounts.

Second, we explicitly extend the analysis to the negative domain of growth rates and discount rates. Consideration of negative growth rates is quite plausible, but the inclusion of negative discount rates might seem less obvious and require some justification. A negative discount rate corresponds to an asset with negative expected return – such investment opportunities can be considered as "hedges" or "insurance" against asset pricing factors. In the terminology of the most widely used model for estimation of the cost of capital, the Capital Asset Pricing Model (CAPM), this means assets that have such a large negative beta that the risk premium, in absolute value, exceeds the risk-free rate, yielding a negative discount rate. In fact, empirical evidence has recently shown that this may be the case with, e.g., energy efficiency and renewable energy projects (e.g., [2, 4, 5, 7, 8, 11, 25, 28]), as it is argued that energy prices exhibit contra-cyclical behavior (e.g., [6, 9, 26, 30, 34]). In this paper, we consider the theoretical range of both the growth and discount rates in the –1 to 1 open interval.

Third, we provide a thorough comparison of the discrete and the continuous case. Such a comparative study, to the best of our knowledge, has not yet been published. Although the two cases bear many similarities, there are essential differences.

Our findings can be summarized as follows for both the discrete and the continuous case, as they are found to bear a close resemblance to one another. If the discount rate exceeds the growth rate, then the conventional approach always overstates the expected present value, with an error that can be at most approximately 30%. However, for any given expected life, there exist several growth rate and discount rate combinations for which such a maximum is attained. In the discrete case, as the main point of distinction with the continuous case, the value of this maximum depends on the expected life and is lower for shorter lives but cannot be lower than approximately 13%. If the discount rate is equal to the growth rate, then the error is always zero; thus, in this case, the conventional approach is perfectly accurate. If the discount rate is less than the growth rate, the conventional approach always understates the expected present value, with an error that can reach the theoretical limit of 100% in magnitude, for any expected life. We highlight that, in this case, the error is very sensitive to the difference between the discount rate and the growth rate. As a rule of thumb, we can establish that for a given discount rate and growth rate combination, a longer expected life – or, conversely, for a given expected life, a larger difference in absolute value between the two rates – results in a larger error. In particular, we find that for realistic cases of 10 to 20 years of expected life and a 2% to 1% difference between rates, the error may exceed 10%, which can be considered significant. These results also call attention to the importance of precision in growth rate and discount rate estimation.

The remainder of this paper is structured as follows. The next two main sections discuss the continuous case and the discrete case separately. The continuous case is taken first because its results are simpler and more tractable. These main sections are also divided into two subsections: mathematical derivations are followed by evaluations of errors. A discussion section consolidates the results also in graphical form, and provides a simple illustrative numerical example. A final section concludes.

2 Continuous case

2.1 Derivation of present value formulas

Recall from the introduction that we assume a continuous exponential cash flow pattern. Thus, the evolution of cash flows is mathematically represented as

$$F(t) = C\,\mathrm{e}^{jt} \tag{7}$$

where j is a non-stochastic and time-invariant continuous growth rate, and C is a constant.

The present value of such a pattern, following from Eq. (2), is computed as (e.g., [32, 33])

$$P = \int_0^T C\,\mathrm{e}^{-(r-j)t}\,\mathrm{d}t = \frac{C}{r-j}\left(1-\mathrm{e}^{-(r-j)T}\right) \tag{8}$$

(We have dropped the c index, as we focus solely on the continuous case in this section.)

Recall that the life of the asset, T, is a random variable. Thus, under the conventional approach, the present value is calculated as

$$\hat{P} = \frac{C}{r-j}\left(1-\mathrm{e}^{-(r-j)E(T)}\right) \tag{9}$$

The correct calculation, by contrast, is

$$E(P) = E\left(\frac{C}{r-j}\left(1-\mathrm{e}^{-(r-j)T}\right)\right) \tag{10}$$

Notice that the present value in Eq. (8) is a function, in other words, a transform of the random variable T. Assuming a positive C, it can be established from the second (partial) derivatives with respect to T that the present value function in Eq. 8 is strictly convex if and only if $r < j$, and it is strictly concave if and only if $r > j$. Now, based on Jensen's inequality, the direction of the bias between Eqs. (9) and (10) can be established. That is, $\hat{P} < E(P)$ if and only if $r < j$; then, the conventional approach provides an understated present value. Conversely, $\hat{P} > E(P)$ if and only if $r > j$; then, the conventional approach provides an overstated present value. (In the case of negative C, i.e., negative present values, just the opposite is true.) Note that equality occurs when $r = j$ (for then the second [partial] derivative is zero, so that the function in Eq. (8) is both convex and concave). In this case, the exponent in Eq. (8) is zero and the integral becomes

$$P = \int_0^T C\,\mathrm{d}t = CT \tag{11}$$

Because C is a constant, it is apparent that $CE(T) = E(CT)$, that is, $\hat{P} = E(P)$. Thus, the conventional approach is perfectly accurate if and only if $r = j$.

It is important to note that the findings thus far are independent of the actual probability distribution of T.

If we rewrite the expected present value in Eq. (10) as

$$E(P) = \frac{C}{r-j}\left(1-E\left(\mathrm{e}^{-(r-j)T}\right)\right) \tag{12}$$

we notice that the expectation term is, in fact, the moment-generating function of T. Thus, knowing the distribution of T, and provided that its moment-generating function exists, the exact formula for the expected present value can be given. Alternatively, a general approximation based on a Taylor series expansion (at $T = E(T)$) can be used, usually by retaining only first and second order terms, as follows (e.g., [15, 39]):

$$E\left(\mathrm{e}^{-(r-j)T}\right) \approx \mathrm{e}^{-(r-j)E(T)} + \frac{(r-j)^2}{2}\,\mathrm{e}^{-(r-j)E(T)}\,V(T) \tag{13}$$

where $V(T)$ is the variance of T. Note that Eq. (13) can also be interpreted as an adjustment to the conventional approach, $\hat{P}$ (cf. [39]), namely

$$E(P) \approx \hat{P} - \frac{C(r-j)}{2} e^{-(r-j)E(T)} V(T) \tag{14}$$

We note another possible formulation for computing the expected present value, $E(P)$. Building on the definition of expected value, we seek the solution of

$$\int_{-\infty}^{+\infty} P f_P(P) \, dP = E(P) \tag{15}$$

Where $f_P(P)$ is the probability density function (pdf) of the present value P.

