PERIODICA POLYTECHNICA SER. TRANSP. ENG. VOL. 27, NO. 1–2, PP. 3–16 (1999)

ROBUST SERVO CONTROL DESIGN USING THE H_∞/μ METHOD[1]

István SZÁSZI* and Péter GÁSPÁR**

*Technical University of Budapest
Műegyetem rkp. 3.
H–1521 Budapest, Hungary
Tel: (+36 1) 463 3089
Fax: (+361) 463 3087
E-mail: szaszi@kaut.kka.bme.hu
** Computer and Automation Research Institute
Kende u. 13-17
H–1111 Budapest, Hungary
Tel: (+36 1) 466 7483
Fax:(+36 1) 466 75 03
E-mail: gaspar@sztaki.hu

Received: June 30, 1999

Abstract

This paper presents a case study of a servo control synthesis to design either a one degree-of-freedom (ODF) controller or a two-degree-of-freedom (TDF) one. In order to achieve nominal performance and meet robust stability specifications, the H_∞/μ synthesis is applied for controller design to take the structured uncertainty of the plant into consideration. In this way, the controller can be designed to provide the track of the predefined reference signal, reduces the effects of the disturbances and the uncertainties on performances. In the paper, the design strategy is illustrated for an inverted pendulum device. It is demonstrated that ODF controllers do not satisfy requirements of practical application and therefore TDF controllers are required.

Keywords: robust control, servo control, two degree-of–freedom controller, H_∞/μ synthesis, uncertain linear systems.

Introduction

In the last decade, the application of the ODF controller for servo problem has been widely used in practice. For the control synthesis, both linear quadratic Gaussian (LQG) methods and H_∞ optimization procedures are proposed. The advantages of these controllers are simple construction and easy implementation. In the last few years, the TDF controllers have also been proposed. They comprise two components, a prefilter and a feedback component. With TDF controllers, the designer has more scopes to satisfy the different design specifications. An important design

[1]This project has been supported by the Hungarian National Science Foundation through grant no. T-030182.

criterion is the robust performance, i.e. the controller should be designed in a way to be suitable not only for the nominal model but also for the actual plant. The controller design is based on the H_∞/μ method, which extends the standard H_∞ method with μ analysis and synthesis, DOYLE et al., (1989); BALAS et al., (1991); ZHOU and DOYLE, (1996).

There are several methods to design TDF controllers, YOULA and BONGIORNO (1985); LIMEBEER et al., (1993); LUNDSTRÖM et al., (1999) etc. LIMEBEER et al., (1993) proposed two different design methods based on normalized coprime factors of the plant. In the first approach the feedback controller and the prefilter are designed in a single step, and in the second method the feedback controller and the prefilter are designed in two separate design stages. The first approach is also proposed by HOYLE et al., (1991).

KEVICZKY and BÁNYÁSZ, (1994) followed a different path to servo system design through an iterative scheme. In their scheme the model identification and the controller design steps are performed in a sequential way to improve the performance properties of the controlled system. This scheme has a number of excellent properties, BOKOR et al., (1999).

The aim of this paper is to present the servo control design methodology based on H_∞/μ, which provides nominal performance and robust stability for an inverted pendulum device. The nominal model is derived from an identification process. This model deviates from the actual plant, because of the numerical problems of the identification algorithm, measuring error, and uncertain components of the actual plant to be controlled. The aim is to design a model-based controller, which guarantees not only the nominal performance, but also the robust performance.

The organization of the paper is as follows. Section 2 discusses the robust servo control design both in ODF and in TDF cases. Section 3 presents the problem setup, i.e. the performance and the robust specifications of the servo design for an inverted pendulum. Section 4 and Section 5 demonstrate the examination of the H_∞/μ synthesis, and give some comparison results. Finally, Section 6 contains some concluding remarks.

