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RESEARCH ARTICLE

Application of Markov Chains for Modeling and Managing Industrial Electronic Repair Processes

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Abstract

This paper presents a research of Markov chain based modeling possibilities of electronic repair processes provided by electronics manufacturing service (EMS) companies. These stochastic processes are considered as business-like, industrialized activities that are typically complex with a high number of process states and many possible paths from the start state to the absorbing end states. Two models based on absorbing and acyclic absorbing Markov chains are introduced in order to model these processes. The presented method provides a quick tool for determining the most important operational and statistical parameters of the process and mapping the paths that contribute the most to the total load of the process. These results support several managerial applications concerning e.g. process improvement, quality control and resource allocation. The proposed model is illustrated with an industrial application.

Keywords

service provider · electronic repair process · stochastic process · process modeling · absorbing Markov chain · acyclic absorbing Markov chain

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1 Introduction

During the last decades, due to the changing economic environment the improvement of business processes to enhance organizational performance has become an important issue resulting in a wide range of management initiatives (TQM, Six Sigma, Lean, BPR, etc.). For this reason, the analysis of processes and the enhancement of process quality has become an important research area in the academic field and a leading managerial issue for the practitioners. Resulting that companies are collecting huge amounts of data, perform complex data and statistical analysis and use sophisticated models; they are competing on analytics (Davenport, 2006; de Vries, 1999).

Processes of manufacturing and service companies are described and analyzed with a wide range of tools. The most frequently used techniques include graphical tools (charts and diagrams), matrices, graphical, object-oriented and workflow-oriented techniques and generic methodologies as simulation (Aguilar-Savén, 2004). In this paper, Markov chains are applied to analyze business processes. The benefits of the application are demonstrated through a specific business process of industrial electronic repair services.

Markov chain based modeling provides process managers with *a fast modeling tool*. Rapid modeling techniques ensure fast results both for planning and for what-if analysis (Suri and Tomsicek, 1998) which is crucial for companies working in the highly competitive markets of the electronic industry. The requirement of time-based competition and the highly variable processes of Electronic Repair Services (ERPs) (demanding frequent managerial decisions) necessitate quick information for the current operation of any related processes.

The proposed Markov chain based modeling results in important *information for process improvement and capacity planning*. The steps of electronic repair processes are stochastic in respect of their sequence and their length. Consequently, the paths (the sequence of the visited process steps) are also variable for the different items delivered for repair. With the help of the proposed model, the likelihood and the lead time of each possible path and its contribution to the total load of the process can be calculated. This can support the management to identify the potential focus areas for process improvement and, by executing sensitivity analysis and simulations, the planning of resources of electronic repair processes can be supported as well.

In such a rapidly changing environment with such complex tasks, with which these companies need to deal in their day-to-day operations, flexible and quick modeling tools are needed to describe and analyze their processes. We found that Markov chain based modeling exceedingly serves these purposes and seems to be one of the most suitable tools to describe ERPs for many reasons. First, the characteristics of electronic repair processes show several similarities with the characteristics of Markov processes. Second, models based on Markov chains have extensive application scope. Markov chain models can be used to study different processes; they have proved in many managerial fields to analyze various problems from financial issues (Wu and Chuang, 2010) through human processes (Guerry, 2011;) to reliability issues (Verlinden et al., 2012). Third, Markov chains and processes are widely used for process improvement. For example, based on Markov processes, the optimal number of repairs was determined (Castro and Pérez-Ocón, 2006), an optimal production-maintenance policy was applied to minimize expected average cost of operation (Wang and Sheu, 2003), a Markov chain model was used to determine optimal replacement policies (Zhang and Love, 2000) and for finding the optimal values of control limits (Zempléni et al., 2004). Accordingly, modeling based on Markov chains not only agrees with the objective of modeling ERPs but fits to the object of the analysis as well.

In the possession of a vast amount of data about ERPs, developing a rapid and simple Markov chain modeling tool of electronic repair processes of EMS providers stood in the focus of our research. Our main objective was to understand and improve these stochastic industrial repair service processes. For this reason, based on the real operation of these repair processes, an *absorbing Markov chain model* was developed. Later, this model has been adjusted according to the characteristics of an actual ERP, and the original model was converted into an *acyclic absorbing Markov chain model*. This approach enables determination of the probability distribution of the ERP lead time (as a random variable) as well. Based on this information, the likelihood and the lead time of each possible path and its contribution to the total load of the process can be calculated.

Accordingly, the remainder of this paper is organized as follows. First, in Section 2, industrial repair services and electronic repair processes are introduced. In Section 3, the process models developed to describe the electronic repair processes are presented and the results and advantages deriving from the practical use of the models are discussed. Section 4 introduces the application of the presented method for a real electronic repair process. Finally, in Section 5, the paper is closed with drawing a number of key conclusions and presenting important managerial implications.

2 Business processes of industrial repair services

2.1 Industrial repair services

Growing faster than the world GDP, electronic industry has become the cornerstone of the 21st century's industrial revolution. This sector has been undergoing a momentous transformation during the last decades in many ways. The complexity and sophistication of electronic products and product functionality has greatly improved in the electronic sector. Customers of electronic products increasingly require highly customized and high-quality products at a competitive price (Kita, 2001). These tendencies have generated a growing demand for high-level manufacturing capabilities, design, engineering and aftermarket services (Zhai et al., 2007). Those manufacturers are preferred who can provide a wide range of additional services (e.g. guarantee, free loaner device) besides supplying highly reliable products.

As these services are not necessarily the core competences of Original Equipment Manufacturers (OEMs), they turn to subcontracting in order to spread risks and to reduce costs. This kind of outsourcing trend in the electronics industry has led to the significant growth of the electronics manufacturing services (EMS) industry (Salvador et al., 2002). EMS providers offer assistance for OEMs in design and product development, in quality-assured and low-cost manufacturing, in access to global distribution network and in support services (Srihari and Vichare, 2001).

One kind of service that EMS companies provide for OEMs is the industrial repair service. These aftermarket services are developed to repair different types of electronic products, modules and parts. In this way, EMS providers offer assistance to OEMs to operate smoothly by providing spare parts for maintenance, to enhance the satisfaction of end users by enabling warranty repairs, etc. Aftermarket services are of outstanding importance from the viewpoint of end customers, which have necessitated the standardization and efficient modeling of these repair processes.

A company providing industrial repair is usually an independent organization without managing an own brand but having a wide range of customers by handling many of their products. Accordingly, electronic repair processes (ERP) are highly variable and the content of these processes is always changing. There are multiple demands on these processes. On the one hand, due to industrial requirements a certain level of standardization needs to be achieved. On the other hand, the applied tools and methods should be able to provide a simplified reflection of complex stochastic processes. Therefore, for planning the activities of ERPs along with their resource and time aspects, a quick, widely usable and easily variable tool is needed.