Because P is a transform of the random variable T, Eq. (15) can be rewritten by substituting Eq. (8) as follows (see, e.g., [10]):

$$\begin{aligned} \int_{-\infty}^{+\infty} P f_T(T) \, dT &= \int_{-\infty}^{+\infty} \frac{C}{r-j}\left(1-e^{-(r-j)T}\right) f_T(T) \, dT \\ &= \frac{C}{r-j}\left(1-\int_{-\infty}^{+\infty} e^{-(r-j)T} f_T(T) \, dT\right) \\ &= E(P) \end{aligned} \tag{16}$$

Where $f_T(T)$ is the pdf of asset life T. Notice that the integral expression on the left is actually the moment-generating function; thus, this formula is equivalent to Eq. (12). The parameters in Eq. (16) may, of course, be required to meet some restrictive conditions in order to achieve convergence of the integral, that is, for the moment-generating function, and thus the expected present value, to exist.

We further note that the technique of the Laplace transform can also be used to compute the expected present value. For a discussion of this approach see, e.g., [12, 14, 22].

Now, recalling our assumption that asset life is exponentially distributed, we substitute the moment-generating function of the exponential distribution into Eq. (12) to obtain the closed form solution for the expected present value (see, e.g., [15, 41]):

$$E(P) = \frac{C}{r-j}\left(1-\frac{1}{1-\frac{-(r-j)}{\lambda}}\right) = \frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right) \tag{17}$$

where λ is the parameter of the exponential distribution, and θ denotes the mean of the exponential distribution, that is, the expected life of the asset (in other words, $E(T) = \theta$), which is related to the distribution parameter via $\theta = 1/\lambda$.

Verifying the convergence criteria from Eq. (16), we find that, assuming positive expected asset life, $\theta(r-j) > -1$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. (If θ is zero, then $E(P)$ is also zero, and the relative error is undetermined due to division by zero.)

2.2 Evaluation of relative error

Dividing Eq. (9) by Eq. (17) and subtracting 1, the relative error is

$$\begin{aligned} \varepsilon &= \frac{\frac{C}{r-j}\left(1-e^{-\theta(r-j)}\right)}{\frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right)} - 1 \\ &= \left(1-e^{-\theta(r-j)}\right)\left(1+\frac{1}{\theta(r-j)}\right) - 1 \end{aligned} \tag{18}$$

Notice that the relative error is independent of the cash flow parameter C. It should also be noticed that Eq. (18) is, in fact, a function of a single variable: substituting $x = \theta(r-j)$, the error can be expressed as

$$\varepsilon = (1-e^{-x})\left(1+\frac{1}{x}\right) - 1 \tag{19}$$

It follows from the previous discussion that if $x \leq -1$, then, because $E(P)$ equals infinity, the error is –100%, and if $x = 0$, which occurs when $r = j$, the error is zero. ($x = 0$ occurs also in the theoretical case where $\theta = 0$, but then we have a degenerate distribution, and the error is undetermined due to division by zero, see earlier. We assume a positive expected life.) Fig. 1 shows the plot of the absolute value of the error function in Eq. (19). We take the absolute value of the error to express the magnitude. The issue of sign has been discussed above in reference to Jensen's inequality.

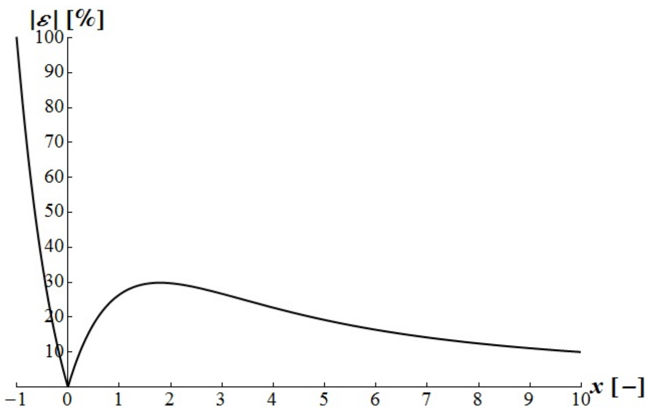

Fig. 1. Function of the absolute value of the relative error in variable $x = \theta(r-j)$

As Fig. 1 shows, the relative error has a local maximum. Differentiating Eq. (19) with respect to x and solving for zero produces only one real root, at $x \approx 1.79$, where $\varepsilon \approx 29.84\%$. Because we take the absolute value of the error, the global maximum is 100%. It can be seen in Fig. 1 that in the negative domain, the error is very sensitive to x. We can also establish that if x converges to infinity, then the error converges to zero. However, if we examine the composition of the variable x, the interpretations of the limiting cases become more subtle. First, consider what is probably the least unrealistic of the limiting cases, namely, that the expected life is infinity. This limiting case is meaningful if and only if $r > j$, which is the convergence criterion of the present value integrals when θ is infinity, and if this criterion holds, the error is indeed zero. Otherwise, neither the conventional nor the correct present value exist (i.e., both are infinity). Plausibly, uncertainty does not matter in the case of an asset that is expected to be infinitely lived (provided, of course, that its present value exists). Convergence of either the discount rate or the growth rate to either positive or negative infinity yield meaningless results. If the discount rate is infinitely large or the growth rate converges to negative infinity, both the conventional and the correct present values are zero; thus, the relative error, which is a ratio, is not meaningful. If the discount rate converges to negative infinity or the growth rate is infinitely large, neither the conventional nor the correct present value exists. (Thus, convergence of x to negative infinity is not meaningful.)