1. Robust Control Design for a Servo Problem

Consider a closed-loop system, which includes the feedback structure of the plant G_{nom}, controller K with the feedback part K_y and prefilter part K_r, and elements associated with the uncertainty models and performance objectives (*Fig. 1*). In the diagram r is the reference, u is the control input, y is the output, n is the measurement noise, and e is the deviation of the output from the required one. The structure of the controller K may be either ODF or TDF. The TDF controller may be partitioned into two parts:

$$K = [K_r \ K_y].$$

In the case of ODF controller $K_r = K_y$. The required transfer function T_{yr} from r to y is defined by the designer. Inside the dashed box are W_R and Δ_M, which represent the uncertainties between the nominal model and the actual plant. Let the type of uncertainty be multiplicative at the plant input. It is assumed that the transfer function W_R is known, and it reflects the amount of uncertainty in the model. The transfer function Δ_M is assumed to be stable and unknown, except for the norm condition, $\|\Delta_M\|_\infty < 1$. In the diagram, z is the input of the perturbation, w its output. The main performance objective is that the transfer function from r to e be small, in the $\|\cdot\|_\infty$ sense, for all possible uncertainty transfer functions Δ_M. The weighting function W_e reflects the relative importance of the different frequency domains in terms of the tracking error. The weighting function W_n represents impact of the different frequency domains in terms of the sensor noise.

Necessary and sufficient conditions for robust stability and robust performance can be formulated in terms of the structured singular value denoted as μ, DOYLE, (1985). Applying the weighting functions and the TDF controller, the augmented plant P with inputs w, r, n, u, and outputs z, e, r, y, respectively, can be formalized as follows:

$$P(s) = \left[\begin{array}{ccc|c} 0 & 0 & 0 & W_R \\ W_e G_{\text{nom}} & -W_e T_{yr} & 0 & W_e G_{\text{nom}} \\ \hline 0 & I & 0 & 0 \\ -G_{\text{nom}} & 0 & -W_n & -G_{\text{nom}} \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right].$$

Note that the form of P in the case of ODF is the same except that $P_{21} = [G_{\text{nom}}\ I\ -W_n]$ and $P_{22} = G_{\text{nom}}$, see LIMEBEER et al., (1993).

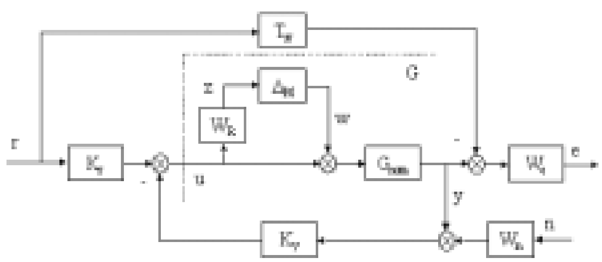

Fig. 1. Closed-loop interconnection structure

The system M is the 2×2 block-structured transfer function matrix, which is derived by the lower linear fractional transformation, $F_\ell(P, K)$ (*Fig. 2*):

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right],$$

where

$$M_{11} = -W_R K_y (I + G_{\text{nom}} K_y)^{-1} G_{\text{nom}}, \quad M_{12} = \begin{bmatrix} W_R (I + K_y G_{\text{nom}})^{-1} K_r \\ -W_R K_y (I + G_{\text{nom}} K_y)^{-1} W_n \end{bmatrix}^T$$

$$M_{21} = W_e (I + G_{\text{nom}} K_y)^{-1} G_{\text{nom}} \qquad M_{22} = \begin{bmatrix} W_e (I + G_{\text{nom}} K_y)^{-1} G_{\text{nom}} K_r - W_e T_{yr} \\ -W_e (I + G_{\text{nom}} K_y)^{-1} G_{\text{nom}} K_y W_n \end{bmatrix}.$$

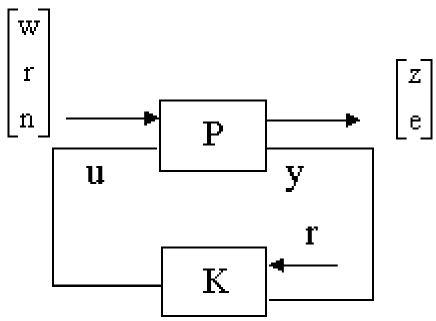

Fig. 2. Generalized closed-loop structure

Note that the form of M in the case of the ODF is the same except that $K_r = K_y$.