2.2 Electronic repair processes

Companies working in the repair industry, independently of that they are repairing items for customers or for their own operation, have to go through a three-step process when a product needs to be repaired (Zuo et al., 2000). To fix the problem on the level of the *product*, the failed component has to be replaced either with a new or with a previously repaired item. To fix the problem on the level of the *component*, a decision has to be made if the failed item can be (or should be) repaired or it is better to discard it. It has to be decided whether repair or replacement is economically justifiable. If repairing the item is reasonable, to fix the *problem itself*, the necessary and suitable type of repair has to be determined as well. These steps, however, are not independent of each other. As the third step can be repeated several times if the repair was unsuccessful, in a well-structured repair process, the second and third steps follow each other in all iterations. This paper focuses on this iterative characteristic of repair processes of electronic items.

In a cyclic repair process, the repair-or-replace decision has to be made several times and it depends on the type of the product, on the type of the failure and on the type of the repair activity how many attempts can be considered reasonable. At some point, however, even if the product is still malfunctioning, the process must be ended. To improve the operation of an electronic repair process, these decisions have to be made efficiently. For this purpose, models of ERPs can provide managers a useful tool.

The graph in Fig. 1 shows the simplified process flow of an ERP. It does not cover the entire product flow as we take the repair process in a narrowed sense, however, it can be used to introduce the most important properties of an ERP.

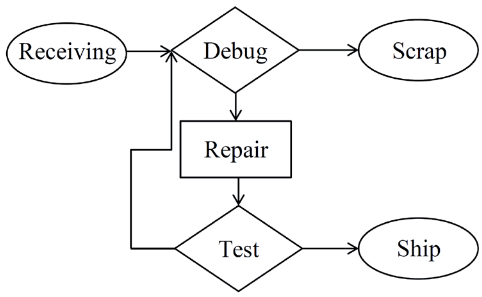

Fig. 1. Process flow of a simplified ERP

Every product that comes back from the end users (or customers) with the purpose to repair it enters the repair process in the *receiving* step. We assume that the products entering the repair process do not fulfill the pre-defined functional requirements. Certainly, it can happen in practice that a received unit functions in accordance with the technical and quality requirements. In these No Failure Found (NFF) cases, the product does not need to go through the repair process step. NFF cases are not presented in the example process in Fig. 1, but later on we will show that the introduced methodology allows identifying the likelihood of NFF cases as well.

The goal of the *debug* process step is to identify the causes of product failures, and decide what repair activities are needed to eliminate the defects that are causing the failures. As debugging requires the identification of failures, it typically contains certain functional and electrical tests that are not shown explicitly in Fig. 1. It is important to emphasize that debugging is a decision point. On the one hand, it is decided if repairing of the processed unit is technically possible and economically sensible; and on the other hand, the physical content of repair activities is determined here as well.

If the repair is justified, the suitable repair actions such as component replacements, de-soldering or re-soldering of electrical components are performed in the *repair* process step. Repair activities can be typically categorized into two large groups: modular repair and component level repair. Modular repair commonly means replacement of some modules in a product, while component level repair represents more technical activities such as replacement of passive and active electronic components on printed circuit board assemblies.

Repair activities are typically followed by various functional tests in a real electronic repair process. In our simplified process, the *test* step embodies a decision point at which the functionality of a repaired unit is checked. If the product works properly, it can be shipped. If there are failures still prohibiting its correct functioning, the unit goes back to the debug process step.

If the repair was successful and the product fulfills the functional requirements, it can be delivered to the customer. In this case, the product reaches the *ship* step. If no (more) attempt is made to repair the malfunctioning product, it reaches the *scrap* process step. We note that this simplified ERP does not look at the processes after the product gets shipped or scrapped, these two process steps are considered as the two possible outcomes of the process (a repaired unit or a scrapped item).

From economical perspective, the Receiving-Debug-Repair-Test-Ship path is the ideal one (the happy flow) as each process step is executed only once and, by the end of the process, the product gets repaired. In many cases, however, the process is not that smooth, it may include cycles. It can happen that after the debug, repair and test steps the product needs to go back to the debug step again, and so the process becomes *cyclic*. Having a circle in an ERP is not necessarily due to its inefficiency as in some cases stepwise and successive debug and repair activities are needed to discover and eliminate all the product defects. Therefore, the circles in the process flow of an ERP can be considered as its property that comes from the nature of the process itself.

As from the description it can be seen, the sequence of process steps that a product goes through in an ERP is much less determined than in a classical manufacturing system. One process step can be followed by multiple other steps, so repairing can be considered a routing problem. From this point of view, an ERP operates like a Flexible Manufacturing System (FMS), and, for efficient operation, similar decisions have to be made in the two systems. Determine the capacity of an FMS is essential to make well-grounded managerial decisions (Koltai and Stecke, 2008; Sebestyén and Juhász, 2003; Sawik, 1993). Similarly, one objective of modeling ERPs is to compute the capacity of the process and its utilization. For this, the load of each process step has to be calculated or estimated. In the following, the load of a process step is expressed in terms of time so it is defined as the product of the processing time of the step and the probability of visiting the given step. Determine these values, however, is a complicated task. In an ERP, not only the sequence of the process steps is undefined, but also the circulation of the process steps is possible. It means that an ERP can be considered as a *stochastic process* in which the product goes from each process step to other ones with certain likelihoods. Moreover, ERPs are stochastic in the manner that the content and processing time of some process steps such as debug, repair and test can largely depend on the product failures. Calculating these likelihoods and process times, however complicated it is, allows determining the load of each process step. And knowing the loads makes it possible to justify the amount of different resources needed to perform a given ERP and to support other similar managerial decisions.

3 Methodology

3.1 Electronic repair processes as absorbing Markov chains

Looking at the ERP in Fig. 1 from the perspective of the product, a run of the process is a sequence of product states, where the possible states are the process steps (s_i (i = 1,2,3, ···)). This sequence of states (process steps) can be modeled as a sequence of random variables $\xi_1,\xi_2,\xi_3\cdots$ where $\xi_i \in \mathbf{S}$ (i = 1,2,3, ···) and **S** is the set of possible states. (Notations used in the paper are listed in Table 1.) The likelihood that the process is in state s_j in step n depends only on the state s_i where the process was in step (n-1) $\left(s_i, s_j \in \mathbf{S}\right)$.

Based on this characteristic, ERPs can be considered as Markov chains and the set of possible process steps corresponds with the state space of the Markov chain. Using this approach our sample ERP can be modeled as an *absorbing Markov chain* with two absorbing states (ship, scrap) as it can be seen in Fig. 2.

