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RIESZ BASES IN CONTROL THEORY

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Abstract

In this paper we examine the reachable states of motion of a vibrating string, starting from given initial and boundary conditions and driving the string by an appropriate $u(t)$ control force which is an element of a specified function field. The motion is described using Fourier methodology. The convergence of the series expansion is examined for different function classes. This requires spectral-theoretical studies to become acquainted with the asymptotic behaviour of the eigenfunctions and eigenvalues.

Keywords: vibrating string, Fourier method, Riesz bases.

1. Introduction

Consider the following equation with fixed $0 < a < 1$ and $0 < T < \infty$

$$\varrho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial y(x,t)}{\partial x}\right] + \delta(x-a)u(t) \tag{1}$$

for all $0 < x < 1$ and $0 < t < T$. This equation describes the *oscillatory motion of a string* which is stretched over the $x \in [0,1]$ interval of the $x - y$ plane. We suppose that there is only transversal oscillation, i.e. only the y-coordinate of an individual point of the string changes during oscillation; then $y(x,t)$ denotes the abscissa value belonging to the point with ordinate x at time t.

In *Eq.* (1) $\varrho(x)$ is the mass density, consequently, if the cross-section of the string is q, then the mass of a dx part of the string is $q\,\varrho(x)\,\mathrm{d}x$; $p(x)$ is the elastic modulus, that is the proportion between the drawing force on dx and relative stretching caused by it. If we multiply the left hand side of (1) by $q\,\mathrm{d}x$ then we get the product of mass and acceleration of a dx part of the string, therefore the right hand side of the equation should express the (vertical) force(s) acting on the dx part.

The first member of right hand side is the *internal force* acting on dx, which is the drawing force transmitted by the neighbouring parts of the string, and the second member expresses that at point $x = a$ and time t there is a transversal force $u(t)$.

Function $u(t)$ is the so-called *control force*, since $u(t)$ is altered – under certain conditions – in order to influence the oscillation of the string.

The study of linear discrete-time systems in infinite dimensional spaces has been motivated by the fact that it gives rise to many new problems and results which do not occur in the finite-dimensional case and by the great possibility of application to study continuous-time systems described by classical differential equations, retarded differential equations, partial differential equations, etc. We are also motivated by the fact that vibrating strings and membranes can be found in several problems of vehicle dynamics, not only in structures, but also in components:

- The precise controllability of the membrane of an ABS modulator or a proportional valve is the most important problem of the brake system. The membranes are used instead of pistons in the valves due to the reduced inertia, however, unwanted vibrations of an ABS valve membrane may cause failure in the brake operation.
- Constrained vibrating strings are used for measuring the intensity of the airflow in the intake manifold.
- Traverse gravimeter was applied by Apollo-17 to measure and map the gravitational field of the Moon. It was mounted on the Lunar Roving Vehicle and used a vibrating string accelerometer to measure gravity fields.
- The super conducting vibrating string gradiometer, a device with no moving parts, where the length of the string under tension of a gravitational field is measured by two SQUIDS located at the ends of the string, has recently been developed but is still in research phase (it is not demonstrated that it is mature enough to be fitted onto a moving platform).

2. Fourier Description of Oscillation

We define the state of motion of the string as the function pair

$$\big(y(\cdot,t),\, y_t(\cdot,t)\big),$$

i.e. the actual position and velocity functions. In this paper we examine the admissible states of motion starting from given initial and boundary conditions and driving the string by an appropriate $u(t)$ control force which is an element of a specified function field. We can use concentrated force(s) without any loss of generality as by superposing *Dirac* delta functions any arbitrary force distribution can be produced.

We use the *Fourier* methodology, i.e. let us define

$$y(x,0) = y_0(x), \qquad y_t(x,0) = \hat{y}_0(x), \tag{2}$$

$$U_1(y(\cdot,t)) = U_2(y(\cdot,t)) = 0, \qquad (0 < t < T) \tag{3}$$

and assume that

$$p, \varrho \in C^2[0,1], \qquad p, \varrho > 0.$$

Substituting

$$\hat{y}(x^*,t) := y(\phi(x^*),t)\sqrt[4]{p(\phi(x^*))\varrho(\phi(x^*))},$$
$$\phi := r^{-1},$$
$$r(x) := \int_0^x \sqrt{\frac{\varrho}{p}}$$

into (1), (2) and (3) restoring y and x in place of $\hat{y}$ and x^*, respectively, we get the simpler forms

$$y_{tt} - y_{xx} - q(x)y = \frac{\delta(x-a')}{\alpha(a)}u, \qquad (0 < x < l, 0 < t < T), \tag{4}$$
$$y(x,0) = y_0(x), \qquad y_t(x,0) = \hat{y}_0(x), \qquad (0 < x < l), \tag{5}$$
$$V_1(y(\cdot,t)) = V_2(y(\cdot,t)) = 0, \qquad (0 < t < T), \tag{6}$$

where V_1 and V_2 are the transformed boundary conditions and

$$\alpha(a) := \sqrt[4]{\frac{\varrho^3(a)}{p(a)}}, \qquad a' := \int_0^a \sqrt{\frac{\varrho}{p}}, \qquad l := \int_0^1 \sqrt{\frac{\varrho}{p}}, \qquad q \in C[0,l].$$