Our goal is to guarantee the robust performance of the closed-loop system in the face of nominal plant perturbation. Robust performance is equivalent to $\|F_u(M, \Delta)\|_\infty < 1$, where $F_u(M, \Delta)$ is the upper linear fractional transformation:

$$F_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}.$$

- The closed-loop system achieves nominal performance if the performance objective is satisfied for the nominal plant model, G_{nom}. In this problem, that is equivalent to the following form:

$$\|M_{22}\|_\infty < 1.$$

- The closed-loop system achieves robust stability if the closed-loop system is internally stable for all the possible plant models. In this problem, that is equivalent to a simple norm test on a particular nominal closed-loop transfer function:

$$\|M_{11}\|_\infty < 1.$$

- The closed-loop system achieves robust performance if the closed-loop system is internally stable for all the possible plant models, and in addition to that, the performance objective is satisfied:

$$\sup_\omega \mu(M) < 1 \Leftrightarrow \|\mu(M)\|_\infty < 1.$$

The goal of the μ synthesis is to minimize over all stabilizing controllers K, the peak value $\mu_{\mathbf{\Delta}}(\cdot)$ of the closed loop transfer function $F_L(P, K)$. The formula is as follows:

$$\min_K \sup_\omega \mu_\Delta[F_L(P, K)(j\omega)].$$

In this formula the block structure Δ is defined in the following form:

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in C^{1\times1}, \Delta_2 \in C^{3\times1} \right\} \subset C^{4\times2}.$$

The first block of this structured set with input z and output w corresponds to the scalar-block uncertainty Δ_M which is used to model the uncertainty. The second block, Δ_2 is a fictitious uncertainty block with input e and outputs r, n. This block is used to incorporate the H_∞ performance objective on the weighted output sensitivity transfer function into the μ-framework. At present, there is no direct method to synthesize a μ optimal controller, however, the D-K iteration, which combines μ-analysis and H_μ synthesis yields good results. For a constant matrix M and an uncertainty structure Δ, an upper bound for $\mu_{\mathbf{\Delta}}(M)$ is an optimally scaled maximum singular value:

$$\mu_{\mathbf{\Delta}}(M) < \inf_{D \in D_{\mathbf{\Delta}}} \overline{\sigma}\,(DMD^{-1}),$$

where $D_{\mathbf{\Delta}}$ is the set of matrices with the property that $D\Delta = \Delta D$ for every $D \in D_{\mathbf{\Delta}}$, $\Delta \in \mathbf{\Delta}$.

Using this upper bound, the optimization is reformulated as

$$\min_K \sup_\omega \inf_{D_\omega \in D_{\mathbf{\Delta}}} \overline{\sigma}\,[D_\omega F_L(P, K)(j\omega) D_\omega^{-1}],$$

where D_ω is selected from the set of scaling $D_{\mathbf{\Delta}}$ independently of every ω. The optimization problem can be solved in an iterative way using for K and D. This is the so-called D-K iteration. It is performed with a two-parameter minimization in a sequential way, first minimizing over K with D_ω fixed, then minimizing pointwise over D_ω with K fixed, etc. Although the joint optimization of D and K is not convex and the global convergence is not guaranteed, this approach works well, BALAS et al., (1991); PACKARD and DOYLE, (1993); ZHOU and DOYLE, (1996).

In the following sections, the servo control design will be demonstrated through an inverted pendulum both in ODF and in TDF cases.

