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MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL DESIGN FOR ACTIVE SUSPENSION STRUCTURES

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Abstract

This paper presents the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis for active suspension design. The compensator is designed in such a way that it minimizes a given $\mathcal{H}_2$ performance function, which keeps the maximum supported $\mathcal{H}_\infty$ perturbation below appropriate levels. In this design problem, the aim is to find a set of values for the design variables that yield an optimum value of the objective (or cost) function and which fit a number of constraints. In order to solve this problem a design procedure based on a trade-off curve is presented.

Keywords: mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, robust control, optimal control, flexible structures, uncertain linear systems.

1. Introduction

The aim of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis is to treat the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems as separate problems but in a unified state-space framework. This method provides a compensator that combines the $\mathcal{H}_2$ quadratic performance criterion for disturbance rejection with the $\mathcal{H}_\infty$ performance criterion for maximum robustness against destabilizing uncertainties. It means, the controller which minimizes the $\mathcal{H}_2$ performance index is selected from the suitable $\mathcal{H}_2$ controllers [1, 3, 13, 15]. The solution of the optimization problem leads to three Riccati equations, which are mutually interconnected, and therefore the solution can be reached in an iterative way. There are effective and powerful algorithms to solve this problem, e.g. the homotopy technique, or a quasi-Newton technique [12, 11]. Another solution is based on the matrix inequality method, i.e. the interior point algorithm [10].

The purpose of the active suspension is to eliminate the harmful vibration caused by road irregularities and on-board excitation sources. However, the model contains uncertainties, which are caused by the actuator error, parameters varying around their nominal value, neglected non-linear effects, and uncertain components.

Several methods have been proposed for solving the active suspension design problem. The $\mathcal{H}_2$ control is well suited to the design of nominal performance in terms of disturbance rejection, however, performance cannot be guaranteed in the presence of uncertainties [6, 8]. The $\mathcal{H}_\infty$ control guarantees robust stability and nominal performance in the presence of uncertainties, however, it often results in a conservative controller [14, 7]. In this paper we concentrate on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ method, in which the conservatism of the $\mathcal{H}_\infty$ method is reduced.

The paper is structured as follows. Section 2 presents the suspension structures which are used in the design process. Section 3 shows the principle of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design, and it discusses the solution of the control synthesis. In Section 4 the method is illustrated through simulation results. Finally, Section 5 presents concluding remarks.

2. Models Used in the Suspension Design

Suspension performance criteria include wheel-load variation, static and dynamic attitude control, working space, discomfort, and steering behavior. Using a half-car model, wheel-load variation, suspension working space and body mass acceleration can be calculated. Since the displacement of the body mass can be linked with discomfort, the criteria above can be considered applying a half-car model [2, 4, 5, 9].

The well-known rigid half-car vehicle model, which is shown in *Fig. 1*, is widely used for active suspension design. The model comprises three parts: the sprung mass and two unsprung masses. Let the sprung and unsprung masses be denoted by m_s and m_a, respectively. Both suspensions consist of a linear spring, a damper and an actuator to generate a pushing force between the body and axle. The front and rear suspension stiffness, the front and rear tire stiffness are denoted by k_{sf}, k_{sr} and k_{tf}, k_{tr}, respectively. The front and rear suspension damping are denoted by b_{sf}, b_{sr}. The half-car model has four degrees-of-freedom: The sprung mass is assumed to be a rigid body and has freedoms of motion in the vertical and pitch direction. Each of the unsprung masses has freedom of motion in the vertical direction. Let the front and rear displacement of the sprung and the unsprung mass be denoted by x_{1f}, x_{1r} and x_{2f}, x_{2r}. In the half-car model, the disturbances, w_f, w_r are caused by road irregularities. The input signals, u_f, u_r are generated by the actuators.