To define the distribution-equality (4) there are several possibilities. We use the following

Definition 1 The solution of the system (4)–(6) is such a function

$$y(x,t) \in L^2\big((0,l)\times(0,T)\big)$$

which fullfils the equation

$$\int_0^l \int_0^T y(z_{tt} - z_{xx} - qz)\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^l \big[\hat{y}_0 z(\cdot,0) - y_0 z_t(\cdot,0)\big]\,\mathrm{d}x + \int_0^T \frac{z(a',\cdot)}{\alpha(a)}u\,\mathrm{d}t \tag{7}$$

for all

$$z \in C^2\big([0,l]\times[0,T]\big)$$

to which

$$z(\cdot,T) = z_t(\cdot,T) \equiv 0, \qquad W_1(z(\cdot,t)) = W_2(z(\cdot,t)) = 0, \qquad \forall t \tag{8}$$

holds, where W_1, W_2 are the adjugate boundary conditions to V_1, V_2, respectively, see [7].

We can deduce *Eq.* (7) from (4) through multiplying it by $z(x,t)$ and performing formal partial integrations. Consider the following

$$L_v = v'' + qv, \qquad V_1(v) = V_2(v) = 0$$

and

$$L_w = w'' + qw, \qquad W_1(w) = W_2(w) = 0$$

eigenvalue-problems on interval $[0,l]$. For sufficiently general boundary condition types (e.g. for strictly regular boundary conditions, see [7]) there are countable eigenvalues and eigenfunctions, namely

$$v_n'' + qv_n + \lambda_n v_n = 0, \qquad V_1(v_n) = V_2(v_n) = 0, \tag{9}$$

$$w_n'' + qw_n + \lambda_n w_n = 0, \qquad w_1(w_n) = w_2(w_n) = 0 \tag{10}$$

and for them

$$\langle v_n, w_k \rangle = \delta_{n,k} \tag{11}$$

holds (see more detailed later). If now we set

$$z(x,t) := w_n(x)b(t), \tag{12}$$

where

$$b \in C^2[0,T], \qquad b(T) = b'(T) = 0 \tag{13}$$

and

$$\begin{aligned} y(x,t) &= \sum v_n(x)c_n(t), \\ y_0(x) &= \sum c_n^0 v_n(x), \\ \hat{y}_0(x) &= \sum \hat{c}_n^0 v_n(x), \end{aligned} \tag{14}$$

then from (7) we arrive at

$$\int_0^T c_n \left[b'' + \lambda_n b \right] \mathrm{d}t = c_{n,0}' b(0) - c_{n,0} b'(0) + \frac{\overline{w_n(a')}}{\alpha(a)} \int_0^T bu \, \mathrm{d}t$$

for any b that satisfies (13). In distribution-meaning this is equivalent to the boundary condition problem

$$c_n'' + \lambda_n c_n \frac{\overline{w_n(a')}}{\alpha(a)} u, \qquad c_n(0) = c_n^0, \qquad c_n'(0) = \hat{c}_n^0$$

and for the solution of it

$$c_n(t) = c_n^0 \cos\sqrt{\lambda_n}t + \hat{c}_n^0 \frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} + \frac{\overline{w_n(a')}}{\alpha(a)} \int_0^T u(\tau) \frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}} \,\mathrm{d}\tau \qquad (15)$$

holds.

According to the above we can see that if we use the *Fourier* method then we have to face with two kinds of problems. The first problem is the 'goodness' of the series composed by (9) and (10), consequently, we have to examine the convergence for different function classes. This requires spectral-theory studies, e.g. to become acquainted with the asymptotic behavior of the eigenfunctions and eigenvalues. The second problem can be seen from (15) where the values of the *Fourier* transform of the control force $u(t)$ taken in countable places appear. Since the *Fourier* transform is an entire function, thus we arrive at an interpolation problem in complex function theory. The modern theory of this problem was developed in the last two decades, and in the background of it there is the theory of *Hardy* spaces, see [8]. We discuss this problem in the next chapter, as well.