2. Problem Setup: Servo Control Design for an Inverted Pendulum

The simplified structure of the inverted pendulum is shown in *Fig. 3*. The cart is propelled by a DC servomotor supported by a power amplifier, the cart position and the rod angle is measured by potmeters. Direct digital control can be realized by means of a computer complemented with analog to digital and digital to analog converters. The objective of the experiment is to design a controller which stabilizes

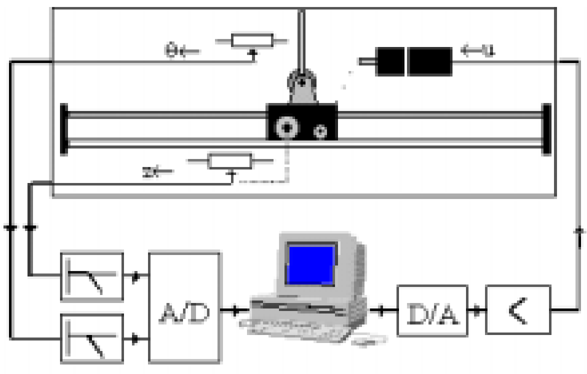

Fig. 3. Schematic diagram of the laboratory experiment

the rod and keeps the cart in a desired position. Further details of the structure of the inverted pendulum can be read in *Soumelidis* et al., (1997).

This is a two degree-of-freedom system, the input F is the force of the cart the outputs are the displacement of the cart, and the angle of the rod. The motion differential equations of the inverted pendulum are as follows:

$$(M+m)\ddot{y} - ml\dot{\theta}^2 \sin\theta + ml\ddot{\theta}\cos\theta = F$$
$$m\ddot{y}\cos\theta + ml\ddot{\theta} = mg\sin\theta,$$

where M is the mass of the cart, m is the mass of the rod, l is the length of the rod. This is a nonlinear system, which is simplified by a linear model in the identification process.

The controller must be designed in such a way that the following criteria are met:

Specification 1: The closed-loop system must be stable.

Specification 2: Let the reference signal for the displacement be a step function with 0.4 steady state value. The output signals, i.e. displacement y (tracking) and rod angle θ (interaction), should satisfy the following specifications:

- The settling time should be less than 10 sec, i.e. $|y(t) - \overline{y}(t)| < 0.04$, $\forall t \geq 10$ sec.
- The overshoot should not exceed 10%, i.e. $y(t) < 0.45, \forall t$.
- The steady-state error should be under 1%, i.e. $|y(t) - \overline{y}(t)| < 0.004$.
- The interaction should be minimal, i.e. $\theta(t) < 0.1, \forall t$.
- The steady-state in the other channel should be minimal, i.e. $\theta(\infty) < 0.1$.

Specification 3: The control voltage must not be more than 10 V.

3. The H_∞/μ Synthesis of the ODF Controller

Let the required transfer function from the reference to the displacement of the cart be the following simple first-order system:

$$T_{yr} = \frac{1}{s+1}.$$

The pendulum uncertainty is modelled as a complex scalar block, multiplicative uncertainty at the plant input. Let the frequency weighting function of the unmodelled dynamic be as follows:

$$W_R = 3\frac{s+200}{s+2000}.$$

It means that in the low frequency domain, the modelling error is about 30% and, in the upper frequency domain it is up to 100%. The frequency functions of the performance and the robust stability are shown in *Fig. 4*. It is assumed that the sensor noise is 2 mm in the cart position and 0.01 rad in the rod angle in the whole frequency domain.

