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LQG/LTR CONTROLLER DESIGN FOR AN AIRCRAFT MODEL

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Abstract

The automation of the operation of aircraft equipment lightens and diminishes the manipulation of the pilot during the flight. This article gives some introduction about the fundamental aspects of the LQG/LTR control theory, and it shows an application option through an example. The movement of an aircraft can be modeled with a linear time invariant dynamic system, which must be controlled by a flight controller. This article contains the synthesis and analysis of the stability and other qualitative control parameters of the flight control.

Keywords: LQG/LTR controller, aircaft.

1. Introduction

In this paper we shall give a short view of the so-called Linear Quadratic Gaussian theory, which can be consulted for more details in [1] and [2]. KWAKERNAAK and SILVAN, ANDERSON and MOORE, DAVIS and VINTER, ASTRÖM and WITTENMARK, FRANKLIN and POWEL and many others worked on this theory. Then we revise the main stages of a Linear Quadratic Gaussian/Loop Transfer Recovery method, which was elaborated by Doyle and Stein [16]. This article dicusses an example of an aircraft flight controller design, using first LQR and LQG, afterwards the LQG/LTR methods. We can refer to [3], [4] and [5], [7] where we can find some basic applications for Linear Quadratic controller design for simplified aircraft models.

First of all we will examine the traditional optimal controller design of the aircraft flight controller system using LQR (Linear Quadratic Regulator) and LQG (Linear Quadratic Gaussian) method. With LQG/LTR method we recover the stability margin of the Kalman filter at the plant output.

In the LQG case we can use the separation principle, which means that we are able to design the LQG controller in two steps. First, the design of the LQR (Linear Quadratic Regulator), and then we have to find a state estimator, an LQE (Linear Quadratic Estimator) applying a modified cost function.
where x is the state vector, r is the referential signal, y is the output vector, $w(t)$ is the state noise, $v(t)$ is the sensor noise, $\hat{x}$ is the estimated state vector, e is the error

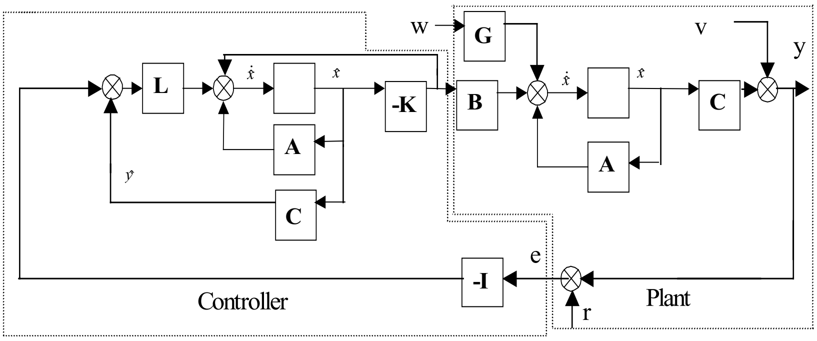

Fig. 1. The LQG controlled plant

signal, $\hat{y}$ is the estimated output vector, A is the state matrix, B is the input matrix, C is the output matrix, G is the disturbance input matrix, K is the static feedback gain matrix, L is the stationary Kalman filter static gain matrix, the observer gain, I is the identity matrix.

There exist, in our case, two types of noises (external, internal). We will calculate with an external stochastic noise (for example the air turbulence), and an internal random measurement noise. The two random (external disturbance and sensor) noises are white, Gaussian (normal) zero-mean stationary vector processes. We know the covariance matrices, and the external disturbance noise has a disturbance input matrix G. To find a controller which stabilizes our plant, we have to solve a stochastic integral problem.

2. Problem Setup

As we know well, both the LQ regulator, and the Kalman filter have good robustness and performance, so the LQG controller would have good properties too. Unfortunately this is not the case.

We have two methods for the LQG design. Either we design the feedback gain before the design of the Kalman filter, or we make first a filter and a feedback gain after. We receive a different solution if we use the two different methods of the LQG design.

Are we able to find a way for tuning the LQG controller with a parameter? The answer is yes. There is a way of recovering either the full state-feedback properties or the state estimator properties.