3. Discussion of Reachable States

Let us investigate the *homogeneous* string, in other words

$$y_{tt}(x,t) = y_{xx}(x,t) + \delta(x-a)u(t), \qquad 0 < x < 1, 0 < t < T, \qquad (16)$$

$$y(0,t) = y(l,t) = 0, \qquad (17)$$

$$y(0,x) = y_t(x,0) = 0. \qquad (18)$$

In this case, for the coefficient functions of the series expansion

$$y(x,t) = \sum c_n(t)v_n(x) = \sum c_n(t)\sqrt{2}\sin n\pi x$$

holds that

$$n\pi c_n(T) + ic_n'(T) = i\sqrt{2}\sin n\pi a \int_0^T u(t)e^{in\pi t}\,\mathrm{d}t \cdot e^{-inT}. \qquad (19)$$

It is known that the transformation

$$H^1(0,1) \oplus L^2(0,1) \to l_2,$$
$$(y_0, \hat{y}_0) \mapsto (n\pi c_n + i\hat{c}_n)_n$$

is an isomorphism – remark that

$$y_0(x) = \sum c_n\sqrt{2}\sin n\pi x, \qquad \hat{y}_0(x) = \sum \hat{c}_n\sqrt{2}\sin n\pi x.$$

Let us define the *reachability set* $\mathcal{D}_a(T)$, the set of the states of motion that can be reached from the state of rest in time T, in the following way:

$$\mathcal{D}_a(T) = \left\{(y(\cdot, T), y_t(\cdot, T)) \in H : u(t) \in L^2(0, T)\right\},$$

where

$$H = \left\{(f_0, f_1) \in H^1(0,1) \oplus L^2(0,1) : f_0(0) = f_0(1) = 0\right\}.$$

Then the following theorem holds:

Theorem 1 *Let $a = p/q$ be a rational number, $(p, q) = 1$. Then*
1. $\mathcal{D}_a(T_1) = \mathcal{D}_a(T_2)$, if $2(q-1)/q \leq T_1 < T_2$
2. $\mathcal{D}_a(T_1) \subsetneq \mathcal{D}_a(T_2)$, if $T_1 < T_2 \leq 2(q-1)/q$
3. $\mathcal{D}_a(T) \subset H$ is closed for every T.

Proof. (1) and (2) can easily be shown using the orthogonal decomposition

$$L^2(0,2) = H_1 \oplus H_2,$$

where

$$\begin{aligned} H_1 &= V\left\{\sin n\pi x, \cos n\pi x : q \nmid n\right\} = \\ &= \left\{u \in L^2(0,2) : u(x) + u\left(x + \frac{2}{q}\right) + \ldots + u\left(x + 2\frac{q-1}{q}\right) = 0 \quad \text{a.e.}\right\}, \\ H_2 &= V\left\{\sin n\pi x, \cos n\pi x : q \mid n\right\} = \\ &= \left\{u \in L^2(0,2) : u(x) + u\left(x + \frac{2}{q}\right) + \ldots + u\left(x + 2\frac{q-1}{q}\right) = 0 \quad \text{a.e.}\right\}, \end{aligned}$$

here $V\{.\}$ denotes the closed linear shell in $L^2(0,2)$ of the functions of $\{.\}$. For proving (3) let us suppose that

$$\left(\langle u_n, e^{ik\pi x}\rangle\right)_{q \nmid k} \to (a_k) \qquad (n \to \infty) \tag{20}$$

in l_2 sense. It has to be shown that there exists a $u \in L^2(0, T)$ for which

$$(\langle u, e^{ik\pi x}\rangle) = a_k \qquad (q \nmid k).$$

We can suppose that $T \leq 2(q-1)/q$, because – for a T larger than that – $\mathcal{D}_a(T)$ does not change any more. It is sufficient to show that (u_n) is limited, because it has a weakly convergent sub-series then. It can be supposed that for the u_n series, extended with 0, $u_n \in L^2(0,2)$ holds. Let us consider the following decomposition according to $H_1 \oplus H_2$:

$$u_n = u_{n,1} + u_{n,2}.$$

From the convergence (20) it follows that $(u_{n,1})$ is a limited series. And then, because of

$$\|u_n - u_{n,2}\|_{L^2(2(q-1)/q,2)} = \|u_{n,2}\|_{L^2(2(q-1)/q,2)} = q\|u_{n,2}\|_{L^2(0,2)}$$

$(u_{n,2})$ and so (u_n) are limited series. Thus theorem 1 is proved. □

Theorem 2 *Let* $0 < a < 1$ *be an irrational number. Then*
1. $\mathcal{D}_a(T_1) = \mathcal{D}_a(T_2)$, *if* $2 \le T_1 < T_2$
2. $\mathcal{D}_a(T_1) \subsetneq \mathcal{D}_a(T_2)$, *if* $T_1 < T_2 \le 2$
3. $\mathcal{D}_a(T)$ *is closed* $\iff T < 2$.

Proof. We only consider the closeness in case of $T < 2$. Now it is sufficient to show that the set

$$\mathcal{B}_a(T) = \left\{ \left(\sin n\pi a \int_0^T u(t) e^{in\pi t}\, \mathrm{d}t \right) : u \in L^2(0,T) \right\}$$

is closed in l_2. Let

$$\left(\sin n\pi a \int_0^T u_k(t) e^{in\pi t}\, \mathrm{d}t \right)_n$$

be convergent in l_2 for $k \to \infty$. Then for any $\varepsilon > 0$, the series

$$\left(\int_0^T u_k(t) e^{in\pi t}\, \mathrm{d}t \right)_{n \in \mathbb{Z}(\varepsilon)}$$

is also convergent in l_2, if

$$\mathbb{Z}(\varepsilon) = \{ n \in \mathbb{Z} : |\sin n\pi a| > \varepsilon \}.$$