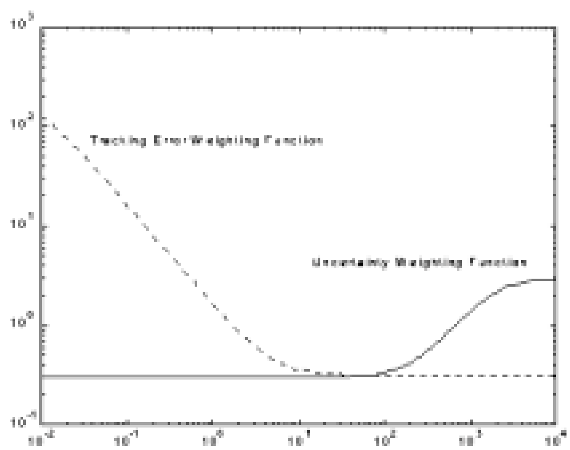

Fig. 4. Uncertainty and performance weighting functions

Using the weighting functions of the nominal performance and the robust stability specifications, the optimal H_∞ controller is designed using the standard gamma iteration. The gamma value achieved is 20.74. The M_{11} and M_{22} transfer functions associated with robust stability and nominal performance may be evaluated separately. The controlled system achieves robust stability, however, it does not achieve nominal performance. This conclusion follows from the singular value plots as it is shown in *Fig. 5*.

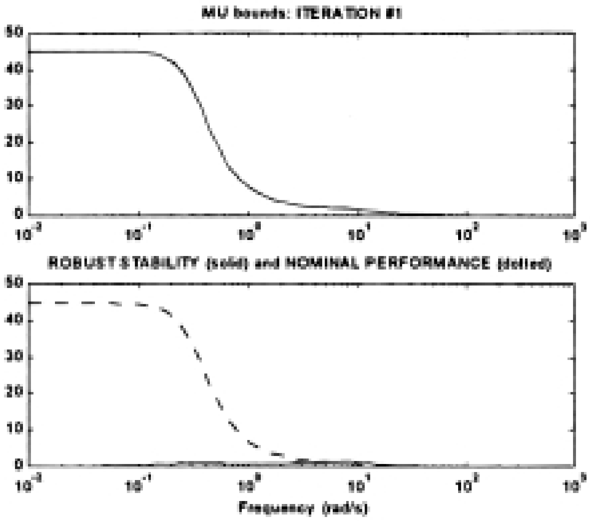

Fig. 5. Robust stability and nominal performance with H_∞ controller

In the next step, the D-K iteration is performed. The results of Step 4 of the D-K iteration are shown in *Fig. 6*. It is claimed that both the nominal performance and the robust stability requirements are fulfilled. Moreover, robust performance is also achieved, because the value of μ is under 1. The important values of the steps of the D-K iterations are shown in *Table 1*.

Table 1. Summary of the D-K iteration

Iteration	#1	#2	#3	#4
Controller Order	7	9	17	23
D-Scale Order	0	2	10	16
Gamma Achieved	45.000	2.867	1.226	0.991
Peak μ Value	44.999	2.867	1.226	0.995

The step responses of the cart position and of the rod angle with the control input are shown in *Fig. 7*. The steady state of the cart position meets the requirement, however, the transient properties of it do not meet other requirement. The oscillation of the step response is significant and the overshoot exceeds the required value. The interaction between the signals also exceeds the defined limit. The most serious problem is that the control input increases significantly in a short time period. It means, that the controlled system does not meet the 3rd specification, and therefore, the controller is not realizable.

It is noted that theoretically a weighting function can be applied to the control