3. Loop Transfer Recovery (LTR) at the Plant Output

The state space representation of the plant is given by

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw, \\ y &= Cx + v, \end{aligned} \tag{1}$$

and its transfer function is the following

$$G(s) = C(sI - A)^{-1}B, \tag{2}$$

where $G(s)$ is the Laplace transform of the transfer function from control input to the output. The state equations of the LQG controller become

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(r + \hat{y}), \\ \hat{y} &= C\hat{x}, \\ u_{\text{opt}} &= -K\hat{x}, \end{aligned} \tag{3}$$

and the controller transfer function is

$$G_C(s) = K(sI - A + BK + LC)^{-1}L. \tag{4}$$

The controlled LQG closed loop can be written as

$$\begin{aligned} \varepsilon &= x - \hat{x}, \\ \dot{x} &= Ax + Bu = Ax - BKx + BK\varepsilon, \\ \dot{\varepsilon} &= Ax + Bu - A\hat{x} - Bu + L(r - C\hat{x}) = (A - LC)\varepsilon + Lr, \\ \begin{bmatrix} \dot{x} \\ \dot{\varepsilon} \end{bmatrix} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ L \end{bmatrix} r, \\ y &= [C \; 0] \begin{bmatrix} x \\ \varepsilon \end{bmatrix}, \end{aligned} \tag{5}$$

where: ε is the state error or the difference between the real and the estimated state. The transfer functions of the open loop for input is

$$G_H(s) = G_C(s)G(s) \tag{6}$$

and for the output

$$G_H(s) = G(s)G_C(s). \tag{7}$$

We shall design and recover the return ratio of Kalman filter. The stability margins of the Kalman filter are guaranteed to be good. Now, the key of the recover is the Q weighting matrix.

First, we will write the open loop transfer function

$$G_H(s) = (C(sI - A)^{-1}BK(sI - A + BK + LC)^{-1}L. \tag{8}$$

Using the notations

$$\begin{aligned}\Phi(s) &= (sI - A)^{-1},\\ \Psi(s) &= (sI - A + LC)^{-1}\end{aligned} \tag{9}$$

and applying the matrix inversion lemma

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \tag{10}$$

one obtains for the open loop transfer function

$$\begin{aligned}G_H(s) &= C\Phi(s)BK(\Psi(s)^{-1} + BK)^{-1}LG_H(s)\\ &= C\Phi(s)BK[\Psi(s) - \Psi(s)B(K\Psi(s)B + I))^{-1}K\Psi(s)]L\\ &= C\Phi(s)BK\Psi(s)[I - (BK\Psi(s) + I)^{-1}BK\Psi(s)]L\\ &= C\Phi(s)BK\Psi(s)(BK\Psi(s) + I)^{-1}L.\end{aligned} \tag{11}$$

The feedback gain K for LQ design is given by

$$K = R^{-1}B^T P, \tag{12}$$

where $P \geq 0$ is the maximal solution of the Control Algebraic Ricatti Equation (CARE)

$$AP + PA^T + Q - PBR^{-1}B^T P = 0. \tag{13}$$

Suppose that the weighting matrix Q can be written as

$$Q = Q_0 + qM. \tag{14}$$

We will examine the situation while $q \to \infty$:

$$\lim_{q\to\infty}\left(\frac{AP}{q} + \frac{PA^T}{q} + \frac{Q_0}{q} + M - \frac{PBR^{-1}B^T P}{q}\right) = 0 \tag{15}$$

and [16]

$$\lim_{q\to\infty}\frac{P}{q} = 0. \tag{16}$$

If $q \to \infty$ K can be obtained from the equation

$$\begin{aligned}M &= \frac{PBR^{-1}B^T P}{q},\\ q^{1/2}M^{1/2} &= R^{-1/2}K,\\ \lim_{q\to\infty}(K) &= \lim_{q\to\infty}(q^{1/2}R^{-1/2}M^{1/2}).\end{aligned} \tag{17}$$

Choosing $M = I$! the open loop transfer function becomes

$$\lim_{q\to\infty}(G_H(s)) = \lim_{q\to\infty}(C\Phi(s)Bq^{1/2}R^{-1/2}\Psi(s)(Bq^{1/2}R^{-1/2}\Psi(s)+I)^{-1}L) =$$
$$= \lim_{q\to\infty}(C\Phi(s)BR^{-1/2}\Psi(s)\Psi^{-1}(s)R^{1/2}B^{-1}L), \quad (18)$$
$$\lim_{q\to\infty}(G_H(s)) = C\Psi(s)L.$$

This result says that with a q very high we can approach the return ratio of the Kalman filter.