Closeness will be proved if we show that

$$\exists \mathbb{Z}^T \subset \mathbb{Z}(\varepsilon) \text{ such that } \left(e^{in\pi t}\right)_{n \in \mathbb{Z}^T} \text{ is a Riesz basis in } L^2(0,T). \tag{21}$$

This can easily be proved with Theorem 11 of AVDONIN [1]. Let

$$\lambda_n = \frac{2\pi}{T} n.$$

This is the root system of the function $\sin(T/2)x$, and the indicator diagram of the function is $[-iT/2, iT/2]$. These λ_n's have to be moved into various elements of the set $\pi\mathbb{Z}(\varepsilon)$

$$\lambda_n + \delta_n \in \pi\mathbb{Z}(\varepsilon)$$

so that the condition (b) of Theorem 11 should be satisfied. In fact, (b) can be guaranteed with any small constant, instead of 1/4. Let us see how. Since $\sin n\pi a$ has a uniform distribution for an irrational a on $[-1, 1]$, it follows that for a sufficiently small ε the adjacent elements of the series

$$\mathbb{Z} \setminus \mathbb{Z}(\varepsilon) = \{u : |\sin n\pi a| \leq \varepsilon\}$$

follow each other with a place greater than any prescribed distance. So if we choose ε properly small, then, because of $T/2 < 1$, we can achieve that there is a d, so that in any section with length d, there are at least $1 + \delta$ times as many from the elements of $\pi\mathbb{Z}(\varepsilon)$ as from λ_n. We would like to use this surplus in such a way that we divide $\mathbb{R}$ into sections with length d, and on every even-th section the λ_n values are shifted to the right (that is $\delta_n > 0$), and on every odd-th section to the left ($\delta_n < 0$). With this, $\sum \delta_n$ breaks up into detail sums with alternating signs, so we expect that $|\sum \delta_n|$ can be kept under a given limit on any section with arbitrary length. Although the procedure above does not give this result yet, but once the basic idea is known, the necessary modifications can easily be found; we leave it to the reader. The proof is completed. □

Theorem 3 *Let us consider the following system:*

$$\begin{aligned} \varrho(x) y_{tt}(x,t) &= y_{xx}(x,t) + \delta(x-a)u(t), \\ y(0,t) &= y(1,t) = 0, \\ y(x,0) &= y_t(x,0) = 0, \end{aligned} \tag{22}$$

where $0 < \varrho \in C^2[0, 1]$.

Then for all $0 < a < 1$ *– with countable exceptions – the following statements hold:*

1. $\mathcal{D}_a(T_1) = \mathcal{D}_a(T_2)$, *if* $\hat{T} \leq T_1 \leq T_2$, $\hat{T} = 2\int_0^1 \sqrt{\varrho}$

2. $\mathcal{D}_a(T_1) \subsetneq \mathcal{D}_a(T_2)$, *if* $T_1 < T_2 \leq \hat{T}$

3. $\mathcal{D}_a(T)$ *closed* $\iff T < \hat{T}$

Proof. With the transformation used in Section 2 and on the grounds of the asymptoticism given in [7] p. 58, Theorem 1, and of [9] p. 118 and p. 172, we obtain that the asymptotic behavior of the system

$$\begin{aligned} v_n'' + \lambda_n \varrho v_n &= 0, \\ v_n(0) = v_n(1) &= 0 \end{aligned} \tag{23}$$

is the following:

$$\lambda_n = \left(2n\frac{\pi}{\hat{T}}\right)^2 + \mathcal{O}(1), \tag{24}$$

$$v_n(x) = \varrho^{1/4}(x) \sin\left(\frac{2\pi n}{\hat{T}} \int_0^x \sqrt{\varrho}\right) + \mathcal{O}\left(\frac{1}{n}\right) \tag{25}$$

uniformly in $x \in [0, \hat{T}/2]$. With the (24) and (25) estimations the proof of (3) can be obtained from *Avdonin's* theorem, in a similar way as in the previous theorem. The proof of (1) and (2) depends on whether the system

$$\{1\} \cup \left(e^{\pm i\sqrt{\lambda_n}x}\right)_{n=1}^{\infty} \tag{26}$$

is a *Riesz* basis in $L^2(0, \hat{T})$. For we know, that

$$\sqrt{\lambda_n}c_n(T) + ic_n'(T) = iv_n(a)e^{-i\sqrt{\lambda_n}T}\int_0^T u(t)e^{i\sqrt{\lambda_n}t}\,\mathrm{d}t, \tag{27}$$

therefore if (26) is a *Riesz* basis on $(0, \hat{T})$ then for any $T \geq \hat{T}$

$$\left(\int_0^T u(t)e^{i\sqrt{\lambda_n}t}\,\mathrm{d}t\right)_{n=1}^{\infty}$$

runs the (complex) l_2 while u runs the (real) $L^2(0, T)$. Thus (1) is shown.