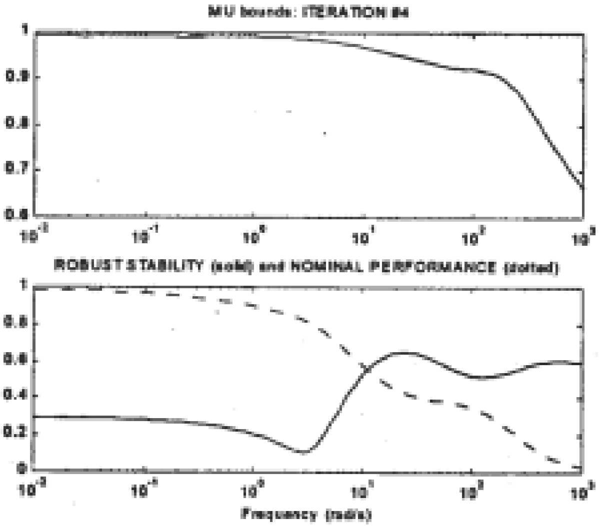

Fig. 6. Robust stability and nominal performance after the D-K iteration

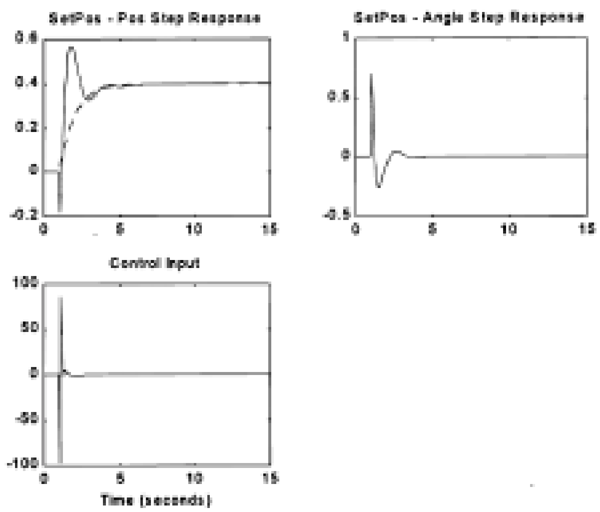

Fig. 7. Step responses of the cart position and of the rod angle with control input

input to decrease its amplitude. Experiment results show, however, that in this case, nominal performance and robust stability specifications cannot be met using H_∞/μ synthesis. Moreover, to achieve robust performance these specifications would have been modified in a way, which would not satisfy requirements of practical application. Consequently, the design specifications do not fulfill if ODF servo controller is used.

4. The H_∞/μ Synthesis of the TDF Controller

The conclusion of the previous section leads to the application of the TDF servo controller. This structure provides the weighting of the control input, which is important in terms of the controller realization. The role of the weighting function W_u is to emphasize the different frequency domains of the input effort. The modified closed-loop structure is shown in *Fig. 8*. In this case, the other design specifications are also improved. E.g. the designed controller meets the robust stability requirement, however larger modelling error is allowed. The frequency functions of the performance and the robust stability are shown in *Fig. 4*. Let the frequency weighting function of the control input be as follows:

$$W_u = \frac{1}{20}.$$

It means, that the effect of the reference signal on the control input do not exceed 26 dB.

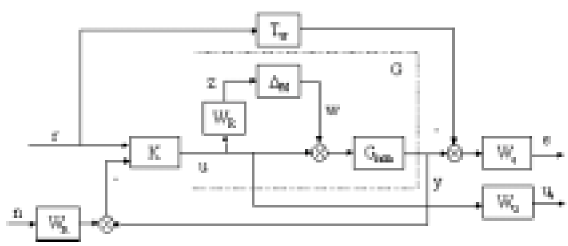

Fig. 8. Closed-loop interconnection structure

Using the weighting functions of the nominal performance and the robust stability specifications, the optimal H_∞ controller is designed using the standard gamma iteration. The gamma value achieved is 2.34. The controlled system achieves robust stability, however, it does not achieve nominal performance, as it shown in *Fig. 9*.

The results of Step 2 of the D-K iteration are shown in *Fig. 10*. It is claimed that both the nominal performance and the robust stability requirements are met. Moreover, robust performance is also achieved, because the value of μ is under 1. The important values of the steps of the D-K iterations are shown in *Table 2*.

Using a simulation procedure the step responses of the cart position and of the rod angle with the control input are shown in *Fig. 11*. The tracking of the reference