The first table represents the effect of the LQG loop transfer recovery.

Table 1. Loop transfer recovery

The poles of the $G_C(s)_{q\to\infty}$	$\to$	The zeros of the plant $C\Phi(s)L$
The zeros of the $G_C(s)_{q\to\infty}$	$\to$	The zeros of the $C\Phi(s)B$

The stages of the design LQG/LTR for output

1. We must find the required Kalman filter gain L.
2. Afterwards we determine the LQ feedback gain, using the substitutions: $M = I$, $Q = Q_0 + qM$. We want to find the K feedback gain while $q \to \infty$.

Remarks

1. We must take care of the type of LQG/LTR tuning at the plant input, because we enhance the state noises while we approximate the required LQ gain. Our aim with increasing q is to find a compensated plant which converges sufficiently closely to (18) in a large range of frequency.
2. In the case of a non-minimal phase system the optimal loop cannot be rebuilt, because the zeros of the open loop transfer function are not all negatives.

4. Linear Quadratic Gaussian/ Loop Transfer Recovery Design at the Plant Output of a Hypothetical Fighter Aircraft

During the LQG/LTR controller design it is supposed that our aircraft is cruising at a constant altitude, and with a constant velocity. The linearized longitudinal equations are simple, ordinary linear differential equations with constant coefficients. The coefficients in the differential equations are made up of aerodynamic stability derivatives, mass and inertia characteristics of the aircraft.

The longitudinal dynamic model of the fighter aircraft, in linearized form equations is [17]:

$$\begin{aligned}
\dot{q}(t) &= -0.8q(t) - 0.0006u(t) - 13.2\alpha(t) - 19\delta_E(t) - 2.5\delta_f(t),\\
\dot{u}(t) &= -0.014u(t) - 16.64\alpha(t) - 32.2\theta(t) - 0.66\delta_E(t) - 0.5\delta_f(t),\\
\dot{\alpha}(t) &= q(t) - 0.0001u(t) - 1.65\alpha(t) - 0.16\delta_E(t) - 0.6\delta_f(t),\\
\dot{\theta}(t) &= q(t),
\end{aligned} \tag{19}$$

where for control inputs $\delta_f(t)$ is the perturbed flapperon angle deflection, $\delta_E(t)$ is the perturbed elevator angle deflection, and for states $\alpha(t)$ is the perturbed angle of attack, $\theta(t)$ is the perturbed pitch angle, $q(t)$ is the pitch rate, $u(t)$ is the perturbed horizontal velocity.

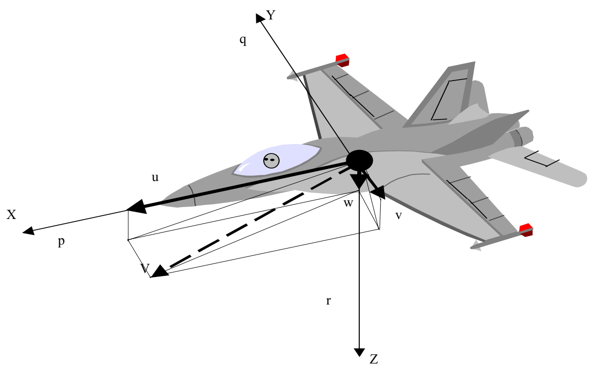

Fig. 2. The aircraft body coordinate system (ABC).