Now, further examining the composition of the variable x and knowing that, as θ is positive, x is positive if and only if $r > j$, and x is negative if and only if $r < j$, the results can be interpreted as follows. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but this maximum can be attained for every expected life θ, as there exist several (in fact, infinitely many) $r - j$ combinations for which the extremum condition is satisfied. This is because the parametric equation $x = \theta(r - j)$ by itself is an underdetermined system of equations and thus has infinitely many solutions. In other words, the real number x can be written as the product of two real numbers in infinitely many ways. If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which again, for the reason just mentioned, can be attained for every expected life. Fig. 1 clearly shows that, in this case, the error is very sensitive to the difference between the discount rate and the growth rate. When the two rates are equal, the error is zero, as discussed earlier.

It is also worthwhile to more closely examine the monotonicity of the error function depicted in Fig. 1. In the negative domain of x, the function is strictly decreasing (more precisely, on the −1 to 0 open interval because below −1, the error is identically 100%; see earlier), and in the positive domain, it is strictly increasing up to the local maximum at $x \approx 2$, beyond which the function is strictly decreasing. As x is the product of two variables, if we take one of them as fixed, x changes in magnitude in the same direction as the other variable changes in magnitude. That is, for a given rate difference, $r - j$, x increases/decreases in absolute value as the expected life, θ (which is assumed to be always positive), increases/decreases. Similarly, for a given expected life, x increases/decreases in absolute value as the rate difference increases/decreases in absolute value. We can argue that, for realistic cases, the rate difference may rarely exceed 10% in absolute value (e.g., a 10% discount rate paired with a 0% growth rate) and the planning horizon (i.e., the expected life) may rarely exceed 20 years. These practical constraints correspond to a limit of $x = 2$ in the positive domain, approximately the point of the local maximum. In fact, this implies that the local error maximum is unlikely to be attained in practice. Under these tentative assumptions, the relevant range of x can be confined to −1 to 2. (The range of −2 to −1 can be neglected, as the error is already at the theoretical maximum of 100% in that region.) Thus, recalling our monotonicity observations, the following rule of thumb can be stated: for a given rate difference, the larger the expected life – or, alternatively, for a given expected life, the larger the rate difference in absolute value – the larger is the magnitude of the error. A caveat must be added, however, for the sake of precision. Because the absolute value of the error function is not symmetric around $x = 0$ (i.e., in the negative domain of x, it is "steeper"), it may occur that, given an expected life, a negative rate difference has a larger error magnitude than what a more positive rate difference (i.e., the absolute value of which is larger than the absolute value of the negative difference) has. In other words, for a given expected life, a decrease/increase in the rate difference to the extent that it changes sign may result in an increase/decrease in the magnitude of the error even though the absolute value of the rate difference decreases/increases. Therefore, we make the clarification that by "a larger rate difference in absolute value (or magnitude)" we mean henceforth that a positive (negative) rate difference is made more positive (negative); thus, a change of sign is ruled out. (We note that this change of sign issue does not occur when the rate difference is given because the expected life is assumed to be always positive. Thus, that kind of formulation of our rule of thumb requires no such caveat.)

Fig. 2 plots error functions for specific expected lives, to better illustrate the above points (regarding, for example, local error maximum attainable for every expected life and monotonicity under realistic constraints).

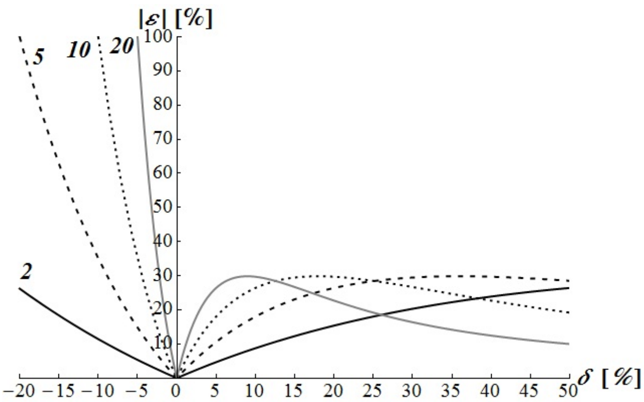

Fig. 2. Absolute value of the relative error as a function of the rate difference $\delta = r - j$ for expected lives of $\theta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray)

As a side-observation, we note that similar conclusions can be drawn using the Taylor series approximation formula in Eq. (13). Knowing that the variance of an exponentially distributed random variable is the square of its expected value, we can approximate the error function (omitting here tedious algebraic manipulations) as

$$\varepsilon \approx \frac{x^2}{2\,\mathrm{e}^x - 2 - x^2} \tag{20}$$

which behaves similarly to the exact function in Eq. (19) and also has a single local maximum in the positive domain, at $x \approx 1.59$, with an approximate value of the local error maximum of 48%. These results can be seen as close to those obtained via the precise analysis.

Finally, we identify the advantage of working with the relative error, or more precisely, why the local maximum cannot be obtained using the absolute error. The absolute error can be written as (following Chen and Manes [15])

$$\begin{aligned}\phi &= \hat{P} - E(P) \\ &= \frac{C}{r-j}\left(1-\mathrm{e}^{-\theta(r-j)}\right) - \frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right) \\ &= C\frac{1-(1+\theta(r-j))\,\mathrm{e}^{-\theta(r-j)}}{(1+\theta(r-j))(r-j)}\end{aligned} \tag{21}$$

It is obvious that Eq. (21) cannot be rewritten as a function of a single variable; in fact, it is a function of at least three variables because the cash flow parameter cannot be eliminated. The system of equations in which the partial derivatives are all zero cannot be solved; therefore, the necessary conditions for the existence of a local extremum are not fulfilled. This is plausible because several absolute errors exist for which the same relative error is attained, and thus, although the relative error is maximal, the absolute error can be arbitrarily large.