For the proof of (2) we have to consider countable $0 < a < 1$ values (it is in fact necessary for (3) too), the ones in which one of $v_n(a) = 0$, because then the n-th *Fourier* coefficient drops out in (27). It is known from [9] that any eigenfunction of the *Sturm–Liouville* operator has only a finite number of roots, so we really excluded only a countable number of values (in case of $\varrho \equiv 1$ these are exactly the rational numbers).

Let now $u_2 \in L^2(0, T_2 - T_1)$ for some $T_1 < T_2 \leq \hat{T}$. If $\mathcal{D}_a(T_1) = \mathcal{D}_a(T_2)$ be true then there would exist a $u_1 \in L^2(0, T_1)$ such that the momentum of $u_1(T_1 - t) - u_2(T_2 - t)$ to any $e^{i\sqrt{\lambda_n}t}$ is zero; because it is real, it follows that its momentums to $e^{-i\sqrt{\lambda_n}t}$ are zero, too. Since (26) is a *Riesz* basis, it follows that $u_1(T_1 - t) - u_2(T_2 - t)$ has to be a constant multiple of the function according to 1 in the bi-orthogonal system of (26). But this is impossible, because the $u_1(T_1 - t) \in L^2(0, T_1)$ and $u_2(T_2 - t) \in L^2(0, T_2)$ functions are arbitrary ones.

So what is left from the proof of Theorem 3 is to show so that (26) is a *Riesz* basis in $L^2(0, \hat{T})$. From the (24) asymptoticism it can be seen that we need a stability theorem which is about a system at a distance according to l_2 from an orthonormal system.

Lemma 1 (Bari [2]) *If (ϕ_n) is an orthonormal basis in a* Hilbert *space H, further, $\psi_n \in H$, $\|\psi_n\| = 1$ and*

$$\sum \|\phi_n - \psi_n\|^2 < 1 \tag{28}$$

then (ψ_n) is a Riesz *basis in H.*

Lemma 2 (Bari [2]) *If, instead of condition (28) of Lemma 1, we only know the weaker estimation*

$$\sum \|\phi_n - \psi_n\|^2 < \infty \tag{29}$$

then system (ψ_n) *has a bi-orthogonal system exactly if it is complete in* H*, and in this case* (ψ_n) *will already be a* Riesz *basis in* H *too. Besides, it is sufficient to assume about* (ϕ_n) *that it is a* Riesz *basis, instead of orthonormality.*

Lemma 3 (Levin [5]) *Let the* $(e^{i\mu_n x})$ *system be complete in* $L^2(0, T)$, $\mu_n \in \mathbb{C}$, $0 < T < \infty$. *Let us replace a finite number of* $e^{i\mu_n x}$ *terms for* $e^{i\mu'_n x}$*, using some* $\mu'_n \in \mathbb{C}$. *If the exponents are different in the newly obtained system then the new system will also be complete in* $L^2(0, T)$.

Lemmas 2 and 3 immediately lead to

Lemma 4 (Replacement theorem) *Let the* $(e^{i\mu_n x})$ *system be a* Riesz *basis in* $L^2(0, T)$. *Let us replace a finite number of* $e^{i\mu_n x}$ *terms for arbitrary* $e^{i\mu'_n x}$ *new terms with* $\mu'_n \in \mathbb{C}$. *Then the new system will also be a* Riesz *basis in* $L^2(0, T)$, *assuming that it consists of different functions.*

Going back to the proof of Theorem 3, the

$$\int_0^{\hat{T}} \left| e^{i\sqrt{\lambda_n}x} - e^{i(2n\pi/\hat{T})x} \right|^2 \mathrm{d}x = \int_0^{\hat{T}} \left| e^{i\mathcal{O}(1/n)x} - 1 \right|^2 \mathrm{d}x =$$
$$= 2\hat{T}\left(1 - \frac{\sin \mathcal{O}(1/n)\hat{T}}{\mathcal{O}(1/n)\hat{T}}\right) = \mathcal{O}(1/n^2)$$

estimation shows, on the grounds of Lemma 1, that for a sufficiently large N the system

$$\left(e^{i(2n\pi/\hat{T})x}\right)_{n=-N}^{N} \cup \left(e^{\pm i\sqrt{\lambda_n}x}\right)_{n=N+1}^{\infty}$$

is a *Riesz* basis in $L^2(0, \hat{T})$. The replacement theorem and $\lambda_n \neq 0$ $(n = 1, 2, \ldots)$ prove that (26) is indeed a *Riesz* basis in $L^2(0, \hat{T})$.