We can write these equations in state-space form:

$$\begin{aligned}
\begin{bmatrix}\dot{q}\\ \dot{u}\\ \dot{\alpha}\\ \dot{\theta}\end{bmatrix} &= \begin{bmatrix}-0.8 & -0.0006 & -13.2 & 0\\ 0 & -0.014 & -16.64 & -32.2\\ 1 & -0.0001 & -1.65 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}q\\ u\\ \alpha\\ \theta\end{bmatrix} + \begin{bmatrix}-19 & -2.5\\ -0.66 & -0.5\\ -0.16 & -0.6\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_E\\ \delta_f\end{bmatrix},\\
\begin{bmatrix}\theta\\ \gamma\end{bmatrix} &= \begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix} + \begin{bmatrix}q\\ u\\ \alpha\\ \theta\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_E\\ \delta_f\end{bmatrix},
\end{aligned} \tag{20}$$

where γ is the flight path angle $(\theta - \alpha)$.

4.1. Time Domain Analysis of the Uncontrolled Aircraft

Before we design compensator for our hypothetical aircraft dynamics it is expected to analyse the pure dynamics. In this section we study the transient response to specific test signals. The step function and Dirac's delta or impulse function are chosen as testing inputs. We can conclude some information about the transient behaviour. We neglect disturbance. We want to control the pitch angle with the elevator deflection (input 1 – output 1), and the flight path angle will be controlled with the variation of the flap (input 2 – output 2).

The impulse and step response functions can be seen in *Fig. 3*. The short and long period (phugoid) oscillations of the uncontrolled aircraft dynamics are two special characteristics of the aircraft movement. They are caused by the special placement of the poles. In this case we have $p_1 = -1.1512 + 3.4464i$, $p_2 = -1.1512 - 3.4464i$, $p_3 = -0.0058 + 0.0264i$, $p_4 = -0.0058 - 0.0264i$. The poles, which are close to the imaginary axes, simulate the long period movement, and the other pole-pair conjugated causes the short period dynamics. The periods are readily obtained once the eigenvalues are known. The duration of the short period movement is $t_s = 1.826$ s and the phugoid is $t_L = 241.6$ s.

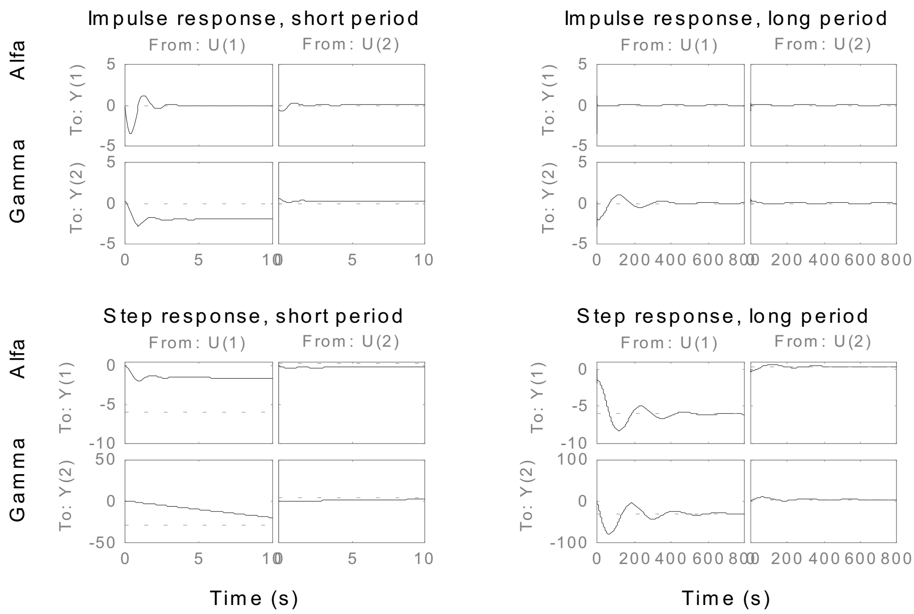

Fig. 3. The behaviour of uncontrolled dynamics in time domain

4.2. LQR Controller Design for a Hypothetical Aircraft

Our linear system is controllable and observable, and because of the equality of the Kalman rank of the observability and controllability matrix it is minimal too. We are going to design a LQR controller and we suppose that all states can be measured for the first approach. We are able to compare the results of the pure LQ controller with a LQG controller (see the next section), and see how the stochastic observer modifies the LQ results.

$$\begin{aligned} &\text{rang}\,(O_4(C, A)) = \text{rang}\,[C\; CA\; CA^2\; CA^3]^T \\ &= \text{rang}\,(C_4(A, B)) = \text{rang}\,[B\; AB\; A^2B\; A^3B] = \dim x(t) = 4. \end{aligned} \tag{21}$$