3 Discrete case

3.1 Derivation of present value formulas

We recall from the introduction that, in the discrete case, we assume a geometric gradient series cash flow pattern. Thus, the evolution of cash flows is defined mathematically as

$$F_n = F_1(1+g)^{n-1} \tag{22}$$

where g is a non-stochastic and time-invariant discrete growth rate.

We also note an interesting similarity between the continuous and discrete growth patterns used in our analysis: the "period-wise" integrals of the exponential cash flow function in Eq. (7), that is, the series of integrals over successive periods, produces a geometric series (i.e., also an exponentially growing series, but in discrete time). To show this, we first compute F_n as a period-wise integral of Eq. (7) as

$$F_n = \int_{n-1}^{n} C\,\mathrm{e}^{jt}\,\mathrm{d}t = \mathrm{e}^{j(n-1)}\frac{C}{j}\left(\mathrm{e}^{j}-1\right) \tag{23}$$

From this, we can evaluate the discrete growth rate (for any n) as

$$1+g = \frac{F_n}{F_{n-1}} = \frac{\mathrm{e}^{j(n-1)}}{\mathrm{e}^{j(n-2)}} = \mathrm{e}^{j} \tag{24}$$

which completes the demonstration. Notice that, not surprisingly, we arrived at the same relationship between g and j as that between i and r in Eq. (3).

Returning to Eq. (22), the present value of F_n, as of time zero, is computed as

$$P_n = F_n(1+i)^{-n} = \frac{F_1}{1+g}\left(\frac{1+g}{1+i}\right)^{n} \tag{25}$$

where P_n denotes the present value of F_n.

Thus, P_n represents a geometric series, and using the well-known formula for the sum of a geometric series, the present value of the series of cash flows (shown previously in Eq. (1)) can be given as (e.g., [32, 33])

$$P = \frac{F_1}{1+i}\frac{\left(\frac{1+g}{1+i}\right)^{N}-1}{\frac{1+g}{1+i}-1} = \frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{N}\right) \tag{26}$$

(We have omitted the d index, as it is clear that we are discussing the discrete case in this section.)

Recalling that the life of the asset, N, is a random variable, under the conventional approach, the present value is calculated as

$$\hat{P} = \frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{E(N)}\right) \tag{27}$$

The correct calculation, by contrast, is

$$E(P) = E\left(\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{N}\right)\right) \tag{28}$$

Note that the present value in Eq. (26) is a function, i.e., a transform of the random variable N. As both g and i are assumed to be greater than -1, the quotient $(1+g)/(1+i)$ must be positive. Thus, assuming a positive F_1, it can be established from the second (partial) derivatives with respect to N (assuming for the moment that N is a continuous variable) that the present value function in Eq. (26) is strictly convex if and only if $i < g$ and strictly concave if and only if $i > g$. Now, based on Jensen's inequality, the direction of the bias between Eqs. (27) and (28) can be established. That is, $\hat{P} < E(P)$ if and only if $i < g$; then, the conventional approach provides an understated present value. Conversely, $\hat{P} > E(P)$ if and only if $i > g$; then, the conventional approach provides an overstated present value. (In the case of negative F_1, i.e., negative present values, just the opposite is true.) Equality occurs when $i = g$, although then the second (partial) derivative and the closed form present value formula in Eq. (26) are both undetermined due to division by zero. This is actually the result of the violation of the convergence criterion for the sum of the geometric series, namely, that the ratio of successive terms must be less than 1 (in absolute value). Recalling the present value formula for the individual terms of the series in Eq. (25), we obtain in this case, also by substituting i for g (see, e.g., [32, 33])

$$P_n = \frac{F_1}{1+i}\ \forall n \geq 1\text{, from which } P = N\frac{F_1}{1+i} \tag{29}$$

Because both F_1 and i are non-stochastic, it is apparent that $\frac{F_1}{1+i}E(N) = E\left(N\frac{F_1}{1+i}\right)$, that is, $\hat{P} = E(P)$. Thus, the conventional approach is perfectly accurate if and only if $i = g$.

It is important to note that the findings thus far are the same as those that have been established in the continuous case and are again independent of the actual probability distribution of asset life.

If we rewrite the expected present value in Eq. (28) as

$$E(P) = \frac{F_1}{i-g}\left(1 - E\left(\left(\frac{1+g}{1+i}\right)^N\right)\right) \tag{30}$$

we see that the expectation term is actually the probability-generating function of N. Thus, knowing the distribution of N, and provided that the power series representation corresponding to its distribution converges, the exact formula for the expected present value can be given.

Alternatively, we can again retreat to an approximation based on a Taylor series. Assuming for the moment that N is a continuous variable and expanding the expectation term about $N = E(N)$ and retaining only first and second order terms, we have

$$E\left(\left(\frac{1+g}{1+i}\right)^N\right) \approx \left(\frac{1+g}{1+i}\right)^{E(N)} + \left(\frac{1+g}{1+i}\right)^{E(N)} \frac{\ln^2\left(\frac{1+g}{1+i}\right)}{2} V(N) \tag{31}$$

where $V(N)$ is the variance of N. Notice that Eq. (31) can, again, be interpreted as an adjustment to the conventional approach, $\hat{P}$ (similar to Eq. (14) in the continuous case). Specifically,

$$E(P) \approx \hat{P} - \frac{F_1}{i-g}\left(\frac{1+g}{1+i}\right)^{E(N)} \frac{\ln^2\left(\frac{1+g}{1+i}\right)}{2} V(N) \tag{32}$$

Similarly to the continuous case, an alternative formulation for computing the expected present value, $E(P)$, can be stated. Building on the definition of expected value for a discrete random variable, we seek the solution of

$$\sum_{n=0}^{\infty} p_P(n)\, P|_{N=n} = E(P) \tag{33}$$

where $p_P(n)$ is the probability mass function (pmf) of the present value P.