Theorem 3 is thus completely proved. □

4. Strictly Regular Boundary Conditions

Horváth [3] investigated the (1)–(3) system with the conditions $0 < p, \varrho \in C^2[0, 1]$, if U_1 and U_2 are so-called *strictly regular* boundary conditions. These can belong to three categories:

(I)

$$U_1 y = y_0 = 0,$$
$$U_2 y = y_1 = 0.$$

(II)

$$U_1 y = a_1 y_0' + b_1 y_1' + a_0 y_0 + b_0 y_1 = 0,$$
$$U_1 y = \qquad\qquad + c_0 y_0 + d_0 y_1 = 0,$$

if

$$b_1 c_0 + a_1 d_0 \neq 0, \qquad a_1 \neq \pm b_1, \qquad c_0 \neq \pm d_0.$$

(III)

$$U_1 y = y_0' + \alpha_{11} y_0 + \alpha_{12} y_1 = 0,$$
$$U_1 y = y_1' + \alpha_{21} y_0 + \alpha_{22} y_1 = 0.$$

When investigating this string, the first step here is also the substitution described in Section 2 which makes available the spectral theory that had been properly worked out for *Schrödinger* operators. (This necessitates the $p, \varrho \in C^2[0,1]$ condition too.) Using the transformation, our equations will become of the form of (4)–(6). If we suppose that $p(0) = p(1)$, $\varrho(0) = \varrho(1)$ then the transforms V_1, V_2 of the boundary conditions will also be strictly regular. Since the strict regularity is preserved at creating the adjoint operator, the W_1, W_2 adjoint boundary conditions are also strictly regular. Let us consider the

$$Lv = v'' + qv, \qquad V_1(v) = V_2(v) = 0 \tag{30}$$

and the

$$Lw = w'' + qw, \qquad W_1(w) = W_2(w) = 0 \tag{31}$$

boundary value problems. The eigenfunctions of (30), do not necessarily constitute a complete system in $L^2(0,l)$ if the V_1, V_2 boundary conditions are not self-adjoint. In fact, they constitute a finite co-dimensional sub-space, and we can constitute the missing dimensions with the higher order eigenfunctions of (30). The customary eigenfunctions, which are also called zero order eigenfunctions, are the $v \in C^2[0,l]$ solutions that satisfy the

$$Lv + \lambda v = 0, \qquad V_1(v) = V_2(v) = 0$$

equations. The $i > 0$ order eigenfunctions (belonging to the λ eigenvalues) are functions $v_i \in C^2[0,l]$ that satisfy

$$Lv_i + \lambda v_i = v_{i-1}, \qquad V_1(v_i) = V_2(v_i) = 0,$$

where v_{i-1} is an $i-1$ order eigenfunction with eigenvalue λ.

Theorem 4 (Mihailov [6], Kesselman [4]) *The zero and the higher order eigenfunctions of the boundary value problem (30) constitute a Riesz basis in $L^2(0,l)$. The bi-orthogonal system consists of the zero and higher order eigenfunctions of the adjoint problem (31).*

In detail: if in the system (30) a chain with length k (consisting of zero and higher order eigenfunctions) belongs to an eigenvalue λ, then in the dual system (31) a chain with length k belongs to $\overline{\lambda}$, and according to the bi-orthogonal correspondence the zero order element of the chain of (30) has to be paired with the $k-1$ order element of the chain of (31), the 1 order element with the $k-2$ order element, …, the $k-1$ order element with the zero order element.

Let us return to the investigation of the vibrating string.

Lemma 5 (Horváth [3]) *For a sufficiently large N the*

$$\left(\frac{v_n'(x)}{\sqrt{\lambda_n}}\right)_{n=N}^{\infty}$$

system is a Riesz *basis in its closed linear shell in* $L^2(0,l)$.

Proof. Let us write the eigenfunctions in the form

$$y = y_1 V_1(y_2) - y_2 V_1(y_1) \quad \text{and} \quad y = y_1 V_2(y_2) - y_2 V_2(y_1),$$

where y_1 and y_2 are the basic solutions defined in [7] Chapter II, 4.5. Then the asymptoticisms of [7] Chapter II, 4.9 give the following estimations:

In case (I)

$$\begin{aligned}\sqrt{\lambda_n} &= \frac{n\pi}{l} + \mathcal{O}(1/n),\\ v_n(x) &= \sin\frac{n\pi}{l} + \mathcal{O}(1/n),\\ \frac{v_n'(x)}{\sqrt{\lambda_n}} &= \cos\frac{n\pi}{l} + \mathcal{O}(1/n).\end{aligned} \tag{32}$$

In case (II)

$$\begin{aligned}\sqrt{\lambda_n} &= \alpha_n + \mathcal{O}(1/n),\\ v_n(x) &= c_0 \sin\alpha_n x + d_0 \sin\alpha_n(x-l) + \mathcal{O}(1/n),\\ \frac{v_n'(x)}{\sqrt{\lambda_n}} &= c_0\cos\alpha_n x + d_0\cos\alpha_n(x-l) + \mathcal{O}(1/n),\end{aligned} \tag{33}$$

where

$$\alpha_n = \frac{2[n/2]\pi + (-1)^n(\ln s/i)}{l}$$

and $[n/2]$ denotes the integer part of $n/2$, s is one of the roots of the equation

$$(b_1c_0 + a_1d_0)(s + 1/s) + 2(a_1c_0 + b_1d_0) = 0.$$

From the other form of the eigenfunctions the following asymptoticisms derive:

$$\begin{aligned} v_n(x) &= a_1 \cos \alpha_n x + b_1 \cos \alpha_n (x - l) + \mathcal{O}(1/n), \\ \frac{v_n'(x)}{\sqrt{\lambda_n}} &= a_1 \sin \alpha_n x + b_1 \sin \alpha_n (x - l) + \mathcal{O}(1/n). \end{aligned} \tag{34}$$