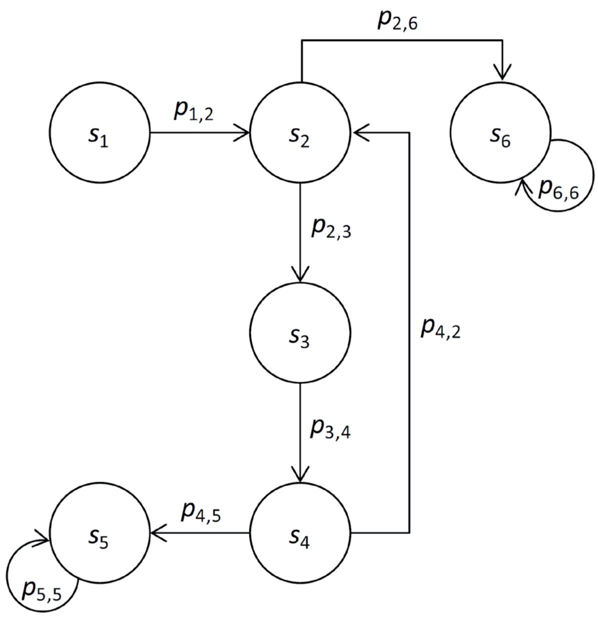

Fig. 2. The simplified ERP as an absorbing Markov chain (s_1 -receiving, s_2-debug, s_3-repair, s_4- test, s_5- ship, s_6- scrap)

Tab. 1. Notation used in the paper

Process parameters		
s_i	–	node i of the graph, step i of the process,
S	–	set of possible process steps
$e_{i,j}$	–	directed edge from s_i to s_j (s_i and s_j are neighboring states)
$R^n_{i,j}$	–	path n from process step i to process step j (consisting directed edges)
$\mathbf{R}_{i,j}$	–	set of all paths from process step i to process step j
$p_{i,j}$	–	the likelihood that the process state changes in one step from s_i to s_j (transition probability)
P	–	transition probability matrix
Operational parameters		
$L(s_i)$	–	the load of process step s_i
$P\left(s_j^{(k)}\right)$	–	the likelihood of the k th visit to state s_j
$\mu(s_i)$	–	the process multiplier of process step s_i
$T(s_i)$	–	the processing time of process step s_i
$\mathbf{T_S}$	–	set of processing times
$L[Proc(\mathbf{S}, \mathbf{P}, \mathbf{T_S})]$	–	the total load of the electronic repair process with parameters **S**, **P**, $\mathbf{T_S}$
$T\left(R^n_{i,j}\right)$	–	path lead time of path n
$P\left(R^n_{i,j}\right)$	–	path probability of path n

The directed graph in Fig. 2 is the formal model of our sample ERP. This graph represents an absorbing Markov chain with the following notations and properties.

The nodes of the graph are the possible process states (process steps),

$$\mathbf{S} = \{s_1, s_2, s_3, s_4, s_5, s_6\}. \tag{1}$$

The state transition probability $p_{i,j}$ $(i, j = 1,2, \cdots ,6)$ specifies the likelihood that the process state changes from s_i to s_j in one step. Each edge of the process graph in Fig. 2 is labeled with the corresponding transition probability. These probabilities are considered being time independent, so this model is a *time homogeneous Markov chain*. The state transition probability matrix **P** of the model in Fig. 2 is

$$\mathrm{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{2,3} & 0 & 0 & p_{2,6} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & p_{4,2} & 0 & 0 & p_{4,5} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where $p_{2,3}$, $p_{2,6}$, $p_{4,5}$, $p_{4,6}$, are transition probabilities.

Based on the properties of Markov chains, the sum of transition probabilities from a state to all the other states is 1 for each state, that is,

$$\sum_{j=1}^{6} p_{i,j} = 1 \quad (i = 1,2,\ldots,6). \tag{3}$$

State s_1 is the start state of the process, that is, the initial distribution of the Markov chain is

$$P(\xi_1 = s_i) = \begin{cases} 1, & i = 1 \\ 0, & i = 2,3,\cdots,6 \end{cases} \tag{4}$$

and only state s_2 is connected with s_1, therefore $p_{1,2} = 1$.

Once the process reaches one of the s_5 and s_6 states, it remains there, and the process ends. Therefore, the nodes (states) s_5 and s_6 are absorbing ones, that is, $p_{5,5} = p_{6,6} = 1$ and so $p_{5,j} = 0$ for any $j \neq 5$, and $p_{6,k} = 0$ for any $k \neq 6$.

3.1.1 Main operational parameters

As mentioned before, one objective of this study is to determine the load of each process step in an ERP. Assuming that an ERP is given by its set of process states **S** and its state transition probability matrix **P**, we are about to calculate the

$$L(s_i) = \mu(s_i) T(s_i) \tag{5}$$

quantity for each $s_i \in \mathbf{S}$ state, where $L(s_i)$ is the *load*, $\mu(s_i)$ is the *process multiplier*, and $T(s_i)$ is the *processing time* of process step s_i. The $\mu(s_i)$ process multiplier represents the *number of expected visits* to s_i in the Markov chain, that is, practically, it expresses how many times the products go through process step s_i on average until they reach one of the end steps of the process (theoretically $0<\mu(s_i)<\infty$). The $T(s_i)$ quantity represents the time that a single execution of process step s_i requires on average. For example, if the processing time of the repair process step is 10 minutes, and the process multiplier of this step is 0.8, then the load of it is 0.8·10 = 8 minutes. It means that 8 minutes of repair activities are spent on a product on average until the product gets either repaired or scrapped.

From this point onwards, we use the *Proc* (**S, P, T**$_s$) notation to denote an electronic repair process that has the set of process steps (states) **S**, state transition probability matrix **P**, and set of processing times $\mathbf{T_s}$, where $\mathbf{T_S} = \{T(s_i) | s_i \in \mathbf{S}\}$. Knowing **S** and **P** allows us to calculate the $\mu(s_i)$ process multipliers, and knowing the process multipliers and the $T(s_i)$ processing times, the $L(s_i)$ process loads can be determined as well. It means that **S**, **P** and $\mathbf{T_s}$ unambiguously define the ERP and, consequently, the load of each process step.

The $L[Proc(\mathbf{S}, \mathbf{P}, \mathbf{T_S})]$ total load of the ERP can be calculated as

$$L\left[Proc(\mathbf{S}, \mathbf{P}, \mathbf{T_S})\right] = \sum_{s_i \in \mathbf{S}} \mu(s_i) T(s_i). \tag{6}$$

This total load, in practice, is known as the Minutes Per Unit (MPU) or Hours Per Unit (HPU) metric.