The state space representation (SSR) of the half-car model can be formalized as follows:

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{1}$$

where the state, disturbance and input force vectors and the system matrices are the following:

$$x = \begin{bmatrix} x_1 & \theta & x_{2f} & x_{2r} & \dot{x}_1 & \dot{\theta} & \dot{x}_{2f} & \dot{x}_{2r} \end{bmatrix}^T; \; w = \begin{bmatrix} w_f & w_r \end{bmatrix}^T; u = \begin{bmatrix} u_f & u_r \end{bmatrix}^T,$$

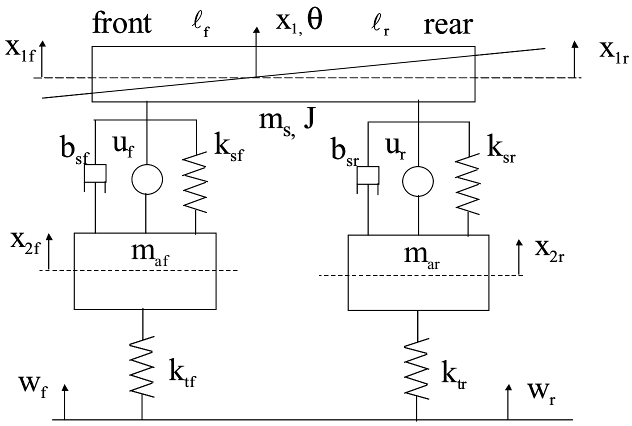


Fig. 1. Half-car model with rigid body structure

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}B \end{bmatrix}; \; B_1 = \begin{bmatrix} 0 \\ -M^{-1}K_r \end{bmatrix}; \; B_2 = \begin{bmatrix} 0 \\ -M^{-1}G_a \end{bmatrix}.$$

Here the sprung mass, the unsprung mass, the suspension stiffness, the tire stiffness, and the suspension damping matrices are as follows:

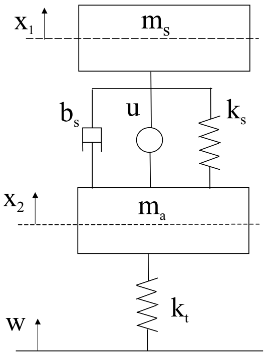
$$M = \begin{bmatrix} M_s & 0 \\ 0 & M_u \end{bmatrix}; \quad B = \begin{bmatrix} GB_sG^T & -GB_s \\ -B_sG^T & B_s \end{bmatrix};$$

$$K = \begin{bmatrix} GK_sG^T & -GK_s \\ -K_sG^T & K_s + K_t \end{bmatrix}; \quad K_r = \begin{bmatrix} 0 \\ K_t \end{bmatrix}; \quad G_a = \begin{bmatrix} -G \\ I \end{bmatrix},$$

where

$$M_s = \begin{bmatrix} m_s & 0 \\ 0 & J \end{bmatrix}; \quad B_s = \begin{bmatrix} b_{sf} & 0 \\ 0 & b_{sr} \end{bmatrix}; \quad K_s = \begin{bmatrix} k_{sf} & 0 \\ 0 & k_{sr} \end{bmatrix}$$
$$M_u = \begin{bmatrix} m_{af} & 0 \\ 0 & m_{ar} \end{bmatrix}; \quad K_t = \begin{bmatrix} k_{tf} & 0 \\ 0 & k_{tr} \end{bmatrix}; \quad G = \begin{bmatrix} 1 & 1 \\ l_f & -l_r \end{bmatrix}.$$

If other suspension models are applied, their SSR can be formalized in a similar way. A simpler structure is the so-called quarter-car model, which is shown on the left hand side of *Fig. 2*, in which the pitch angle cannot be taken into consideration. Another way to model the suspension structure is to use a flexible model, in which the real situations can be analyzed. The flexible model is shown on the right hand side of *Fig. 2*.



[Quarter-car model with rigid body structure]

[Half-car model with flexible body structure]

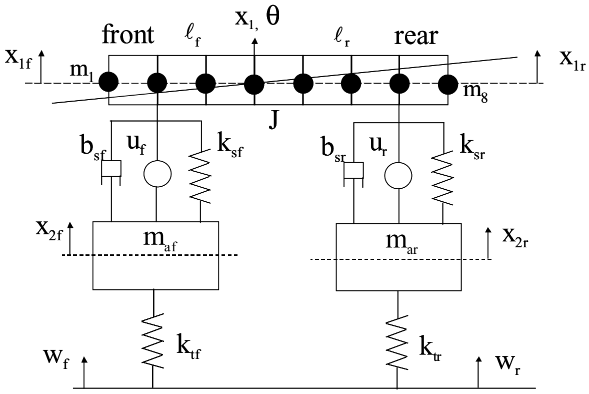


Fig. 2. Model structures for active suspension design

3. Control design based on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ method

Consider the linear plant G with input u, disturbance r, performance outputs z_∞ and z_2, feedback output y. The input is generated by output feedback, using the controller K. The signal z_∞ is the performance associated with the $\mathcal{H}_\infty$ constraint, the signal z_2 is the performance associated with the $\mathcal{H}_2$ criterion. The state space representation (SSR) of the controlled system can be written as follows:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z_\infty &= C_1 x + D_{12} u \\ z_2 &= C_2 x + D_{22} u \\ y &= C_3 x + D_{31} w, \end{aligned} \tag{2}$$

