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ON THE NATURE OF FLUTTER AND DIVERGENCE MATERIAL INSTABILITIES

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Abstract

In the first studies on material instability two main types were distinguished: the divergence and the flutter. While divergence was treated as strain localization the nature and physical explanation of flutter remained an open question. In this paper by using the theory of dynamical systems both mathematical and physical interpretations for these instability modes are proposed.

Keywords: strain localization, dynamical systems.

1. Introduction

Material instability problems like the localization of plastic deformation use different stability definitions. Most of them are generalizations of HILL's concept [9] or the DRUCKER postulate [6]. When a solid body is considered as a dynamical system [15], [16] and a state of the body is a solution of it, the stability of this state means the stability of the solution. In this case the obvious stability definition is the one of the theory of dynamical systems, the so called Liapunov stability [14]. This is a kinematic definition, quite similar to the one used by ERINGEN [7].

The loss of material stability is in close connection with the singularity of the acoustic tensor [4]. Then there is a change in the nature of the acceleration wave speeds. One possibility is that one of them is zero, the other is the appearance of a complex conjugate pair. In case of a zero Wave speed there is a stationary discontinuity, and when the squares of the two wave speed are complex conjugates, it is called a flutter [10], [11]. A similar classification is known for ways of the loss of stability of a solution of dynamical systems [16]. It is called a static bifurcation when the linearized part of the differential equation representing the system has a zero eigenvalue. The case, when there is a pair of pure imaginary eigenvalues, is the dynamic bifurcation. Now, the question is how these concepts relate to each other.

The aim of this paper is to investigate and interpret divergence and flutter instabilities as a loss of stability of an appropriate dynamical system.

2. The Basic Equations of Material Instability and Plastic Localization Problems

Denoting the position of a material point in the reference and the current configurations by X_J and x_i the position vectors are $\mathbf{R} = X_J \mathbf{G}_J$, and $\mathbf{r} = x_j \mathbf{g}_j$. As usual the Cartesian tensor notation and the implied summation of the repeated subscripts are used. The deformation gradient $\mathbf{F}$ is

$$F_{jJ} = \frac{\partial x_j}{\partial X_J}.$$

The equation of motion without volume force is

$$S_{jK,K} = \rho \ddot{u}_j, \tag{1}$$

where S_{jK} is the first Piola–Kirchhoff stress tensor, $S_{jK,K}$ is the divergence of it and $\mathbf{u} = \mathbf{R} - \mathbf{r}$ is the displacement. The classical setting of the equations of material instability problems [5], [13] uses a simplified rate constitutive equation in the form

$$\dot{S}_{jK} = K_{jKlM} \dot{F}_{lM}, \tag{2}$$

where F_{lM} is the deformation gradient and K_{jKlM} is the fourth-order tangent modulus tensor. By substituting (2) into the rate form of (1) the motion of the continuum can be described by

$$\rho \ddot{v}_j = \left(K_{jKlM} v_{l,M}\right)_{,K}. \tag{3}$$

The coefficients K_{jKlM} are considered here as piecewise constants [13]. Now two kinds of questions can be asked. One on the existence of strain localization and the other on the stability of the material.

To answer the first one means to search for the condition of the existence of a thin band in the material, in which the rate field quantities differ from the uniform values outside [12], [13]. By denoting $(\ )^b$ and $(\ )^\circ$ the values inside and outside of the band are

$$\left(\dot{F}_{lM}\right)^b = \left(\dot{F}_{lM}\right)^\circ + q_l n_M, \tag{4}$$

n_M are the coordinates of the vector showing the orientation of the band and q is the amplitude of the jump on the band. The rate of stress equilibrium implies

$$n_K \left(\left(\dot{S}_{jK}\right)^b - \left(\dot{S}_{jK}\right)^\circ\right) = 0,$$

that is, with (2) and (4)

$$\left(n_K \left(K_{jKlM}\right) n_M\right) q_l = 0. \tag{5}$$

There are nonzero amplitudes in (5), if and only if

$$\det\left[n_K \left(K_{jKlM}\right) n_M\right] = 0. \tag{6}$$

For the second question the stability of a state of the material should be investigated. In dynamics a state of a system is said to be stable if its motion remains in an arbitrary small neighbourhood of it by applying sufficiently small perturbations [14]. The same concept of stability is used by [8] for continua. Thus for dynamic stability the role of perturbations and the role of the propagation of disturbances are essential ones. It means that in stability investigations one should concentrate on the wave propagation.

