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RESEARCH ARTICLE

On dynamics of the track/vehicle system in presence of inhomogeneous rail supporting parameters

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Abstract

We study the dynamics of the train/track system in case of an inhomogeneous longitudinal subgrade stiffness/damping distribution. Our model consists of a Bernoulli-Euler beam, fixed at infinity, laying on a viscoelastic Winkler foundation of continuously varying stiffness/damping parameters, and a damped oscillatory load moving along the beam at a constant velocity. In order to obtain an approximate, semianalytical solution we build up a new discretization method based on the approximation of the discretized stiffness/damping values by generalized functions. The approximate solutions tend to continuous functions represented in a closed-form, analytical fashion.

Keywords

railway track dynamics · beam equation · inhomogeneous supporting field

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1 Introduction

In our simple model we consider a viscoelastic Winkler foundation of continuously varying stiffness/damping parameters given by functions $s_0 + s(x),\ k_0 + k(x)$, a Bernoulli-Euler beam of parameters EI, ρA laying on the subgrade, and a load of weight G moving along the beam at a longitudinal velocity v, and vibrating dampedly at complex frequency $w = \alpha + i\omega$. In case $\alpha = 0$ we have a harmonic load, while $w = 0$ stands for the case of a constant load.

The motion of the system is governed by the Bernoulli-Euler beam equation

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + (k_0 + k(x))\frac{\partial z}{\partial t} + (s_0 + s(x))z = G\exp(wt)\delta(x - vt), \tag{1}$$

where δ stands for Dirac's unit impulse distribution, while continuous functions s and k vanish outside the finite interval $[x_0, y_0]$.

Eq. (1) satisfies boundary conditions

$$\lim_{|x|\to\infty} z(x, t) = 0 \tag{2}$$

at $\pm\infty$.