3.1.2 Calculation of process multipliers

Knowing the $P(\xi_1 = s_j)$ initial distribution of the Markov chain that represents an ERP, the $P(\xi_n = s_j)$ steady-state probability, the likelihood that the process is in its $s_j \in \mathbf{S}$ state in step n can be calculated as

$$P(\xi_n = s_j) = \sum_{s_i \in \mathbf{S}} P(\xi_{n-1} = s_i) p_{i,j}, \tag{7}$$

and the process multiplier $\mu(s_j)$ of s_j is

$$\mu(s_j) = \sum_{n=1}^{\infty} P(\xi_n = s_j). \tag{8}$$

For example, the process multiplier of s_2 in the sample ERP in Fig. 2 can be calculated as follows. Let $P\left(s_j^{(k)}\right)$ represent the likelihood of the k th visit to state s_j. Using this notation

$$\begin{aligned} \mu(s_2) &= \sum_{k=1}^{\infty} P\left(s_2^{(k)}\right) = \sum_{k=1}^{\infty} p_{1,2} \left(p_{2,3} p_{3,4} p_{4,2}\right)^{k-1} \\ &= p_{1,2} \sum_{k=0}^{\infty} \left(p_{2,3} p_{3,4} p_{4,2}\right)^{k} = \\ &= p_{1,2} \lim_{k \to \infty} \frac{\left(p_{2,3} p_{3,4} p_{4,2}\right)^{k} - 1}{p_{2,3} p_{3,4} p_{4,2} - 1} \\ &= p_{1,2} \frac{1}{1 - p_{2,3} p_{3,4} p_{4,2}}. \end{aligned} \tag{9}$$

As $p_{1,2} = p_{3,4} = 1$,

$$\mu(s_2) = \frac{1}{1 - p_{2,3}p_{4,2}}. \tag{10}$$

Applying the same approach, the process multiplier of s_4 is

$$\mu(s_4) = p_{2,3}\mu(s_2) = p_{2,3}\frac{1}{1 - p_{2,3}p_{4,2}}. \tag{11}$$

From these results, it can be seen that $\mu(s_2) > 1$ while $\mu(s_4) > 0$, which corresponds to the process in Fig. 1 (the debug state cannot be omitted while NFF products do not need repair and therefore test).

As it can be seen from the results above, to determine the process multiplier of the process step, only statistical data, transition probabilities are needed. These values can easily be collected in the case of an existing process or can be estimated in the case of planning. Knowing the values of process multipliers and the processing time of each process step, the load of the ERP can be easily calculated. This information can support for example the planning of a process and its capacity requirements or the evaluation of process efficiency.

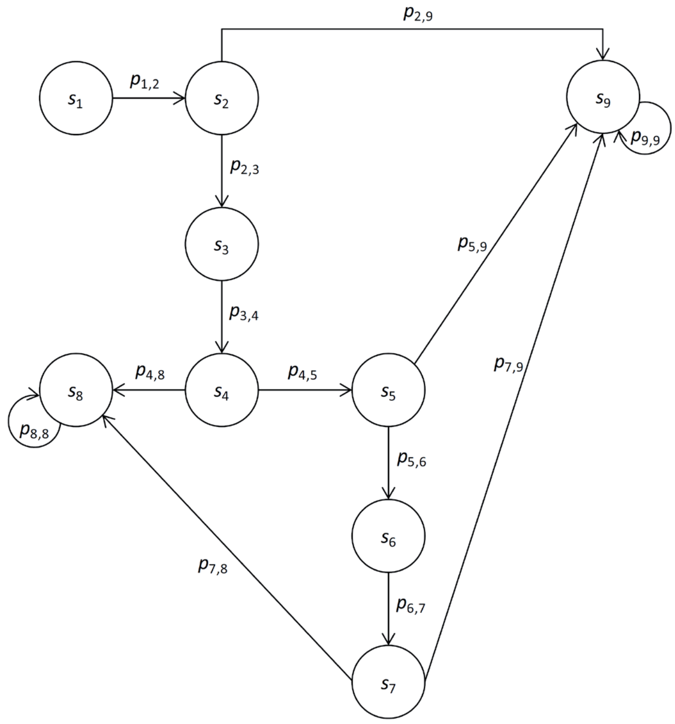

Fig. 3. The simplified ERP as an acyclic absorbing Markov chain (s_1 - receiving, s_2 - 1st debug, s_3 - 1st repair, s_4 - 1st test, s_5 - 2nd repair, s_7 - 2nd test, s_8-ship, s_9 - scrap

3.2 Electronic repair processes as acyclic absorbing Markov chains

When we looked at some specific electronic repair processes and took some certain economic aspects into consideration as well, we found three issues that have to be addressed for the sake of the practical aspects of modeling and analysis.

1) In an absorbing Markov chain model, the Markov chain may include circles and it theoretically allows an *infinite number of circulations* among certain process states (e.g. the $[s_2, s_3, s_4],[s_2, s_3, s_4], \cdots ,[s_2, s_3, s_4] \cdots$ infinite circulation is theoretically possible in the Markov chain in Fig. 2). Practically, based on economic considerations, only finite number of circulations is allowed, that is, during one run of the process until it reaches one of the absorbing states, the number of visits to any state of the process is limited. In practice, the maximum number of such circulations is typically as low as 3, 4 or 5.

2) The *state transition probabilities are time independent* in the time homogeneous absorbing Markov chain model, that is, regardless how many times the transition from state s_i to state s_j happens during a run of the process, the likelihood of such a transition is constantly $p_{i,j}$. In practice, the physical conditions of the product being repaired often are changing due to the different actions performed on it. As the likelihood of process transition from a state to another one can be influenced by the product conditions, the state transition probabilities can be time dependent. The presented absorbing Markov chain cannot consider this property of the process by assuming time homogeneity of the transition probabilities.

3) *The processing time is a time independent value* in the presented absorbing Markov chain model for each process step. In reality, there are often differences, for example, between the technical contents of first, second, third, etc. debugging activities on the same product, and so their processing times may differ as well. This conclusion suggests that the first, second, third, etc. processing times of the same states of the process should be differentiated.

If we modify the approach of the absorbing Markov chain model so that we consider the first, second, third, etc. executions of the same process as different process steps, then, except the absorbing loops, the graph of an ERP can be made circuit free. For example, if maximum two s_2, s_3, s_4, s_2 circulations are allowed in the process in Fig. 2, then this "limited" process can be represented by the graph in Fig. 3.

When the process is considered as an acyclic absorbing Markov chain, the expected number of visits to each state is equal to the probability that the process visits that state. For the s_i state, this probability is denoted with $P(s_i)$, and in this representation, it takes over the role of the $\mu(s_i)$ process multiplier.