In case (III)

$$\begin{aligned} \sqrt{\lambda_n} &= \frac{n\pi}{l} + \mathcal{O}(1/n), \\ v_n(x) &= \cos \frac{n\pi}{l} x + \mathcal{O}(1/n), \\ -\frac{v_n'(x)}{\sqrt{\lambda_n}} &= \sin \frac{n\pi}{l} + \mathcal{O}(1/n). \end{aligned} \tag{35}$$

In cases (I) and (III) Lemma 5 follows immediately from these asymptoticisms, it is sufficient to refer to the following variant of Bari's Lemma 1: If $\phi_1, \phi_2, \ldots$ constitute a *Riesz* basis in an H *Hilbert* space in the $V(\phi_n)$ closed linear shell, and $\sum \|\phi_n - \psi_n\|^2 < \infty$, then for a sufficiently large N $\psi_N, \psi_{N+1}, \ldots$ will also constitute a *Riesz* basis. The problem in (II) is the following: it is not clear whether the main term of the asymptoticism in (33) constitutes a *Riesz* basis, or not. We will show that, in any case, it is at an l_2-distance from a *Riesz* basis. Indeed, the

$$\hat{V}_1 = c_0 y_0' + d_0 y_1' = 0, \qquad \hat{V}_2 = a_1 y_0 + b_1 y_1 = 0$$

boundary conditions are strictly regular, therefore, according to Theorem 4, their system of eigenfunctions constitutes a *Riesz* basis, on the other hand, according to (34), it is at an l_2-distance from the system $(v_n'/\sqrt{\lambda_n})$. In the end, also in case (II), our Lemma 5 can be proved with the above modification of Lemma 1. □

Let

$$H = \begin{cases} \{f \in H^1(0,l) : f(0) = f(l) = 0\} & \text{in case of (I)} \\ \{f \in H^1(0,l) : c_0 f(0) + d_0 f(l) = 0\} & \text{in case of (II)} \\ H^1(0,l) & \text{in case of (III)} \end{cases}$$

that is, from $U_1 f = 0$ and $U_2 f = 0$, the ones that are intelligible for $f \in H^1(0,l)$ have to be satisfied. Now holds the following:

Lemma 6 (Horváth [3]) *The following statements for function* $\sum c_n v_n \in H^1(0,l)$ *are equivalent:*

(i) $\sum c_n v_n \in H$,

(ii) $\left(\sum c_n v_n\right)' = \sum c_n v_n'$,

(iii) $\sum |\lambda_n| \cdot |c_n|^2 < \infty$.

Proof. (ii) $\iff$ (iii) can be simply shown from Lemma 5. For (ii) is true exactly if $\sum c_n v_n'$ is convergent in norm, and this just means (iii), according to Lemma 5.

For the proof of (i) $\iff$ (iii) let us consider the eigenfunctions of the system (31):

$$w_n'' + r w_n + \overline{\lambda}_n w_n = \theta_n w_{n-1} \qquad (n = 1, 2, \ldots),$$

where θ_n equals 0 or 1. Let $\phi = \sum c_n v_n \in H^1(0, l)$. Then, in case of $\lambda_n \neq 0$

$$\sqrt{\lambda_n} c_n = \frac{1}{\sqrt{\lambda_n}} \langle \phi, \theta_n w_{n-1} - q w_n \rangle + \left\langle \phi', \frac{w_n'}{\sqrt{\lambda_n}} \right\rangle - \left[\phi \frac{w_n'}{\sqrt{\lambda_n}} \right]_0^l .$$

Therefore

$$\text{(iii)} \iff \left(\sqrt{\lambda_n} c_n \right) \in l_2 \iff \left(\left[\phi \frac{w_n'}{\sqrt{\lambda_n}} \right]_0^l \right) \in l_2. \tag{36}$$

Let us employ the asymptoticisms of Lemma 5 for this.