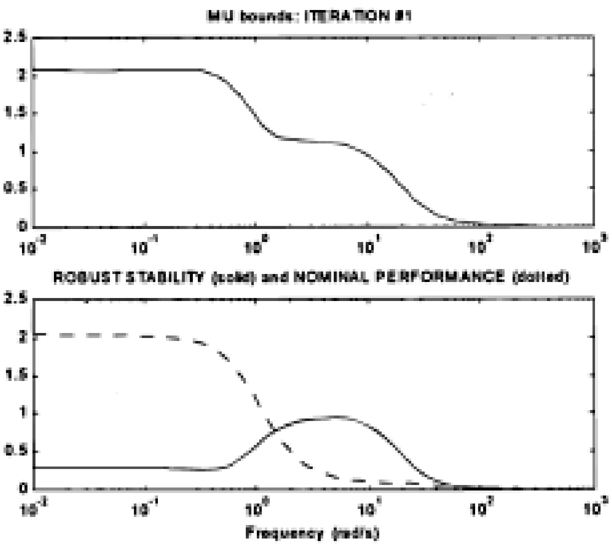

Fig. 9. Robust stability and nominal performance with H_∞ controller

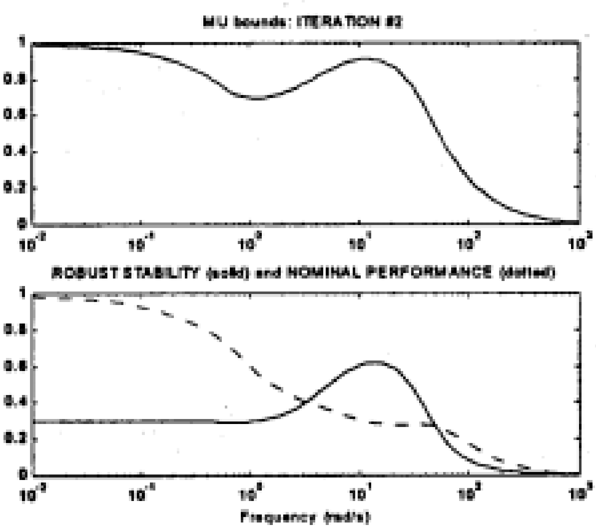

Fig. 10. Robust stability and nominal performance after the D-K iteration

signal meets the requirements both in the transient time domain and in steady state. The interactions between signals are also eliminated accordingly in line with the

Table 2. Summary of the D-K iteration

Iteration	#1	#2
Controller Order	7	9
D-Scale Order	0	2
Gamma Achieved	2.344	1.012
Peak μ Value	2.086	0.991

specification. When the weighting function W_u is applied the control input does not exceed the value set in specification 3. Moreover, the order of the TDF controller significantly lower than the order of the ODF controller.

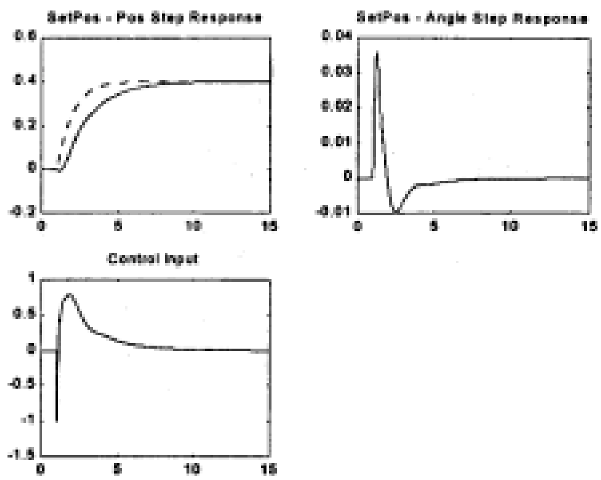

Fig. 11. Step responses of the position and of the angle with control input

The designed controller is used for the real inverted pendulum. The step responses measured show similarly good properties to the simulation results, as it is shown in *Fig. 12*.

5. Conclusion

In this paper the H_∞/μ servo controller design has been presented through the application of an inverted pendulum. It has been demonstrated that the ODF controller is not sufficient and a TDF servo controller is required. The TDF controller was designed in a way that robust performance of the controlled system is guaranteed. In the near future, the focus of the project will be on the development of an iterative