We will use the Inverse Square Rule [4] for determining the weighting matrices Q and R. Also we can write and see the solution in *Fig. 4*:

$$Q = \begin{bmatrix} 0.0365 & 0 & 0 & 0 \\ 0 & 0.000025 & 0 & 0 \\ 0 & 0 & 8.2 & 0 \\ 0 & 0 & 0 & 0.672 \end{bmatrix}, \qquad R = \begin{bmatrix} 3.6 & 0 \\ 0 & 3.6 \end{bmatrix}. \tag{22}$$

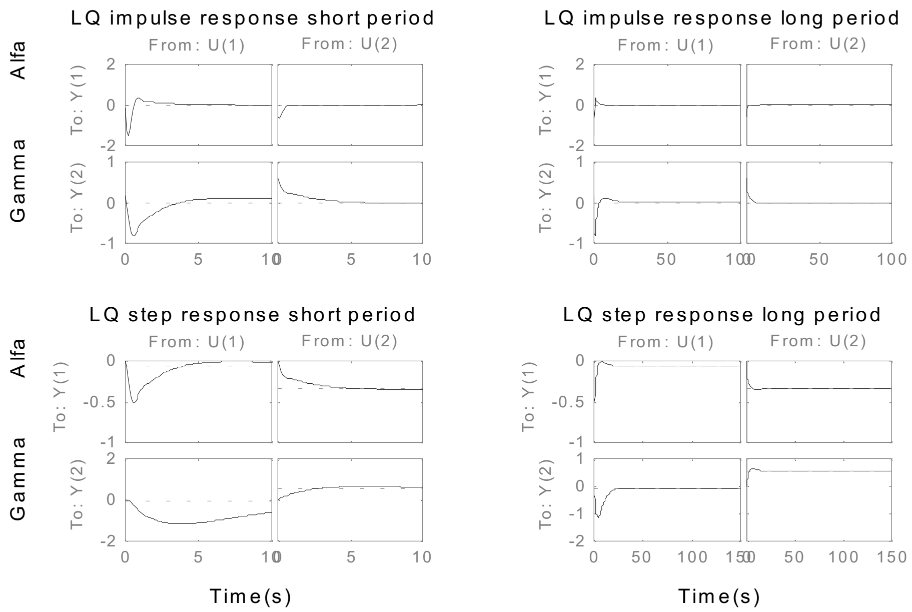

Fig. 4. The behaviour of LQ controlled dynamics in time domain

4.3. LQG Controller Design for a Hypothetical Aircraft

We cannot analyse the dynamical behaviour of our aircraft without external and measurement noise. We suppose an external stochastic turbulent air and an internal noise at the same time.

We will expand the Linear Quadratic Regulator problem to the Linear Quadratic Gaussian problem including the noises. The Separation Principe helps us to design in two different stages the LQG controller. First of all, we shall utilise the weighting matrices Q and R. As for the covariance matrices V and W of the Kalman filter I propose:

$$W = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \qquad V = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}. \tag{23}$$

When we compare the two solutions (LQR in *Fig. 4* and LQG in *Fig. 5*) received by the two different types of controller, we are able to conclude that the Kalman filter always modifies the final solution (*Fig. 4*).

The time simulation of the error function (difference between the real and the estimated state) gives us a convenient result. The error function returns to zero in about 4 s from the initial value 1.

Let us examine the behaviour of the model in frequency domain. *Fig. 6* shows how the open-loop singular values of the uncontrolled, LQG controlled, and LQR controlled loop plant change. In each case the two peaks of the maximal singular value, appropriate to the two couples of complex-conjugated pole are clearly visible.

4.4. LQG/LTR Controller Design for a Hypothetical Aircraft

The aim of the design is to recover the return ratio of the Kalman filter at the plant output. When we design the loop transfer recovery at the plant output method we want to get back the initial diagram of singular values of the open loop transfer function of the observed plant in a relatively large frequency band.