In case (I), (36) yields $(\phi(l) + (-1)^n \phi(0)) \in l_2$, that is $\phi(0) = \phi(l) = 0$, $\phi \in H_l$. In case (III), (36) is satisfied for any $\phi \in H^1(0, l)$. Investigating the case (II), let us remark that the form of the dual W_1, W_2 boundary conditions is

$$\begin{aligned} W_1(y) &= d_0 y_0' + c_0 y_1' + \beta_0 y_0 + \beta_1 y_1 = 0, \\ W_2(y) &= + b_1 y_0 + a_1 y_1 = 0. \end{aligned}$$

Since one of c_0 and d_0, for example c_0, is surely not zero, it follows that

$$\frac{w_n'}{\sqrt{\lambda_n}} = -\frac{d_0}{c_0} \frac{w_n'(0)}{\sqrt{\lambda_n}} + \mathcal{O}(1/n),$$

thus

$$\begin{aligned} c_0 \left[\phi \frac{w_n'}{\sqrt{\lambda_n}} \right]_0^l &= -\frac{w_n'(0)}{\sqrt{\lambda_n}} (d_0 \phi(l) + c_0 \phi(0)) + \mathcal{O}(1/n) = \\ &= (-1)^n c_0 \sin \frac{\ln s}{i} (d_0 \phi(l) + c_0 \phi(0)) + \mathcal{O}(1/n). \end{aligned}$$

Since $s \neq \pm 1$, from this follows $d_0 \phi(l) + c_0 \phi(0) = 0$, that is $\phi \in H$. Thus Lemma 6 is proved. □

Lemma 7 (Horváth [3]) *If $\delta \in \mathbb{C}$, $\lambda_n \neq \delta$ then the transformation*

$$L : H \to l_2$$

$$\sum c_n v_n \mapsto \left(\left(\delta + \sqrt{\lambda_n} \right) c_n \right)_{n=1}^{\infty}$$

is an isomorphism between H and l_2.

Proof. Transformation L is linear and bijective, according to Lemma 6 and Theorem 4. It also follows from Theorem 4 that

$$\left\|\sum c_n v_n\right\|^2 \leq \text{const} \sum |c_n|^2,$$

and because of Lemma 5

$$\left\|\sum c_n v_n'\right\|^2 \leq \text{const} \sum (1 + |\lambda_n|)\, |c_n|^2.$$

For this reason, linear bijection L^{-1} is continuous, and thus, according to the theorem of open transformation, it is an isomorphism, which was to be proved. □

Theorem 5 (Horváth [3]) *In the case of boundary conditions of type (I) and (II), the complete*

$$\left(\frac{v_n'}{1 + |\sqrt{\lambda_n}|}\right)_{n=1}^{\infty} \tag{37}$$

system constitutes a Riesz basis in $L^2(0, l)$, *in its closed linear shell.*

Proof. In accordance with Lemma 5, it is enough to show the linear independence of the system (37), that is, to show that

$$\sum |c_n|^2 < \infty, \quad \sum c_n \frac{v_n'}{1 + |\sqrt{\lambda_n}|} = 0 \implies c_n = 0, \quad \forall n.$$

But in case of $\sum c_n v_n'/(1 + |\sqrt{\lambda_n}|) = 0$, $\sum c_n v_n/(1 + |\sqrt{\lambda_n}|) = \text{const}$, and, according to Lemma 6, from this const $\in H$ follows, but this is only possible in case of const $= 0$, thus $\sum c_n v_n/(1 + |\sqrt{\lambda_n}|) = 0$. And then, it follows from Theorem 4 that $c_n = 0$, $\forall n$. □

Remark 1 *For type (III) the theorem cannot be true in this form, because in the simplest case* ($q \equiv 0$, $y_0' = y_1' = 0$), (v_n) *runs through the set* $\{1, \cos(\pi/l)x, \cos(2\pi/l)x, \ldots\}$, *and so* $v_1' \equiv 0$. *However, the question may arise, if we omit the zero function that might turn up among the derivatives, whether the remaining system constitutes a* Riesz *basis in its closed linear shell, or not. The answer is still* unknown.

Theorem 6 (Horváth [3]) *Let us suppose that* $\lambda_n \neq 0$, $\forall n$. *Then the following statements hold for any* $a \in (0, 1)$ *with a countable number of exceptions*

1. $\mathcal{D}_a(T_1) \subsetneq \mathcal{D}_a(T_2)$ *if* $T_1 < T_2 \leq 2l$

2. $\mathcal{D}_a(T)$ *is closed in* $H \oplus L^2(0, l) \iff T < 2l$.

Proof. The proof can easily be shown on the analogy of Theorem 3, we do not go into details about it. □

5. Further Problems

We could not show yet that $\mathcal{D}_a(T)$ becomes stable in case of $T \geq 2l$. The reason for this is that – because of the higher order eigenfunctions – there appear such *Fourier* coefficients of control $u(t)$ which have higher order exponents, that is, functions in the form of $p(x)e^{i\lambda x}$. This problem can also be formulated in a standardized version in the following way:

> Let $v_1, v_2, \ldots$ be all the zero and higher order eigenfunctions of a *Schrödinger* operator given with strictly regular boundary conditions, let n_k be the number of the eigenfunctions belonging to the eigenvalue λ_n. Is it true that system
>
> $$e(\Lambda) = \left(e^{i\lambda_k x}, xe^{i\lambda_k x}, \ldots, x^{n_k-1}e^{i\lambda_k x}\right)_{k=1}^{\infty}$$
>
> constitutes a (complete) *Riesz* basis in $L^2(0, 2l)$?

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