where (A, B_2) is assumed to be stabilizable and (A, C_3) is assumed to be detectable. These conditions ensure the existence of stabilizing controllers, and the existence of a K that stabilizes the $\mathcal{H}_2$ problem has been shown to be necessary and sufficient for K stabilizing the $\mathcal{H}_\infty$ problem. The desired compensator K can be determined from an optimization problem. The illustration of the controlled system is shown in *Fig. 3*.

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c. \end{aligned} \tag{3}$$

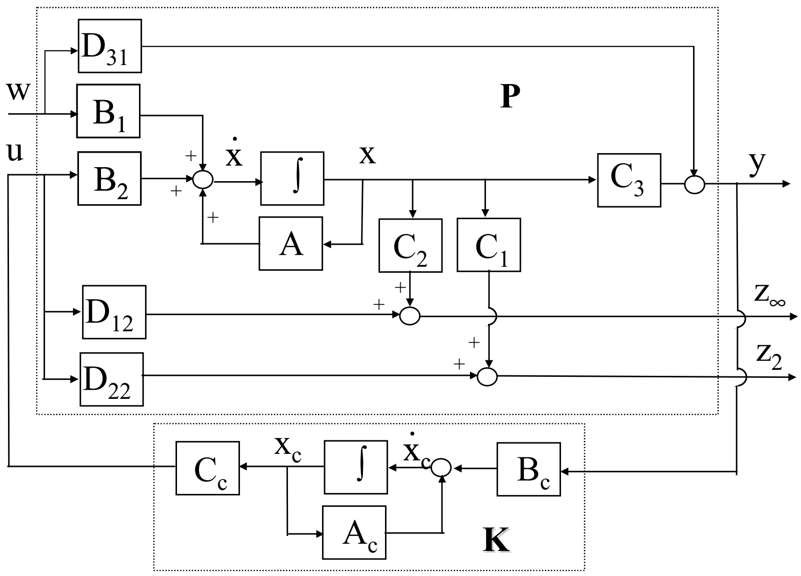


Fig. 3. The closed-loop system for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design

The objective of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is to minimize the 2-norm of the closed-loop transfer function T_{z_2w}, while constraining the inf-norm of the transfer function $T_{z_\infty w}$ to be less than some specified levels. More precisely, the problem can be stated as follows.

For the system P, find an admissible controller K which satisfies the following design criteria:

- the closed-loop system must be asymptotically stable,

- the closed-loop transfer function from w to z_∞ satisfies the constraint:

$$\left\| T_{z_\infty w}(s) \right\|_\infty < \gamma, \tag{4}$$

for a given real positive value γ,

- the closed-loop transfer function from w to z_2 must be minimized

$$\min \left\| T_{z_2 w}(s) \right\|_2 . \tag{5}$$

The task is to parameterize all suboptimal $\mathcal{H}_\infty$ dynamic controllers that stabilize the closed-loop system and satisfy the $\mathcal{H}_\infty$ constraint, and to find among them the controller that minimizes the standard $\mathcal{H}_2$ norm, [1, 3, 15].

Consider the closed-loop system in *Fig. 4*, which includes the suspension system and elements associated with the uncertainty models and performance objectives. Let the performance objectives be the heave acceleration of the sprung mass $\ddot{x}_1$, acceleration of the pitch angle $\ddot{\theta}$, suspension deflections $x_{1f} - x_{2f}$, and $x_{1r} - x_{2r}$, tire deflections of the unsprung masses x_{2f} and x_{2r}, and active forces generated by actuators u_f and u_r. Select the front and rear accelerations, x_{1f}, x_{1r}, of the body as the measured output. The weighting function $W_{p_1}, \ldots, W_{p_5}$ represent the different frequency domains of the performance outputs, namely the heave z_a and pitch z_θ acceleration, the suspension z_{sd} and tire z_{td} deflection and the control input z_u.