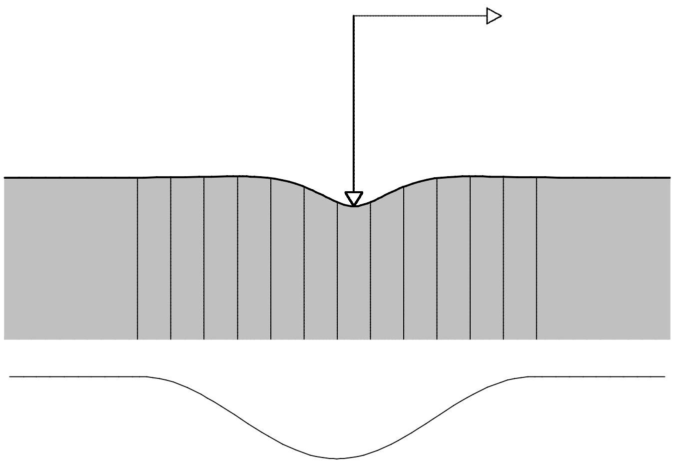

Fig. 1. System model

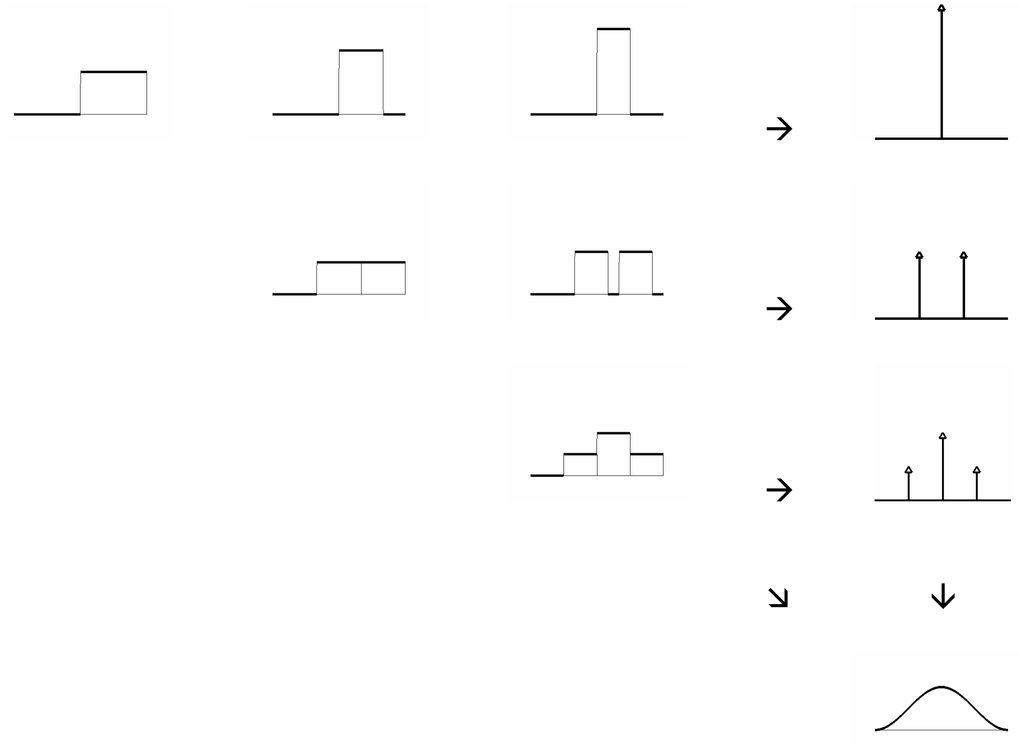

Fig. 2. The limits of the discretized function sequences

2 Approximate boundary problem

First we discretize parameter functions of the subgrade in the following way.

We build up step functions

$$s_{kn}(x) := \begin{cases} s(x_j)\frac{k}{n} \text{ if } x \in [x_j, x_j + \frac{hn}{k}),\ j = 0, 1, \ldots, n-1, \\ 0 \text{ otherwise.} \end{cases}$$

by discretizing continuous function $s(x)$ (or $k(x)$), for $k \geq n$, where $h := (y_0 - x_0)/n$ and $x_j := x_0 + jh,\ j = 0, 1, \ldots, n$ hold.

In this case $\lim\limits_{n \to +\infty} s_{nn}(x) = s(x)$ is satisfied. For the limit, which is a generalized function, in case $k \to +\infty$

$$s_n(x) := \lim_{k \to +\infty} s_{kn}(x) = \sum_{j=1}^{n} s(x_j)\delta(x - x_j)h$$

holds, and for its limit

$$\lim_{n \to +\infty} s_n(x) = \int_{x_0}^{y_0} s(y)\delta(x-y)\mathrm{d}y = s(x)$$

is satisfied, hence we have the following commutative diagram:

$$\begin{array}{ccc} s_{kn} & \xrightarrow{k\to\infty} & s_n \\ & {\scriptstyle k=n\to\infty} \searrow & \downarrow {\scriptstyle n\to\infty} \\ & & s \end{array}$$

The different limits of discretized function sequences are illustrated in Fig. 2.

The beam equation discretized to n parts in case $k \to +\infty$ has the form

$$EI\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} + k_0\frac{\partial u}{\partial t} + s_0 u = G\exp(\omega t)\delta(x - vt) -$$
$$\sum_{j=1}^{n}\left(s(x_j)u(x_j,t) + k(x_j)\frac{\partial}{\partial t}u(x_j,t)\right)h\delta(x - x_j). \qquad (3)$$

Response to the first term is

$$G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}\exp(\omega t + \lambda_i(x - vt))\mathrm{H}(\sigma_i(x - vt))$$

with characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho A v^2\lambda^2 - v(k_0 + 2\rho A\omega)\lambda + (s_0 + k_0\omega + \rho A\omega^2) \qquad (4)$$

and signs $\sigma_i := -sgnRe\lambda_i,\ P(\lambda_i) = 0,\ i = 1, \ldots, 4$, see e.g. [1],[3].

The full response has the form $u(x,t) = \sum\limits_{i=1}^{4}\exp(\omega t + \lambda_i(x - vt))u_i(x)$, and can be given in a recursive way as

$$\sum_{i=1}^{4}\sigma_i\exp(\omega t + \lambda_i(x - vt))\{G\mathrm{H}(\sigma_i(x - vt))/P'(\lambda_i) -$$
$$h\sum_{j=1}^{n-1}c_i(x_j)u_i(x_j)\mathrm{H}(\sigma_i(x - x_j))\}$$

with functions defined by

$$c_i(x) := \sigma_i\exp(\lambda_i x)\frac{s(x) + (\omega - \lambda_i v)k(x)}{4EI\lambda_i^3},$$
$$i = 1, 2, 3, 4, \qquad (5)$$

cf. [2],[4].