3.2.1 Generic model with acyclic absorbing Markov chains

The same idea that was applied for the sample ERP to make its Markov chain representation circuit free can be applied in general to any ERP. The acyclic absorbing Markov chain model of *Proc* (**S, P, T$_s$**) with start state s_i , and end states s_{n-1} , s_n (with several other the states among the start state and the end states) has the following properties:

i.) The set of process states $\mathbf{S}$ is

$$\mathbf{S} = \{s_1, s_2, \cdots, s_{n-1}, s_n\}, \tag{12}$$

where s_1 is the start, and s_{n-1}, s_n are the absorbing end states and by this means

$$P(s_1) = 1, \tag{13}$$

and as the process finally reaches either s_{n-1} or s_n

$$P(s_{n-1}) + P(s_n) = 1. \tag{14}$$

ii.) The probability $P(s_j)$ that the process reaches state s_j is

$$P(s_j) = \sum_{s_i \in \mathbf{S}} P(s_i)\, p_{i,j}. \tag{15}$$

iii.)The sum of transition probabilities from any state $s_i \in \mathbf{S}$ is 1:

$$\sum_{s_j \in \mathbf{S}} p_{i,j} = 1 \tag{16}$$

iv.) For the absorbing end states s_{n-1} and s_n:

$$p_{n-1,n-1} = p_{n,n} = 1. \tag{17}$$

v.) The chain is acyclic, that is, for any $s_{i_1}, s_{i_2}, \cdots, s_{i_k}$ path $s_{i_j} \neq s_{i_l}$ for any $j \neq l, (i_1, i_2, \ldots, i_k \in \{1,2,\ldots,n\}; j,l \in \{1,2,\ldots,k\})$.

3.2.2 Calculating the descriptive statistics of lead time

In the acyclic absorbing Markov chain model of an ERP, there is a finite number of directed paths from the start state to the absorbing end states. For each of these paths, the sum of processing times of states on the path (path lead time), and the product of transition likelihoods on edges of the path (path probability) can be calculated. Let $e_{i,j}$ denote the directed edge from state s_i to s_j, and let us assume that there are l different paths from s_i to s_{n-1},

$$\mathbf{R}_{1,n-1} = \left\{R^1_{1,n-1}, R^2_{1,n-1}, \ldots, R^l_{1,n-1}\right\}$$

and m different paths from s_i to s_n

$$\mathbf{R}_{1,n} = \left\{R^1_{1,n}, R^2_{1,n}, \ldots, R^m_{1,n}\right\}$$

As any product reaches one of the absorbing final states, there are l+m possible paths in the process.

For each $R^i_{1,n-1}$ and $R^j_{1,n}$ path, the lead time of the path is defined as

$$T\left(R^i_{1,n-1}\right) = \sum_{s \in R^i_{1,n-1}} T(s), \tag{18}$$

and

$$T\left(R^j_{1,n}\right) = \sum_{s \in R^j_{1,n}} T(s), \tag{19}$$

respectively, and the $P\left[R^i_{1,n-1}\right]$ and $P\left[R^j_{1,n}\right]$ likelihood of each $R^i_{1,n-1}$ and $R^j_{1,n}$ path is calculated as

$$P\left(R^i_{1,n-1}\right) = \prod_{e_{a,b} \in R^i_{1,n-1}} p_{a,b}, \tag{20}$$

and

$$P\left(R^j_{1,n}\right) = \prod_{e_{a,b} \in R^j_{1,n}} p_{a,b}, \tag{21}$$

respectively (i =1,2, ⋯ , l ; j = 1,2, ⋯ , m).

Let us define the L^* [*Proc* (**S, P, T**$_\mathbf{s}$)] quantity as

$$L^*\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right] = \sum_{i=1}^{l} P\left(R^i_{1,n-1}\right) T\left(R^i_{1,n-1}\right) + \sum_{j=1}^{m} P\left(R^j_{1,n}\right) T\left(R^j_{1,n}\right). \tag{22}$$

The L^* [*Proc* (**S, P, T**$_\mathbf{s}$)] is calculated as the sum of products of processing time and probability of each path from s_1 to s_{n-1} and from s_1 to s_n. Comparing (6) and (22), it can be seen that $P(s_i)$ in the acyclic absorbing Markov chain model has the same role as $\mu(s_i)$ in the absorbing Markov chain model. As the data collection for process multipliers of every process step is demanding and the data of the process paths are more meaningful for managers, it is a useful finding that the L^* [*Proc* (**S, P, T**$_\mathbf{s}$)] quantity of an ERP in (22) is equal to the total load calculated in (6) (for the proof of the this theorem, see Appendix), that is,

$$L^*\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right] = L\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right]. \tag{23}$$

This means that, to calculate the load of an ERP, only the lead times and the likelihood of the different paths are needed which can be determined using the transition probabilities and the processing times of the different activities.

With this method, not only the operation of the process can be studied, but the characteristics of the operational parameters can be analyzed as well. Let us assume that each path from s_1 to the absorbing end states s_{n-1} and s_n has different lead time. As the process follows each path with a certain probability, *the lead time can be considered as a random variable*. By calculating the probability and the processing time of each path, the *probability distribution of the lead time* can be determined. It can be seen that the quantity calculated in (22) is the *expected value of the lead time*. Practically, L^* [*Proc* (**S, P, T**$_\mathbf{s}$)] represents the average load of the ERP. Using the T_1, T_2, ⋯ , T_{1+m} and p_1, p_2, ⋯ , p_{1+m} notations for the lead times and their probabilities, the expected value of the lead time can be written as

$$E(T) = \sum_{i=1}^{l+m} p_i T_i. \tag{24}$$

If there are paths with the same lead time, then their probabilities can be simply summarized and the paths can be considered as a single one. In this case, we differentiate between the lead times of the paths, not between the paths themselves, and the $E(T)$ expected value of lead time can be formally written as

$$E(T) = \sum_{i=1}^{N} p_i T_i, \tag{25}$$

where $T_1, T_2, \cdots, T_N$ are the different lead times, and $p_1, p_2, \cdots, p_N$ are their likelihoods.

The standard deviation of lead time can be calculated as

$$D(T) = \sqrt{\sum_{i=1}^{N} \left[T_i - E(T)\right]^2 p_i}. \tag{26}$$

As it can be seen from these findings, modeling electronic repair processes with acyclic absorbing Markov chains makes possible to analyze the average load of the whole process and of the individual activities and also allows to determine the main descriptive statistics of the lead time of the repair process as well. These parameters can support several managerial decisions as it will be shown in the following sections.