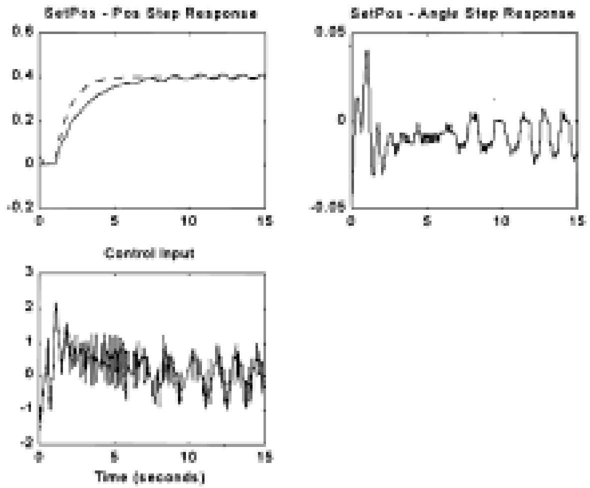

Fig. 12. Step responses of the position and of the angle with control input in real situation

scheme for the model based H_∞/μ controller design. The scheme will be applied to other problems from our practice in the field of vehicle control.

References

[1] BALAS, G. – DOYLE, J. C. – GLOVER, K. – PACKARD, A. SMITH, R.: μ Analysis and Synthesis Toolbox, MUSYN Inc. and The Mathworks Inc., 1991.

[2] BOKOR, J. – KEVICZKY, L. – BÁNYÁSZ, CS.: Identification and Robust Control Based in the K-B Controller Scheme, *Proc. of the European Control Conference*, Karlsruhe, 1999.

[3] DOYLE, J.: Structured Uncertainties in Control System Design, *Proceedings of the 24th Conference on Decision and Control*, Landerdale, pp. 260–265, 1995.

[4] DOYLE, J. C. – GLOVER, K. – KHARGONEKAR, P. P. – FRANCIS, B. A.: State-Space Solutions to Standard H_2 and H_∞ Control Problems, *IEEE Trans. Automatic Control*, Vol. 34, pp. 831–847, 1989.

[5] HOYLE, D. J. – HYDE, R. A. – LIMEBEER, D. J. N.: An H_∞ Approach to Two Degree of Freedom Design, *Proc. of the 31st Conf. on Decision and Control*, Tucson, pp. 1581–1585, 1991.

[6] BOKOR, J. Lecture Notes on H_∞ Control Design, Budapest, 1999.

[7] KEVICZKY, L. – BÁNYÁSZ, CS.: A New Structure to Design Optimal Control Systems, *Proc. of the IFAC Workshop on New Trends in Design of Control Systems*, Smolience, pp. 102–105, 1994.

[8] LIMEBEER, D. J. N. – KASENALLY, E. M. – PERKINS J. D.: On the Design of Robust Two Degree of Freedom Controllers, *Automatica*, Vol. 29, pp. 157–168, 1993.

[9] LUNDSTRÖM, P. – SKOGESTAD, S. – DOYLE, J. C.: Two-degree of Freedom Controller Design for Ill-conditioned Distillation Process Using μ-Synthesis, *IEEE Trans. On Control Systems Technology*, Vol. 7, pp. 12–21, 1999.

[10] PACKARD, A. – DOYLE, J.: The Complex Structured Singular Value, *Automatica*, Vol. 29, pp. 71–109, 1993.

[11] SOUMELIDIS, A. – GÁSPÁR, P. – BOKOR, J.: Inverted Pendulum. An Environment for Intelligent Control Design and Test, *Proc. of the 3rd IFAC Symp. on Intelligent Components and Instruments for Control Applications*, Annecy, pp. 421–425, 1997.
[12] STEIN, G. – DOYLE, J. C.: Beyond Singular Values and Loop Shapes, *AIAA J. Guidance Contr. and Dynamics*, Vol. 14, pp. 5–16, 1991.
[13] YOULA, D. C. – BONGIORNO, J. J.: A Feedback Theory of Two-degree of Freedom Optimal Wiener-Hopf Design, *IEEE Trans. Automatic Control*, Vol. 30, pp. 652–665, 1985.
[14] ZHOU, K. – DOYLE, J. C.: Robust and Optimal Control, Prentice Hall, New Jersey, 1996.