We have already calculated the Kalman filter gain L for the LQG controller. Also the first stage of the design is ready. In the second step increasing the factor q we approximate the initial observed plant. If we choose $q = 1$ the recovery is not acceptable only in a strictly narrow interval of frequency till about 0.07 rad/s. The $q = 3$ factor allows the recovery of the Kalman filtered plant in a rather wide interval (until 0.5 rad/s). Finally, $q = 10$ fits the limit 1 rad/s.

Perfect recovery should not be obtained in the case of a non-minimal-phased system. It is true that the LTR procedure can be applied to unstable plants without difficulty, since unstable poles are shifted by the feedback into the left half-plane

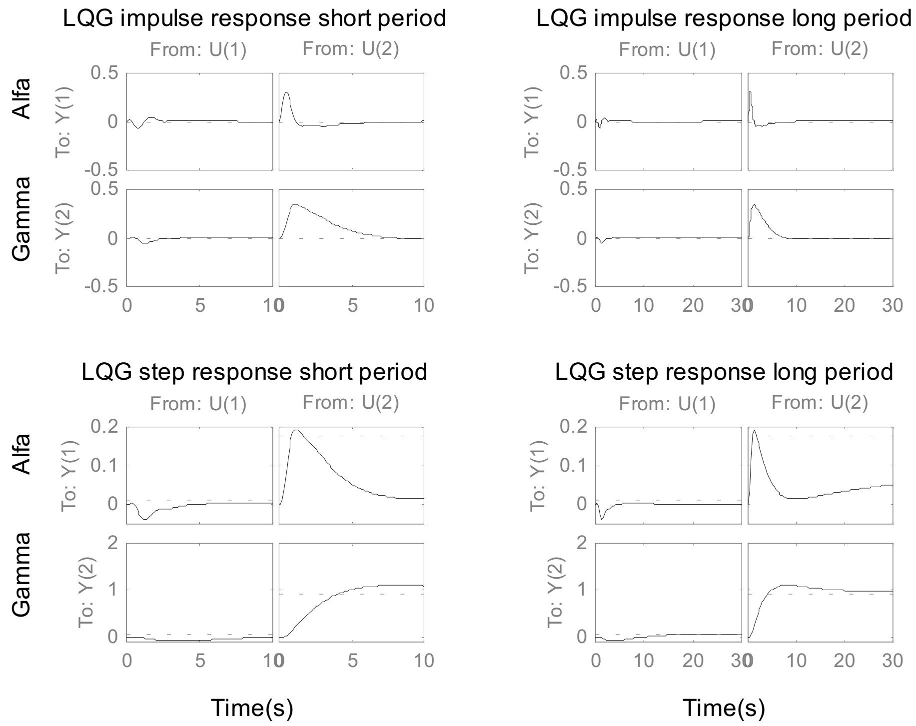

Fig. 5. The behaviour of LQG controlled dynamics in time domain

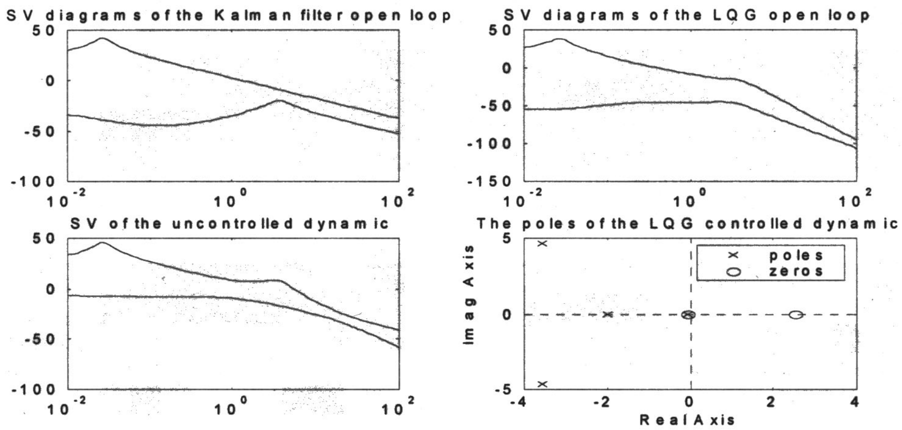

Fig. 6. The behaviour of uncontrolled and controlled dynamics in frequency domain