The measurement noises, which are denoted by n_f and n_r are taken into consideration in the design process. The random disturbances are denoted by w_f and w_r. Because of the effects of the external signals on the system, weighting functions W_w and W_n are applied to the disturbances and noises. The unmodelled dynamics is represented by W_R and Δ_M. It is assumed that the transfer function W_R is known, and it reflects the uncertainty in the model. The transfer function Δ_M is assumed to be stable and unknown with the norm condition, $\|\Delta_M\|_\infty < 1$. In the diagram, e is the input of the perturbation, d is its output. The augmented system to be controlled is shown in *Fig. 4*.

As a consequence, the $\mathcal{H}_2$ performance outputs and the $\mathcal{H}_\infty$ performance outputs are the following:

$$z_2 = \begin{bmatrix} z_a & z_\theta & z_{sd} & z_{td} & z_u \end{bmatrix}^T \tag{6}$$

$$z_\infty = \begin{bmatrix} e & z_a & z_\theta \end{bmatrix}^T . \tag{7}$$

Now, the design setup in *Fig. 4* must be formalized as a standard design problem, as illustrated in *Fig. 5*.

By applying the weighting functions and the compensator, the augmented plant and the closed-loop system can be formalized as the following forms:

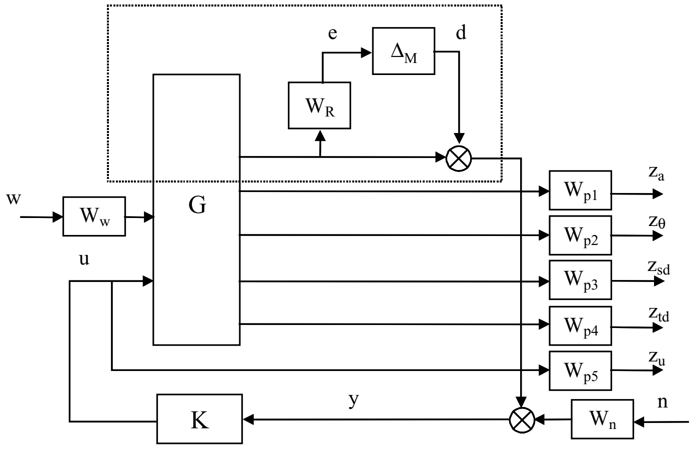


Fig. 4. The closed-loop interconnection structure

$$\begin{bmatrix} e \\ z_\infty \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} v_d \\ u \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} e \\ z_\infty \\ z_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \end{bmatrix} \begin{bmatrix} d \\ v \end{bmatrix}, \tag{9}$$

where

$$v_d = \begin{bmatrix} d & w & n \end{bmatrix}^T$$
$$v = \begin{bmatrix} w & n \end{bmatrix}^T.$$

4. Demonstration Example

The nominal parameters which are used in the design procedure are in the Table 1 in the Appendix. In the first step of the control design, the uncertainty weighting function W_R and the performance weighting function W_P must be selected. It is assumed that in the low frequency domain disturbances at the sprung mass heave and pitch accelerations should be rejected by a factor of 0.5 by using $W_{p_1} = W_{p_2} = 0.2\frac{S+50}{S+200}$, and at the suspension deflection by a factor of 1 by using

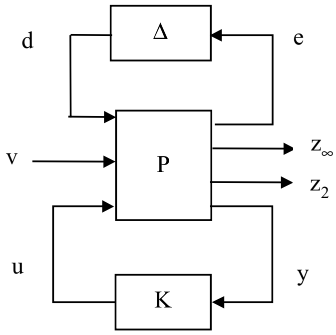


Fig. 5. The (P,K) structure with uncertainty

$W_{p_3} = \text{diag}\left[0.029\frac{S+350}{S+10}, 0.029\frac{S+350}{S+10}\right]$ for front and rear suspension, respectively. Let the frequency weighting function for the tire deflection be $W_{p_4} = \text{diag}\,[1, 1]$, for the control force $W_{p_5} = \text{diag}\left[4 \cdot 10^{-3}, 4 \cdot 10^{-3}\right]$. It is assumed that the sensor noise is 0.001 m/s^2 at the front and rear body acceleration in the whole frequency domain. The weighting function of the unmodelled dynamics is selected as follows: $W_R = 1.875\frac{S+2}{S+25}$. The weighting functions are illustrated in *Fig. 6*.