3 Recurrence formulae

For $Re\lambda_i < 0$ we have $\sigma_i = 1$ and recursion

$$u_i(x_k) = \frac{G}{P'(\lambda_i)}\mathrm{H}(x_k - vt) - h\sum_{j\le k} c_i(x_j)u_i(x_j)$$

with solution

$$u_i(x_k) = \frac{G}{P'(\lambda_i)}\mathrm{H}(x_k - vt)\prod_{j\le k}\frac{1}{1 + c_i(x_j)h}.$$

If $n \to +\infty$ holds, then we obtain

$$\lim_{n\to+\infty}\prod_{x_j\le x}(1 + c_i(x_j)h) =$$

$$\lim_{n\to+\infty}\prod_{x_j\le x}((1 + c_i(x_j)h)^{1/(c_i(x_j)h)})^{c_i(x_j)h} =$$

$$\exp\lim_{n\to+\infty}\sum_{x_j\le x} c_i(x_j)h = \exp\int_{x_0}^{x} c_i(y)\mathrm{d}y.$$

In the case $Re\lambda_i > 0$ we have recurrence formula

$$u_i(x_k) = \frac{-G}{P'(\lambda_i)}\mathrm{H}(vt - x_k) + h\sum_{j>k} c_i(x_j)u_i(x_j)$$

with solution

$$u_i(x) = \frac{-G}{P'(\lambda_i)}\mathrm{H}(vt - x)\prod_{x_j>x}(1 + c_i(x_j)h) \to$$

$$\frac{-G}{P'(\lambda_i)}\mathrm{H}(vt - x)\exp\left(-\int_{y_0}^{x} c_i\right), \quad n \to +\infty.$$

Summarizing the results obtained above we get a finite closed-form integral formula for the continuously supported problem (1-2) in form

$$z(x,t) = G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}$$

$$\exp\left(\omega t + \lambda_i(x - vt) - \int_{l_i}^{x} c_i(y)\mathrm{d}y\right) H(\sigma_i(x - vt)) \qquad (6)$$

with $l_i := \begin{cases} x_0 \text{ if } \mathrm{Re}\lambda_i < 0, \\ y_0 \text{ if } \mathrm{Re}\lambda_i > 0. \end{cases}$

4 Numerical results

In the example, similar to that of [5], the parameters of the beam are $EI = 6 \cdot 10^6$ Nm^2, $\rho A = 60$ kg/m. The weight of the constant load is $G = 6.5 \cdot 10^4$ N, while its horizontal velocity is $v = 40$ m/s. The parameters of the subgrade are given by constants $s_0 = 9 \cdot 10^7$ N/m^2, $k_0 = 4.6 \cdot 10^4$Ns/m^2 and single sinusoidal waves

$$s(x) = \begin{cases} (\cos(\frac{\pi x}{(20\text{ m})} - 1) \cdot 10^7\text{N/m}^2, & \text{if } 0\text{ m} \le x \le 40\text{m}, \\ 0 \text{ otherwise}, \end{cases}$$

$$k(x) = \begin{cases} (\cos(\frac{\pi x}{(20\text{ m}))-1)} \cdot 2500\text{ Ns/m}^2, & \text{if } 0\text{ m} \le x \le 40\text{ m}, \\ 0 \text{ otherwise}. \end{cases}$$

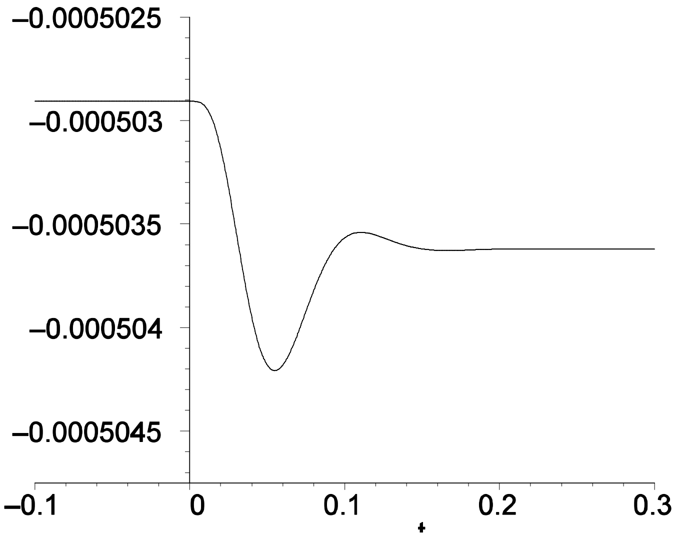

Fig. 3. The vertical position $z(vt, t)$ of the load

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