Again, because P is a transform of the random variable N, Eq. (33) can be rewritten, by substituting Eq. (26) into it and exploiting the fact that the sum of the infinite series of a pmf equals 1 by definition, as follows (see, e.g., [10]):

$$\begin{aligned} &\sum_{n=0}^{\infty} p_N(n) \frac{F_1}{i-g}\left(1 - \left(\frac{1+g}{1+i}\right)^n\right) \\ &= \frac{F_1}{i-g}\left(1 - \sum_{n=0}^{\infty} p_N(n)\left(\frac{1+g}{1+i}\right)^n\right) = E(P) \end{aligned} \tag{34}$$

where $p_N(n)$ is the pmf of life N. The parameters in Eq. (34) may, of course, be required to meet some restrictive conditions in order to achieve convergence of the sum of the infinite series, that is, for the expected present value to exist. Notice that this power series is, in fact, the probability-generating function of N, so we have arrived at a formulation equivalent to Eq. (30).

We further note that the technique of the Z-transform (which can be regarded as the discrete-time equivalent of the Laplace transform) can also be used to compute the expected present value. For a discussion of this approach see, e.g., [13, 35].

Recalling our assumption that asset life is geometrically distributed and substituting the corresponding probability-generating function into Eq. (30), we find the closed form solution for the expected present value (see, e.g., [19]):

$$E(P) = \frac{F_1}{i-g}\left(1 - \frac{\alpha\frac{1+g}{1+i}}{1-(1-\alpha)\frac{1+g}{1+i}}\right) = \frac{F_1}{i-g}\left(\frac{1}{1+\frac{1+g}{\eta(i-g)}}\right) \tag{35}$$

where α is the parameter of the geometric distribution, and η denotes the mean of the geometric distribution, that is, the expected life of the asset (in other words $E(N) = \eta$), which is related to the distribution parameter via $\eta = 1/\alpha$. There are actually two versions of the geometric distribution – we use the one for the domain of positive integers; that is, asset life cannot be zero (see [19]).

Verifying the convergence criteria from Eq. (34), we find, exploiting the fact that expected asset life must be positive, that $\left|\frac{1+g}{1+i}\right| < \frac{\eta}{\eta-1}$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. Because both g and i are assumed to be greater than -1, the absolute value sign can be omitted, as the ratio is always positive. Note also that η cannot be zero, as asset life is assumed to be at least 1 period.

3.2 Evaluation of relative error

Dividing Eq. (27) by Eq. (35) and subtracting 1, we obtain the relative error, after considerable algebraic manipulation:

$$\begin{aligned} \varepsilon &= \frac{\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^\eta\right)}{\frac{F_1}{i-g}\left(\frac{1}{1+\frac{1+g}{\eta(i-g)}}\right)} - 1 \\ &= \left(1-\left(\frac{1+g}{1+i}\right)^\eta\right)\left(1+\frac{1}{\eta\left(\frac{1+i}{1+g}-1\right)}\right) - 1 \end{aligned} \tag{36}$$

Notice that the relative error, similar to that of the continuous case, is independent of the cash flow parameter F_1. More important, however, is the fact that, in contrast to the continuous case in Eq. (18), the error function in Eq. (36) cannot be rewritten as function of a single variable. This is clear from the fact that η is an exponent in one term and a multiplier in another. Due to this limitation, the analysis of the discrete case is more involved than that of the continuous case. However, the above formulation helps us recognize that Eq. (36) can be rewritten as a function of two variables, if we substitute into the expression $y = (1+g)/(1+i)$. Then,

$$\varepsilon = (1-y^\eta)\left(1+\frac{1}{\eta\left(\frac{1}{y}-1\right)}\right) - 1 \tag{37}$$

It follows from the previous discussion that if $y \geq \eta/(\eta-1)$, then, because $E(P)$ equals infinity, the error is –100%; and if $y = 1$, which occurs when $i = g$, the error is zero. Note also that for $\eta = 1$, the error is also always zero (in this case, we have a degenerate distribution similar to $\theta = 0$ in the continuous case).

Although Eq. (37) is a function of two variables, in essence, it is very similar to the single-variable error function Eq. (19)

in the continuous case. To see this, substitute $z = \eta(\frac{1}{y} - 1)$ and apply a first order Taylor series approximation to show that $\mathrm{e}^{-z} = \left(\mathrm{e}^{\frac{1}{y}-1}\right)^{-\eta} \approx \left(1 + \frac{1}{y} - 1\right)^{-\eta} = y^{\eta}$, from which Eq. (37) can be rewritten as

$$\varepsilon \approx (1 - \mathrm{e}^{-z})\left(1 + \frac{1}{z}\right) - 1 \tag{38}$$

which has the same form in z as Eq. (19) has in x. Thus, we can expect results in the discrete case to be similar to those we found in the continuous case.

Fig. 3 shows the plot of the absolute value of the error function of two variables, given in Eq. (37), for specific values of η. Again, we take the absolute value of the error to express the magnitude. The issue of sign has been discussed above, in reference to Jensen's inequality.

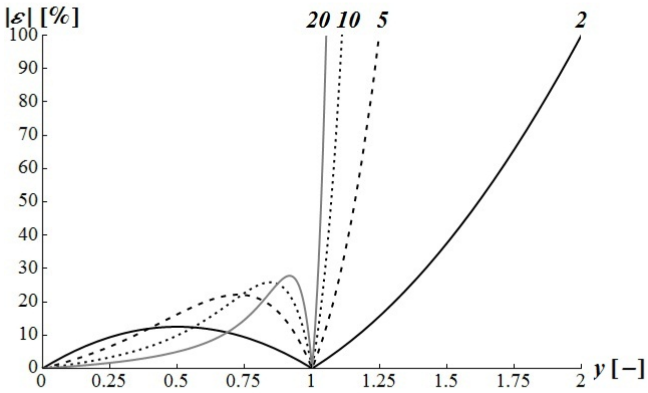

Fig. 3. Function of the absolute value of the relative error in variable $y = (1+g)/(1+i)$ for expected lives of $\eta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray)

As Fig. 3 shows, for every η, for $y < 1$, the relative error has a local maximum, the value of which is conditional on η. Because we take the absolute value of the error, the global maximum is 100% for every η. In the domain where $y > 1$, the error is very sensitive to y. These observations closely accord with those for the continuous case, with the difference that in the discrete case the shape of the error function, more precisely, the value of the local maximum, varies (slightly) with η. The main characteristics, however, are the same – that is, for a given expected life, the discrete and continuous errors behave identically. This is confirmed when we examine more closely the composition of the variable y. If $y < 1$, which occurs if and only if $i > g$, there exists a local error maximum. The error is very sensitive to values of $y > 1$, which occur if and only if $i < g$. The error is zero when $y = 1$, which occurs if and only if $i = g$. (Our assumptions do not allow y to equal 0.)