4 Industrial application

In the following, we will demonstrate the application of the presented method with the help of a real repair process of personal computers. In this process, after *receiving* the items to repair, they are *visually inspected* (VI). This way, the physical and cosmetic conditions of the defective product are checked. If its condition is really poor, the item is *scrapped*. If the results of the inspection are acceptable, a *quick test* (QT) is performed to check the level of the core product functionalities. The test can show that the core functions are satisfactory, and in these NFF cases, there is no need to repair, a *functional test* (FT) is performed. If the quick test shows malfunctions, a *debug* process is started to identify the causes of product failures and to determine the repair activities needed to recover the product. Based on the results of debugging, a *software upgrade* (SU) or a special type of *repair* (L2, L3, L4) is performed. Level 2 repairs are modular repair activities during which complete modules of a product are replaced. Level 3 repairs represent component level repair activities; typically some passive electronic components or connectors are replaced during this step. Level 4 repairs mean the replacement of active components such as integrated circuits with different packaging types. If the replacement of a passive (L3) or active component (L4) is unsuccessful, the item is *scrapped*; in any other cases a *functional test* (FT) is performed. This test is applied to verify that the product is recovered and able to function according to the specifications. If the test is passed, the item is *packed* and *shipped*. If the functional test shows problems, another cycle starts with a new debugging. In a second cycle, a software upload cannot solve the problem, so one of the three types of repairs is needed. If a third cycle is necessary, only complete modules can be replaced (Level 2 repair). The ERP introduced above is shown in Fig. 4 as an acyclic absorbing Markov chain.

In Fig. 4, each node of the process graph is identified with the notation of its name and a number. The number indicates the number of the cycle in which the activity is performed. This notation allows us to differentiate between the first, second, and third execution of the same process activity, e.g. (Debug, 1) is the first time debug and (Debug, 2) is the second time debug.

Table 2 shows the transition probability matrix of the Markov chain. (Please note that the empty cells indicate zero probabilities.) These transition probabilities are approximated by relative frequencies based on the database generated from 1300 products going through the process indicated in Fig. 4.

As it can be seen from the process graph in Fig. 4 and from the transition probability matrix in Table 2, the start state of the process is the receiving step, and its absorbing (terminal) states are the ship and scrap process steps. However, the process model seems to be relatively simple, there are 84 possible directed paths from the receiving state to one of the absorbing states.

With the help of the formulae introduced in Section 3.2.1, the lead time of each path and their probabilities were calculated. Table 3 shows the process paths with the highest probabilities, their lead time and their probability. The states in each path are separated by the "|" character.

The lead time of a path is calculated as the sum of processing times, and the probability of a path equals the product of process step probabilities. The calculation of the first path's lead time:

Lead time (Path 1) = 0.9+2.5+5.7+12.2+3.1+6.3+2.1+6.2=39

Probability (Path 1) = 1·0.98·0.73·0.5·0.92·1=0.3291

The sum of the highest five probabilities (shown in Table 2) is 0.8695, that is, these five paths – out of the possible 84 – are traveled by almost 87% of the products arriving for repair.

Based on the probabilities of the different paths, it can be determined that 92.80% of the products leave the process in the first cycle, 5.67% in the second cycle and only 1.53% enters the third cycle. The probability of a successful repair is the highest in the first cycle. 94.97% of the repairable items can be repaired at the first attempt, which means an 89.17% success rate among the items leaving the process in the first cycle. The second cycle increases the success probability by 4.52%, which means that the repairs in this cycle are successful for 69.51% of the products leaving the process in the second cycle. The third cycle increases the success rate only by 0.51% because in this cycle the repairs are not frequently successful (28.80%). In the analyzed ERP, only 87.13% of the received products can be repaired within 3 cycles, but it can be seen that more cycles cannot significantly increase this ratio. It means that there is no need for more than 3 cycles, and the low success rate of the third cycle doubts the justification of 3 cycles. With more thorough researches, it can be decided whether 2 or 3 is a better repair limit in this process.

As we discussed earlier, the average lead time, i.e. the load of the process, can be calculated as the sum of products of processing

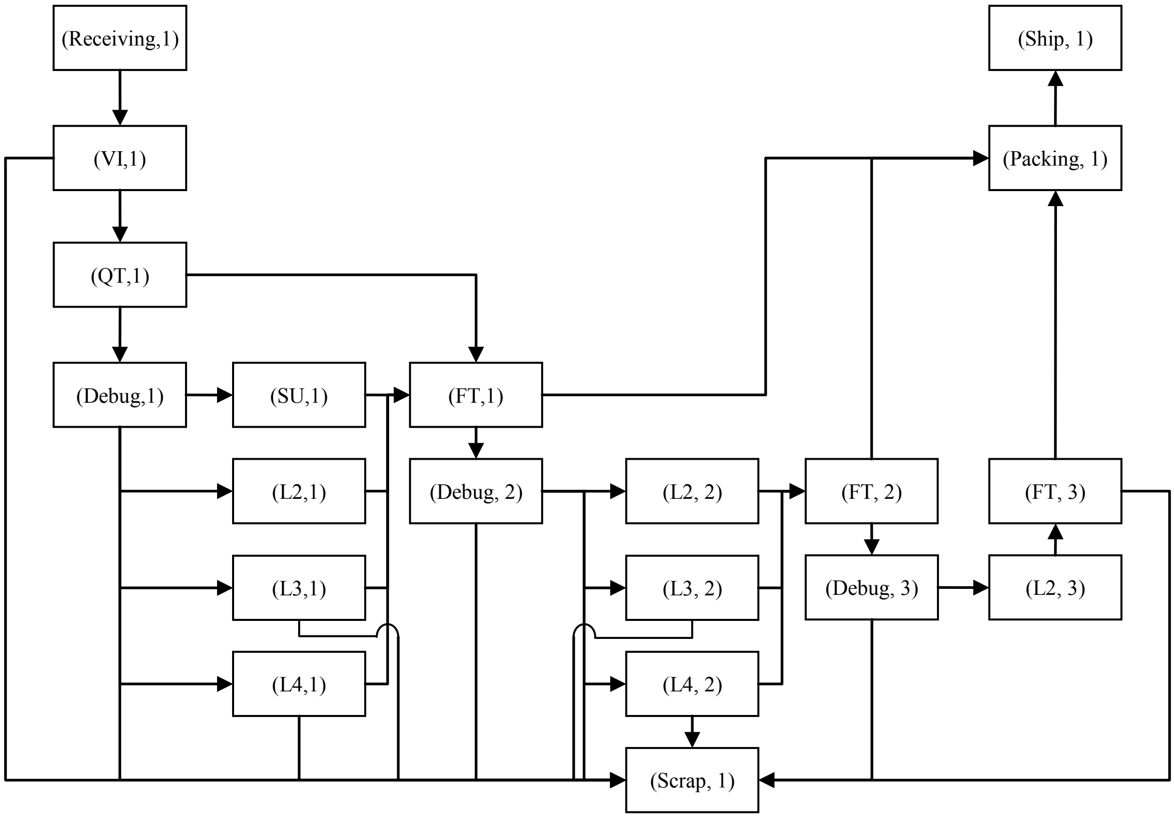

Fig. 4. Acyclic directed graph of an ERP
(VI - Visual Inspection, QT - Quick Test, SU - Software Upgrade, L2 - Level 2 Repair, L3 - Level 3 Repair, L4 - Level 4 Repair, FT - Functional Test