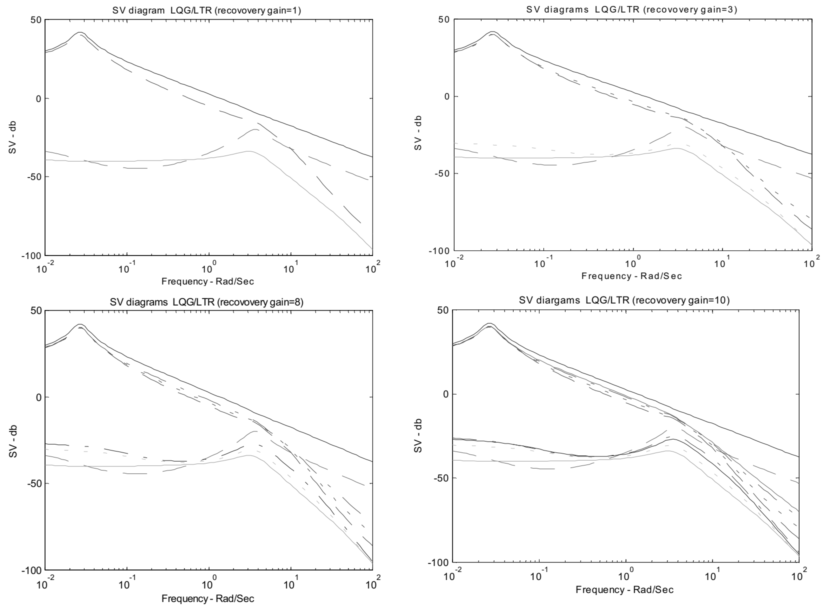

Fig. 7. The behaviour of LQG/LTR controlled dynamics in frequency domain

if all zeros and poles are in the negative real part. In our case the system is non-minimal-phased. So the recovery may be achievable at those frequencies at which the plant's response is very close to that of a minimal-phase plant. We conclude to accept the final stage of recovery when $q = 10$ because it assures the bandwidth of the referenced system. The behaviour of the LQG/LTR in time domain can be seen in *Fig. 8*. We can see the step response function, which is given by the controlled dynamic. Normally one wants to see the return of the step response function to the unit or to zero (unit from input 1 to output 1, and from input 2 to output 2, and in the other case return to zero). We have the output error when we use LQ controller. Also we have to design a referenced signal following controller called LQ servo.

5. Summary

In this paper we analyzed the aircraft dynamic described by a linear equation of the motion in time and frequency domain. We could see the effect of uncontrolled dynamics, and the design LQR, LQG and LQG/LTR controller. With LTR having obtained, we recovered a satisfactory return ratio for the Kalman filter, stochastic state-observer, we succeeded in recovering the return ratio at the plant output. The author utilized the MATLAB and CONTROL Toolbox to obtain these results of the simulations.

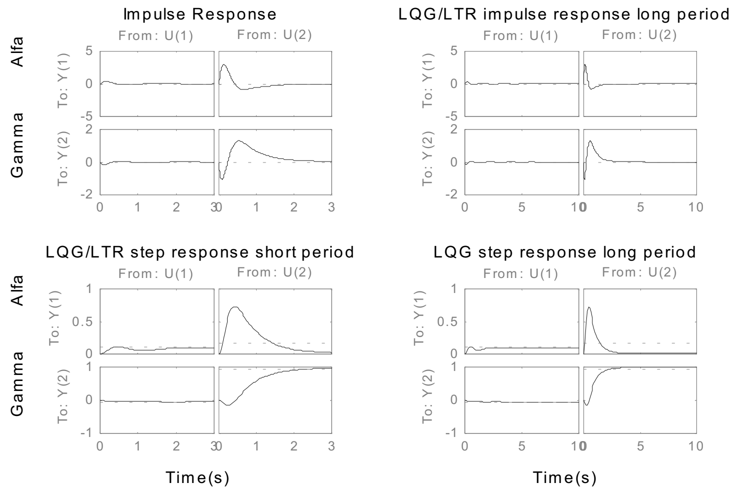

Fig. 8. The behaviour of LQG/LTR controlled dynamics in time domain

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