In the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design, a balance must be set up between the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm, i.e. between nominal performance and robust stability specifications. In order to create a balance between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$, the $\mathcal{H}_\infty$ versus $\mathcal{H}_2$ curve is analyzed. When modifying γ from a relatively high value, in each case the $\mathcal{H}_\infty$ norm and the $\mathcal{H}_2$ norm must be estimated to plot the points of the curve.

A gamma value that reduces the $\mathcal{H}_\infty$ norm significantly with little increase in the $\mathcal{H}_2$ norm should be selected, thus the selected γ gives an appropriate balance to the two norms. Due to the subjectivity of this selection, several controllers are constructed using some acceptable γ values. In these cases, the closed loop transfer functions between the disturbance w and performance outputs z_2 and z_∞ are computed to plot their time and frequency responses. In order to support the selection of γ, the pure $\mathcal{H}_2$ compensator design is performed to examine the closed loop between w and z_2. Moreover, the pure $\mathcal{H}_\infty$ controller is also designed to examine the closed loop between w and z_∞. This is a heuristic comparison possibility between the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ compensators. *Fig. 7* shows the mixed $\mathcal{H}_2$ versus $\mathcal{H}_\infty$ curves by using different suspension structures, i.e. the quarter-car and the half-car models.

The effects of the disturbance on the body mass acceleration, the suspension, tire deflection, on the control force are illustrated in the time domain in *Fig. 8*. In the example, the input signal is simulated as a bump with 0.02 m maximal value. The solid line corresponds to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis, the dashed line to the $\mathcal{H}_\infty$ synthesis, the dotted line to the LQG design, and the dashed-dotted line to the passive system. The frequency responses are shown in*Fig. 9*.

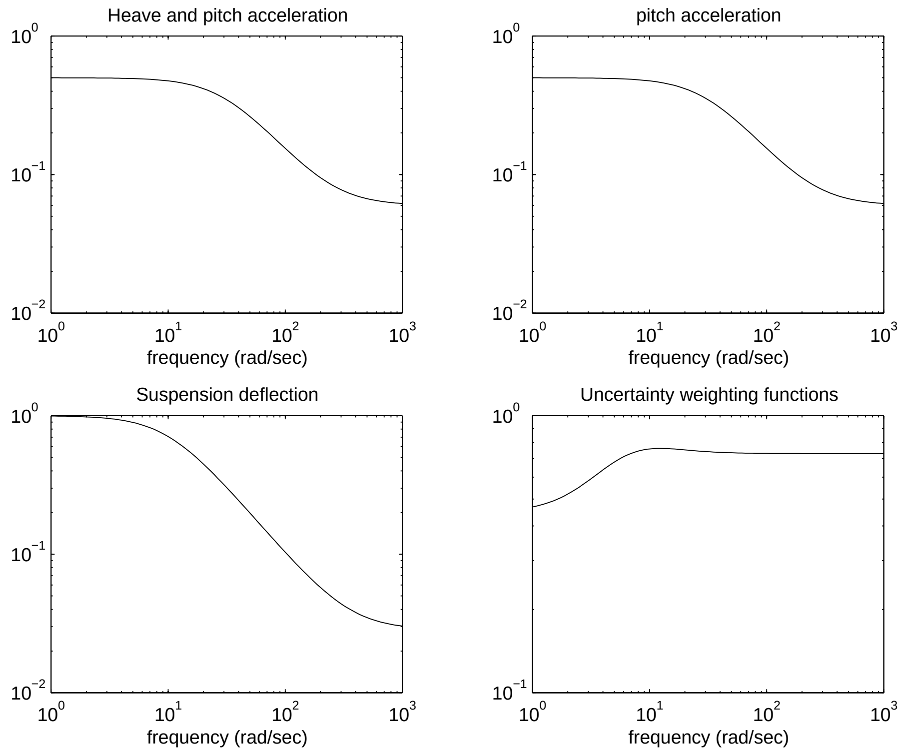


Fig. 6. Uncertainty and performance weighting functions.

5. Conclusions

This paper has presented the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design to solve the active suspension problem. The advantage of this method is that it provides excellent disturbance rejection as a result of the $\mathcal{H}_2$ criterion, and a good performance and stability margin as a result of the $\mathcal{H}_\infty$ criterion. The demonstration example has illustrated that this method is well suited to controllers for active suspension since the technique generates a set of controllers that balance between nominal performance and robust stability requirements.