The main difference between the discrete and the continuous case, which can be observed in Fig. 3, is that in the discrete case, the value of the local error maximum is not the same for every expected life, as was the case in the continuous case. For example, for an expected life of 5 periods, the value of the local error maximum is approximately 22%, in contrast to the approximately 30% attainable in the continuous case. Fig. 3 also shows that (and this can be verified by examining the partial derivatives) the location and the value of the local maximum are both increasing in η. That is, they are lowest for $\eta = 2$, when the value of the local error maximum is 12.5%, obtained at $y = 0.5$, and highest as η approaches infinity. For example, for $\eta = 1000$, the value of the local error maximum is 29.8%, which is the same maximum value we found in the continuous case and is obtained at $y = 0.998$. It is intuitive that the largest possible value of the local error maximum in the discrete case should be the same as that in the continuous case because, from the "view of infinity," the length of the discrete periods is infinitesimal, that is, the discrete case looks to be continuous. Therefore, we can establish that the error in the discrete case cannot exceed 30% if $i > g$. We also note that this theoretical extreme is approached quite rapidly, e.g., for $\eta = 20$, the value of the local error maximum is 27.8%.

Additionally, it is important to recognize that because both i and g are typically small (i.e., close to zero), the ratio y behaves quite analogously to the rate difference $i - g$. (This can be verified, e.g., via a first-order Taylor series approximation of the ratio in two variables.) That is, the larger the difference in magnitude, the further the ratio is away from 1, where the error is zero. Thus, the results so far accord with those established in the continuous case. That is, for a longer expected life, the rate difference must be smaller to attain the local error maximum. Thus, as η increases, the point of the local maximum moves closer to $y = 1$ (see in Fig. 3).

In summary, we can draw conclusions very similar to those we drew in the continuous case. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but a maximum of at least 12.5% can be attained for every expected life η (except for 1), as there exist several $i - g$ combinations for which the extremum condition is satisfied (in fact, infinitely many because both i and g can be real numbers). This is because the parametric equation $y = (1+g)/(1+i)$ by itself is an underdetermined system of equations and thus has infinitely many solutions. In other words, the real number y can be written as the ratio of two real numbers in infinitely many ways. If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which can be attained for every expected life. Fig. 3 clearly shows that, in this case, the error is very sensitive to the ratio of the rates. When the two rates are equal, the error is zero, as discussed earlier.

It is also worthwhile to examine some limiting cases of the error function. If the expected life converges to infinity, the error is meaningful if and only if $i > g$, which becomes the convergence criterion of the series composing the present values, and if this criterion holds, the error is indeed zero. Otherwise, neither the conventional nor the correct present value exists (i.e., they are both infinity). This is the same observation we made in the continuous case, and again it reflects that uncertainty does not matter in case of an asset that is expected to be infinitely lived (provided, of course, that its present value exists). Convergence of either the discount rate or the growth rate to either positive or negative infinity yield meaningless results in the discrete case as well. If the discount rate converges to positive or negative infinity, all terms in the series are zero (see Eq. (25)), so both the conventional and the correct present values are also zero; thus, the relative error, which is a ratio, is not meaningful. If the growth rate converges to positive or negative infinity, all terms in the series are infinitely large, so neither the conventional nor the correct present value exists.

Finally, the same rule of thumb that we formulated for the continuous case can be formulated for the discrete case. Let us now recall the practical boundaries introduced in the evaluation of the continuous case. First, the rate difference may rarely exceed 10% in magnitude. Adding that the individual rates themselves are unlikely to exceed 20% in magnitude, this translates into a minimal y of approximately 0.9. Second, the expected life may rarely exceed 20 years – for $\eta = 20$, the local maximum is obtained at $y \approx 0.9$, and for smaller values of η, it is obtained at smaller values of y (see the previous discussion). Thus, we can establish that, under these practical constraints, the local maximum is unlikely to be attained, and thus it is sufficient to consider only the two monotonic ranges above and below $y = 1$. It can also be observed in Fig. 3 (and verified by examining the partial derivative) that in this range, the error is increasing in η. In conclusion, for a given rate difference, the larger the expected life – or, alternatively, for a given expected life, the larger the magnitude of the rate difference – the larger is the magnitude of the error. This is the same rule as the one formulated above for the continuous case. The same caveat made in the continuous case applies here as well, that is, because the absolute value of the error function is not symmetric around $y = 1$, we clarify that “larger” as pertains to the magnitude of the rate difference means that the rate difference is “more positive (negative)”.

Due to its complexity and limited relevance, we omit the discussion of the similarity of the results obtainable by using the Taylor series approximation formula in Eq. (31) compared to the precise analysis. Additionally, discussion of the difference between absolute and relative errors is omitted for the same reason.