Tab. 2. Transition probability matrix

	(Receiving, 1)	(VI, 1)	(QT, 1)	(Debug, 1)	(SU, 1)	(L2, 1)	(L3, 1)	(L4, 1)	(FT, 1)	(Scrap, 1)	(Packing, 1)	(Ship, 1)	(Debug, 2)	(L2, 2)	(L3, 2)	(L4, 2)	(FT, 2)	(Debug, 3)	(FT, 3)	(L2, 3)
(Receiving, 1)		1																		
(VI, 1)			0.98							0.02										
(QT, 1)				0.73					0.27											
(Debug, 1)					0.5	0.2	0.15	0.05		0.1										
(SU, 1)									1											
(L2, 1)									1											
(L3, 1)									0.95	0.05										
(L4, 1)									0.9	0.1										
(FT, 1)											0.92		0.08							
(Scrap, 1)										1										
(Packing, 1)												1								
(Ship, 1)												1								
(Debug, 2)										0.2				0.4	0.25	0.15				
(L2, 2)																	1			
(L3, 2)										0.1							0.9			
(L4, 2)										0.1							0.9			
(FT, 2)											0.72							0.28		
(Debug, 3)										0.4										0.6
(FT, 3)										0.52	0.48									
(L2, 3)																			1	

Tab. 3. Process paths with the highest probabilities

Path	Lead time	Probability
(Receiving, 1)|(VI, 1)|(QT, 1)|(Debug, 1)|(SU, 1)|(FT, 1)|(Packing, 1)|(Ship, 1)	39	0.3291
(Receiving, 1)|(VI, 1)|(QT, 1)|(FT, 1)|(Packing, 1)|(Ship, 1)	23.7	0.2434
(Receiving, 1)|(VI, 1)|(QT, 1)|(Debug, 1)|(L2, 1)|(FT, 1)|(Packing, 1)|(Ship, 1)	41.1	0.1316
(Receiving, 1)|(VI, 1)|(QT, 1)|(Debug, 1)|(L3, 1)|(FT, 1)|(Packing, 1)|(Ship, 1)	40.6	0.0938
(Receiving, 1)|(VI, 1)|(QT, 1)|(Debug, 1)|(Scrap, 1)	24.3	0.0715

times and probabilities of each possible path. In our case, the total load is 35.44 minutes. The load of the five paths with the highest probabilities is 29.56 minutes, 83.41% of the total load. This information about the process allows the process owners to consider these five paths as dominant ones.

In the analyzed process, there are many paths with the same lead time. The 84 different paths result in 76 different lead times. If we group the paths by their lead times, that is, if a lead time is presented multiple times, their likelihoods are summed, then, as discussed earlier, the probability distribution of the lead time is determined. From the probability distribution, it can be determined that the lead time has relatively large variance; the relative standard deviation is 30.49%. This and the long-tailed characteristic of the lead time distribution are unfavorable for process management and for quality control tasks. The longest lead times, obviously, occur when a third cycle is started; therefore, elimination of these cases could be a reasonable and feasible management objective. It also has to be taken into consideration, however, that the longest times are related to successful repairs.

5 Managerial implications and future research directions

With the help of the presented method, any electronic repairing process (and any other similar process) can be modeled and analyzed. Based on the results of the analyses, several related managerial decisions can be supported.

Modeling repair, manufacturing and business processes as acyclic absorbing Markov chains can ground for many *process management* activities. These analyses enable managers to determine the probability distribution of lead time of any repairing process, which can support several managerial decisions. In practice, ERPs are typically complex processes with a high number of process states and several possible paths from the start state to the absorbing end states. In such a complex process, it is far not obvious which paths are contributing the most to the total load of the process. Using the presented approach, the likelihood and the lead time of each possible path can be determined, and, as the product of these two, the path load and its contribution to the total load of the process can be calculated. The path with the highest load in the analyzed process identifies the potential focus areas for process improvement.

In view of the descriptive statistics, the expected value and the standard deviation of the lead times allows monitoring the real process against these figures. In this way, if the total load of the process is changing over time, the potential causes of the observable shift can be identified. The change of the total load can be caused by variations in certain processing times and in transition likelihoods as well. Through the identification of the possible paths and their loads, managers can conclude on the cause(s). For quality control purposes, however, not only statistical measures matter. Supposing that electric failures follow a fixed pattern and the load of the graph is constant in time, the changing of dominant paths can refer to internal problems and inner inefficiency. These changes, without a suitable modeling method, can easily escape the managers' attention.

Beside of supporting process controlling and process improvement activities, the results of the presented modeling technique contribute to several other managerial considerations as well.

Identifying bottleneck paths and bottleneck activities in a process can ground for *resource allocation* efforts. By exploring the dominant paths of the graph that contribute to the total load of the process to the highest degree, the research results can help the allocation of resources. It identifies the dominant paths of the graph that should be rearranged, and the resources that are to be reallocated. As the presented repair process showed, different activities generally have different loads; moreover, these differences can be intensified by the iterative characteristics of the process (e.g. only Level 2 repair can happen in all iterations). In this way, resource allocation should be based not only on the load of the different paths but on the loads of different activities as well. Analyses based on the presented model can support this type of decisions as well.

Another important characteristic of the analyzed ERPs is that they are low automatized; the most important resource used is human labor. By analyzing the load of the different activities, not only the demand for living labor can be specified, but *training plans* can be formulated as well. These plans can be used in common work circumstances to enhance workforce skills and the type of necessary trainings can also be defined if, for example, the bottleneck of the process changes.

Economic considerations can be drawn as well, if the analyses are complemented with cost data. In this way, based on the research results, the number of economically justifiable repeats can be determined. In the analyzed repair process, for example, 92.80% of the items do not enter the second cycle. Further cycles do not contribute notably to the results, and this finding complemented with cost data can serve as base for modeling the costs of repeats. By drawing a cost line, the number of economical cycles and repeats can be estimated. Based on this approach, a general method can be developed for evaluating service activities performed within the company and for pricing repair services performed for other companies.