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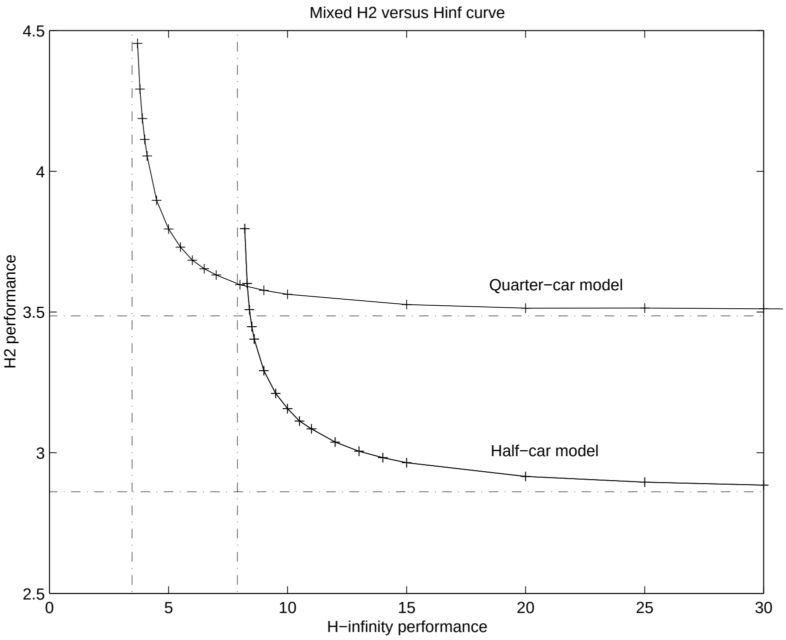

Fig. 7. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances in different model structures.

acknowledged.

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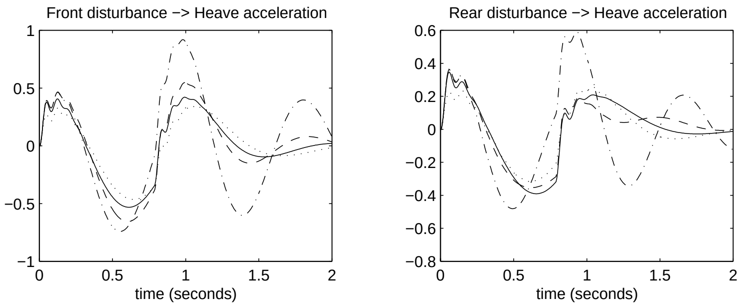
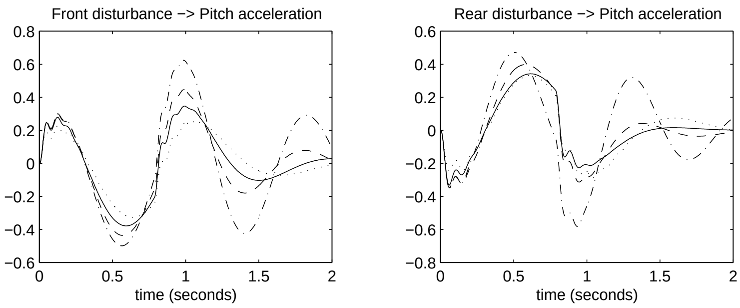
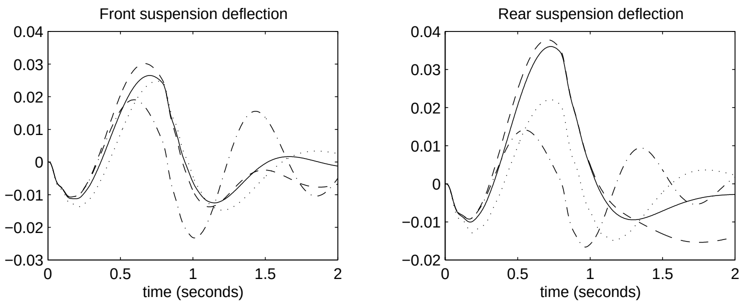