4 Discussion of results

As we have shown, the error characteristics in the continuous and the discrete case are quite similar. It might be argued that a discount rate higher than the growth rate is the typical case in real life. If so, then it is somewhat reassuring to know that, given any rate difference and asset life combination, the error cannot exceed 30%. However, it is discomforting that, at least theoretically, a non-negligible error maximum may occur for any expected life (or any rate difference). Generally, a long (short) expected life associated with a small (large) rate difference may give rise to such an extreme situation. As was argued, however, analysts may rarely encounter this error maximum in practice. If the growth rate exceeds the discount rate, the picture is much more dire. Then, the error has no limit except the theoretical maximum of 100% and, in addition, is more sensitive to the rate difference. The case of negative discount rates is particularly relevant here: then, even for very small, near-zero growth rates, the error may be severe, and for this to occur, the discount rate need not be very negative, as we show below. Perhaps the best way to demonstrate possible computational errors attributable to using expected asset life, rather than computing the expected present value, is via a graphical illustration. Fig. 4 shows nomograms (two-dimensional diagrams) that exhibit the severity of errors for various discount rate and growth rate combinations for expected lives of 2, 5, 10, and 20 periods, in both the continuous and discrete cases. For a more tractable demonstration, we cut off rates above 20% in magnitude.

As the nomograms show, the continuous and discrete cases are very similar, as expected. It is alarming to observe that the error may easily exceed 10%, which can be considered significant, and that for that to happen, the rate difference need only be a few percentage points. For example, in the continuous case, for an expected life of 10 periods, rate differences of 2% and −2% produce errors of 8.8% and −11.4%, respectively. For an expected life of 20 periods, rate differences of 1% and −1% produce the same error percentages. As follows from our previous findings and as seen in Fig. 4, as the expected life increases, the band of rate combinations associated with errors of less than 10% shrinks. That is, a given rate difference (typically) implies a more severe error, the longer is the expected life (as stated in our rule of thumb). These results call attention to the importance of precision in growth rate and discount rate estimation, as even a small percentage point deviation can have serious consequences. That is, for imprecise rate estimates the error may be found to be negligible, so the analyst would wrongly think that it is unnecessary to bother with computing the expected present value and would simply use the expected life. Whereas, the error may be much larger if the correct rates were used, and the computation of the expected present value would be clearly preferred. The nomograms presented here help assess such possible errors.

Finally, we provide a brief illustrative example to present the error characteristics numerically. Consider an energy efficiency project, specifically, an installation of photovoltaic cells on the rooftop of a building, where the returns consist of a single cash flow stream resulting from savings on energy costs. (Other relevant cash flows, e.g., operation and maintenance costs, can usually be neglected for such projects, so consideration of the savings flows alone is a good working approximation.) Such a project can be fairly described as a growing annuity of cash flows, accounting for possibly rising fuel prices. The life of photovoltaic cells may be approximated by a memoryless probability distribution, such as the exponential or geometric distribution (e.g., [29, 40]). We assume in this example a reasonable 20-year expected life for the project. As discussed in the introduction, several papers have argued for the counter-cyclical nature of energy prices and found empirically that related projects

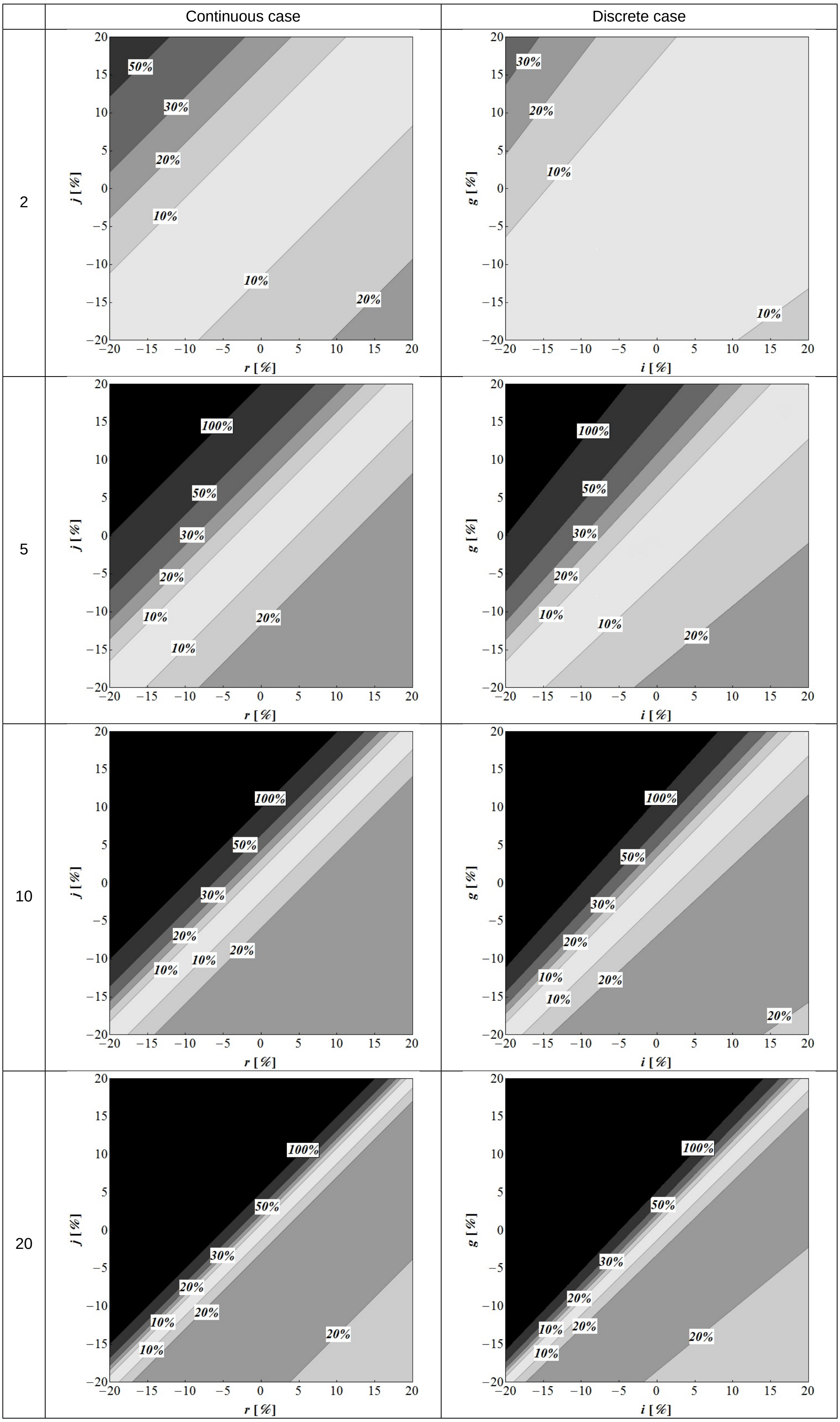

Fig. 4. Comparative evaluation of errors for various expected lives and discount rate – growth rate combinations; darker colors indicate more severe errors