In our following studies, we plan to broaden the horizons of the researches with the mentioned economic aspects and try to make general observations about the operations, problems and pricing of electronic repair processes. Accordingly, modeling the costs of repeats, defining a general method for pricing repair services and determining the economically optimal cycle limit is the topic of our further researches. By complementing our model and research method with suitable sensitivity analyses, we aim at drawing general conclusions and completing the above mentioned management areas with economic aspects (e.g. to analyze the effects of possible cost decreases and process changes).

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Appendix

Theorem

$$L^*\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right]=L\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right]. \tag{27}$$

Proof

Let s_A be an arbitrarily chosen state of the process. The coefficient of $T(s_A)$ in L [$Proc$ (**S, P, T**$_s$)] is P (s_A) Let us assume that the ode s_A is on the

$$R_{1,n-1}^{i_1},R_{1,n-1}^{i_2},\cdots,R_{1,n-1}^{i_q}$$

and

$$R_{1,n}^{j_1},R_{1,n}^{j_2},\cdots,R_{1,n}^{j_r}$$

paths $\left(1\le q\le l,\, 1\le r\le m\right)$.

Based on (18) and (19), L^* [$Proc$ (**S, P, T**$_s$)] can be expanded to

$$L^*\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_s}\right)\right]$$
$$=\sum_{i=1}^{l}\left\{P\left(R_{1,n-1}^{i}\right)\sum_{s\in R_{1,n-1}^{i}}T\left(s\right)\right\}+\sum_{j=1}^{m}\left\{P\left(R_{1,n}^{j}\right)\sum_{s\in R_{1,n}^{j}}T\left(s\right)\right\}. \tag{28}$$

From this form, it can be seen that the coefficient of $T(s_A)$ in L^* [$Proc$ (**S, P, T**$_s$)] is

$$\sum_{u=1}^{q}P\left(R_{1,n-1}^{i_u}\right)+\sum_{v=1}^{r}P\left(R_{1,n}^{j_v}\right). \tag{29}$$

Each path from s_1 to s_{n-1} containing the state s_A can be decomposed into two sub-paths: one from s_1 to s_A, and one from s_A to s_{n-1}. Let us assume that

$$\mathbf{R}_{1,A}=\left\{R_{1,A}^{1},R_{1,A}^{2},\cdots,R_{1,A}^{q_1}\right\},$$

are all the paths from s_1 to s_A and

$$\mathbf{R}_{A,n-1}=\left\{R_{A,n-1}^{1},R_{A,n-1}^{2},\cdots,R_{A,n-1}^{q_2}\right\}$$

are all the paths from s_A to s_{n-1}. Certainly, $q_1q_2=q$ Using this decomposition and (20) and (21), any of the $P\left(R_{1,n-1}^{i_u}\right)$ probabilities can be written as the product of two suitable probabilities, the probability $P\left(R_{1,A}^{g}\right)$ of the corresponding sub-path $R_{1,A}^{g}$ from s_1 to s_A, and the probability $P\left(R_{A,n-1}^{h}\right)$ of the corresponding sub-path $R_{A,n-1}^{h}$ from s_A to s_{n-1}:

$$P\left(R_{1,n-1}^{i_u}\right)=P\left(R_{1,A}^{g}\right)P\left(R_{A,n-1}^{h}\right) \tag{30}$$

where $1\le u\le q, 1\le g\le q_1, 1\le h\le q_2$.

For one $R_{A,n-1}^{h}$ sub-path, there are q_1 sub-paths from s_1 to s_A and q_2 sub-paths from s_A to s_{n-1}, and so the

$$\sum_{u=1}^{q}P\left(R_{1,n-1}^{i_u}\right) \tag{31}$$

sum in (29) can be written as

$$\begin{aligned}\sum_{u=1}^{q}P\left(R_{1,n-1}^{i_u}\right)&=\sum_{h=1}^{q_2}\sum_{g=1}^{q_1}P\left(R_{1,A}^{g}\right)P\left(R_{A,n-1}^{h}\right)\\&=\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)\sum_{g=1}^{q_1}P\left(R_{1,A}^{g}\right)\\&=\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)P\left(s_A\right)\\&=P\left(s_A\right)\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)\end{aligned} \tag{32}$$

because

$$\sum_{g=1}^{q_1}P\left(R_{1,A}^{g}\right)=P\left(s_A\right). \tag{33}$$

Following a similar way of thinking for the paths from s_1 to s_n containing the state s_A the

$$\sum_{v=1}^{r}P\left(R_{1,n}^{j_v}\right) \tag{34}$$

sum in (29) can be written as

$$\sum_{v=1}^{r}P\left(R_{1,n}^{j_v}\right)=P\left(s_A\right)\sum_{h=1}^{r_2}P\left(R_{A,n}^{h}\right), \tag{35}$$

where

$$\mathbf{R}_{A,n}=\left\{R_{A,n}^{1},R_{A,n}^{2},\cdots,R_{A,n}^{r_2}\right\}$$

are all the paths from s_A to s_n.

Based on (32) and (35), the coefficient of $T(s_A)$ in formula (29) of L^* [$Proc$ (**S, P, T**$_s$)] can be written as

$$\begin{aligned}&\sum_{u=1}^{p}P\left(R_{1,n-1}^{i_u}\right)+\sum_{v=1}^{q}P\left(R_{1,n}^{j_v}\right)\\&=P\left(s_A\right)\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)+P\left(s_A\right)\sum_{h=1}^{r_2}P\left(R_{A,n}^{h}\right)\\&=P\left(s_A\right)\left\{\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)+\sum_{h=1}^{r_2}P\left(R_{A,n}^{h}\right)\right\}.\end{aligned} \tag{36}$$

As

$$\sum_{h=1}^{q_2}P\left(R_{A,n-1}^{h}\right)+\sum_{h=1}^{r_2}P\left(R_{A,n}^{h}\right) \tag{37}$$

is the sum of probabilities of all possible paths from s_A to the absorbing end states s_{n-1} and s_n, this sum is equal to 1, and by this means the coefficient of $T(s_A)$ in L^* [$Proc$ (**S, P, T**$_s$)] in formula (38) is $P(s_A)$. It means that the coefficient of $T(s_A)$ is the same in L [$Proc$ (**S, P, T**$_s$)] and L^* [$Proc$ (**S, P, T**$_s$)] , and since s_A is an arbitrary chosen state of the ERP, the coefficients of processing time of any state in L [$Proc$ (**S, P, T**$_s$)] and L^* [$Proc$ (**S, P, T**$_s$)]are equal, which proves that

$$L^*\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right]=L\left[Proc\left(\mathbf{S},\mathbf{P},\mathbf{T_S}\right)\right]. \tag{38}$$