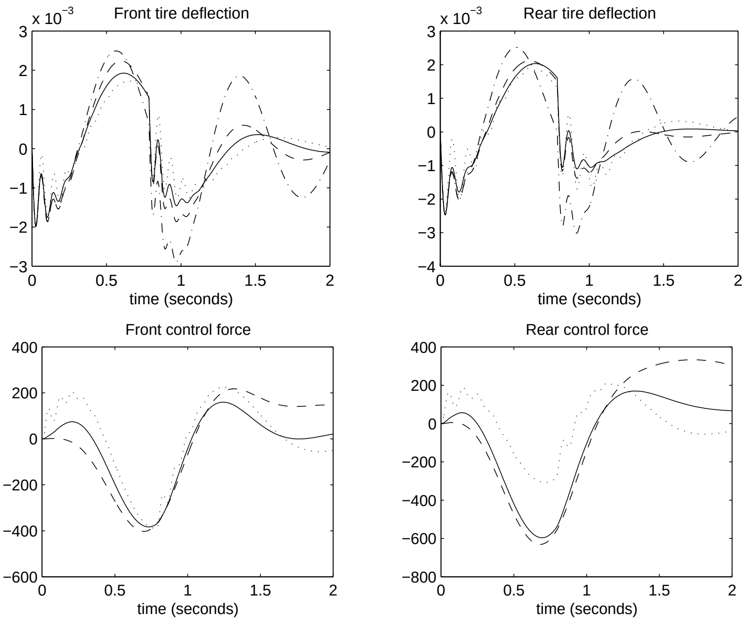
[Heave accelerations]



[Pitch accelerations]



[Suspension deflections]



[Tire deflections]

[Control forces]

Fig. 8. Time responses of the controlled system

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Appendix

[Heave accelerations]

[Pitch accelerations]

[Suspension deflections]

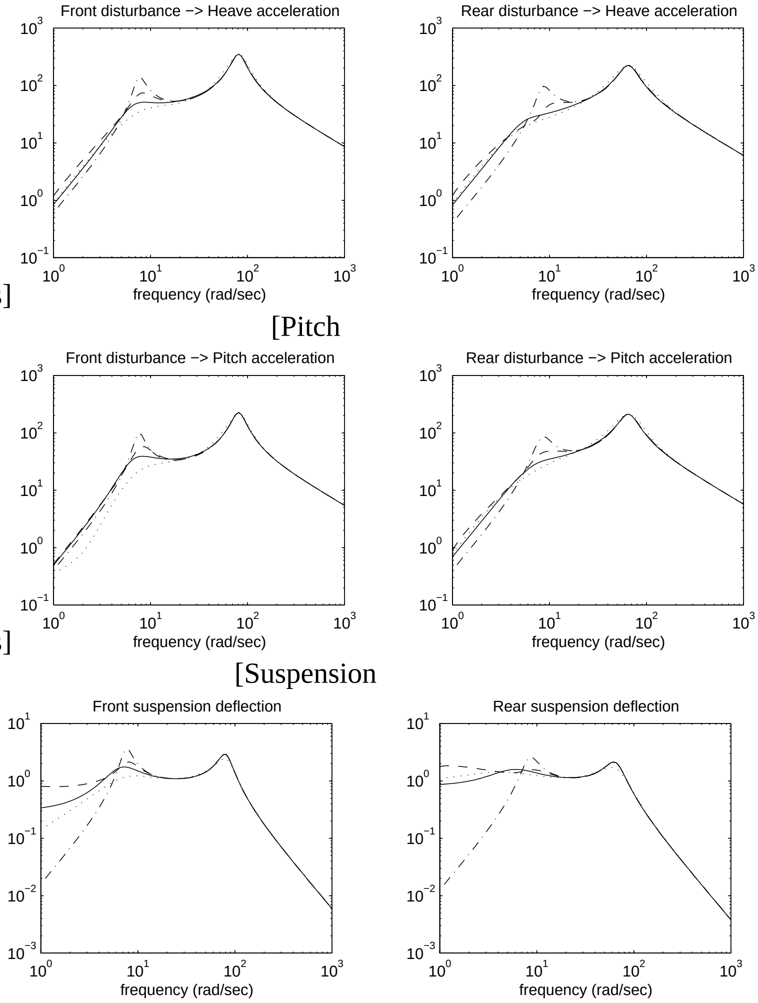

Fig. 9. Frequency responses of the controlled system

Table 1. Parameters of the half car model

Parameters (symbols)	Value
sprung mass (m_s)	580 kg
front unsprung mass (m_{af})	40 kg
rear unsprung mass (m_{ar})	40 kg
front suspension stiffness (k_{sf})	23500 N/m
rear suspension stiffness (k_{sr})	25500 N/m
front tire stiffness (k_{tf})	19900 N/m
rear tire stiffness (k_{tr})	19900 N/m
front suspension damping (b_{sf})	1000 N/m/s
rear suspension damping (b_{sr})	1100 N/m/s