may have a negative beta in the CAPM. A negative beta may imply a negative discount rate, depending on one's assumptions about the market parameters (i.e., the risk-free rate and the expected market risk premium). Because some of these papers (e.g., [2, 11, 28]) observe that the possibility of a positive beta generally cannot be ruled out for energy-saving projects (i.e., they cannot reject the null hypothesis of zero beta, cf. [16]), and because Andor and Dülk [2] conclude that the risk-free rate is generally a fair approximation for the cost of capital for such projects, we consider a narrow range of discount rates around zero. The growth rate is examined in a similarly narrow range, as it is not expected to be large for an extended period of time. For reference, some negative growth rates are also included. We consider one year as the interest period. We note that the intraperiod pattern of cash flows may not be precisely described as an exponential growth in the continuous case, or as an end-of-period lump sum in the discrete case, but the error attributable to this issue is negligible for the small discount rates we use (see [3] for a thorough discussion of this topic). Table 1 summarizes the errors for the various rate combinations.

Tab. 1. Relative errors (rounded to integers) for various discount rate and growth rate combinations for the illustrative project with an expected life of 20 years. A positive error means overstatement, and a negative error means understatement, of the correct present value.

Continuous case							
j \ r	−6%	−4%	−2%	0%	2%	4%	6%
−5%	−11%	9%	20%	26%	29%	30%	29%
−3%	−45%	−11%	9%	20%	26%	29%	30%
−1%	−100%	−45%	−11%	9%	20%	26%	29%
0%	−100%	−69%	−26%	0%	15%	24%	28%
1%	−100%	−100%	−45%	−11%	9%	20%	26%
3%	−100%	−100%	−100%	−45%	−11%	9%	20%
5%	−100%	−100%	−100%	−100%	−45%	−11%	9%
Discrete case							
g \ i	−6%	−4%	−2%	0%	2%	4%	6%
−5%	−12%	9%	20%	25%	27%	28%	27%
−3%	−46%	−11%	8%	19%	25%	27%	28%
−1%	−100%	−45%	−11%	8%	19%	25%	27%
0%	−100%	−68%	−25%	0%	14%	22%	26%
1%	−100%	−98%	−43%	−11%	8%	19%	25%
3%	−100%	−100%	−95%	−42%	−11%	8%	19%
5%	−100%	−100%	−100%	−92%	−41%	−10%	8%

As Table 1 shows, the errors are not negligible and may even be severe for the realistic scenarios examined. This is not surprising in light of our earlier results, as here we are dealing with a long life combined with a small rate difference. Perhaps the most realistic scenario is the 2% risk-free rate as the discount rate and a zero growth rate (if we are thinking in real terms), for which the error is approximately 15%, which is significant. Therefore, computation of the expected present value is desirable in the case of such an energy efficiency project.

As a complement to practical recommendations, we note that for the valuation of companies the issue of expected present value is probably not relevant, as such valuations typically assume an infinite life, along with the condition that the discount rate exceeds the growth rate, and, recalling our related findings, in this case the error is zero.

5 Conclusions

In this paper, we have shown that if the economic life of an asset is uncertain, the expected present value should be calculated instead of simply substituting the expected life of the asset into the present value formula. The computational error attributable to the latter approach depends on the cash flow pattern of the asset, the cost of capital, and the probability distribution of the asset life. We have evaluated relative errors for the case of a continuous exponential cash flow pattern and exponentially distributed life, as well as for their discrete equivalents of a geometric gradient cash flow series and geometrically distributed life, for various costs of capital. By using the relative error we have gained insights that previous studies have not achieved because they were working with absolute, rather than relative, error. Most notably, we found a local error maximum if the discount rate exceeds the growth rate and showed that this maximum can be attained for any expected life. We are the first to present a detailed comparison of the discrete and continuous cases and to demonstrate that very similar conclusions can be drawn in the two cases. The most essential difference concerns the local maximum, the value of which is found to vary with the expected life in the discrete case, while it is invariant in the continuous case. More concretely, we found that if the discount rate exceeds the growth rate, then the error cannot be larger than 30% in either the discrete or the continuous case, and the correct present value is overstated. This value of the local error maximum can be attained for every expected life in the continuous case, but it can be attained only for relatively long (approximately 20 periods or longer) expected lives in the discrete case. In the discrete case, the value of the local error maximum is smaller for a shorter expected life and is found to be at least 12.5%, except for an expected life of one year, when the error is zero. The error is also always zero if the discount rate is exactly equal to the growth rate. We found that the error may easily become severe if the growth rate exceeds the discount rate and may even reach the theoretical maximum of 100%, both in the discrete and continuous case. The correct present value is then understated. Setting practical constraints on the possible range of rates and expected lives, we have formulated the following general rule of thumb: for a given difference between the discount rate and the growth rate, a larger expected life – or, alternatively, for a given expected life, a larger rate difference in magnitude – results in a larger magnitude of the error. Our findings call attention to the need for precision in discount rate and growth rate estimation, as, for realistic cases, even a single percentage point rate difference may result in a non-negligible error. An empirically supported illustrative example was presented to highlight possible errors, including cases with negative discount and/or growth rates. Analyses similar to that presented in this paper could be conducted for other cash flow patterns and probability distributions.

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