Eq. (3) has a wave solution in the form

$$v_j = q_j \exp\left(i\,(n_K X_K - ct)\right), \tag{7}$$

where n_K is the direction of propagation and $i = \sqrt{(-1)}$. In (7) the wave speed c determines the stability. When $c^2 > 0$ (7) is stable, when $c^2 < 0$ it is unstable [13]. By substituting (7) into (3)

$$-\rho c^2 q_j \exp\left(i\,(n_K X_K - ct)\right) = \left(K_{jKlM}\right) n_M n_K q_l \exp\left(i\,(n_K X_K - ct)\right)$$

is obtained. Hence

$$\left(\left(K_{jKlM}\right) n_M n_K - \rho c^2 \delta_{jl}\right) q_l = 0. \tag{8}$$

Thus the condition of the existence of a wave solution of nonzero amplitudes reads

$$\det\left[\left(K_{jKlM}\right) n_M n_K - \rho c^2 \delta_{jl}\right] = 0,$$

that is, the stability depends on the eigenvalues of

$$\left[\left(K_{jKlM}\right) n_M n_K\right].$$

When all of them are real $\left(c^2 > 0\right)$, the material is in a stable state. On the one hand the loss of stability $\left(c^2 = 0\right)$ may take place under the same conditions as the localization and when there is at least one imaginary eigenvalue $\left(c^2 < 0\right)$, there is an unstable state. The loss of stability is connected with the appearance of imaginary eigenvalues. On the other hand [13] introduces the case called flutter, in which eigenvalue c^2 gets complex values. Compared to divergence when the wave solution stops in the body to form a stationary discontinuity flutter is somehow mysterious. How could we imagine complex wave speeds? Introducing the theory of dynamical systems in the next part we will show a possible explanation of such a case.

3. Dynamical Systems

Let us introduce the notations of the theory of dynamical systems into this material instability problem [3]. For the sake of simplicity from now on small displacements are assumed. In an operator form (3) reads

$$\ddot{v} = f(v). \tag{9}$$

Here $v = (v_1, v_2, v_3)$ is a vector of the coordinates of the velocity field satisfying the boundary conditions and $f(v)$ is a differential operator defined by the left hand side of (3). *Eq.* (9) defines an infinite dimensional dynamical system. The stability of a state of the continuum means the Liapunov stability of a solution $v(t)$ of (9), that is, by perturbing the system the velocity field $v'(t)$ is sufficiently close to the unperturbed one $v(t)$. The stability investigation of some solutions of equations like (9) starts with a transformation into a first order equation by introducing new variables

$$w_j^1 = v_j, \qquad w_j^2 = \dot{v}_j, \qquad (j = 1, 2, 3) \tag{10}$$

and with the linearization at a solution, (at $v = 0$ for the sake of simplicity)

$$\dot{w} = Df\, w.$$

The eigenvalues of the linear operator Df show the stability properties. Unfortunately, an equation like (9) cannot even give strict results for stability, because the set of eigenvalues consists of pairs $\pm\sqrt{\lambda}$ and when $\lambda > 0$ there is instability, and when $\lambda < 0$ the real part of the eigenvalues is zero. For conservative or linear systems this would imply stability but even the nonlinearities can ruin it. Moreover, (9) is not structurally stable in the sense of [1], that is, any small perturbations can cause qualitative changes of the solutions. To get structural stability, as the simplest possibility for small strains, a strain rate dependent material is used instead of (2). In a general form [2], [7]

$$\sigma_{jk} = K_{jklm}^1 \epsilon_{lm} + K_{jklm}^2 \dot{\epsilon}_{lm}, \tag{11}$$

where the coefficients K_{jklm}^1, K_{jklm}^2 are considered to be piecewise constants. Then the equation of motion is

$$\rho \ddot{v}_j = K_{njkl} v_{k,ln} + L_{njkl} \dot{v}_{k,ln},$$

where

$$K_{njkl}^1 = \frac{1}{2}\left(K_{njkl} + K_{njlk}\right) \qquad \text{and} \qquad K_{njkl}^2 = \frac{1}{2}\left(L_{njkl} + L_{njlk}\right).$$

Introducing new variables from (10) the equation of motion is

$$\dot{w}_j^1 = w_j^2,$$

$$\dot{w}_j^2 = \frac{1}{\rho}\left(K_{njkl} w_{k,ln}^1 + L_{njkl} w_{k,ln}^2\right).$$

By defining linear differential operators

$$\widehat{K}_{jk} v_k = K_{njkl} \frac{\partial^2}{\partial x_n \partial x_l} v_k, \quad \widehat{L}_{jk} v_k = L_{njkl} \frac{\partial^2}{\partial x_n \partial x_l} v_k$$

and

$$\mathbf{L}\left[w_j^1, w_j^2\right] := \left[w_j^2, \frac{1}{\rho}\left(\widehat{K}_{jk}w_k^1 + \widehat{L}_{jk}w_k^2\right)\right]$$

the equation of motion is

$$\left[\dot{w}_j^1, \dot{w}_j^2\right] = \mathbf{L}\left[w_j^1, w_j^2\right]. \tag{12}$$

The Liapunov stability depends on the real part of the eigenvalues λ of the linear operator $\mathbf{L}$. The eigenvalue equation is

$$\mathbf{L}\left[w_j^1, w_j^2\right] = \lambda\left[w_j^1, w_j^2\right]. \tag{13}$$

When an eigenvector

$$\left[\overline{w}_j^1, \overline{w}_j^2\right]$$

is substituted into the equation of motion (12)

$$\left[\dot{\overline{w}}_j^1, \dot{\overline{w}}_j^2\right] = \overline{\lambda}\left[\overline{w}_j^1, \overline{w}_j^2\right]$$

is obtained with the eigenvalue $\overline{\lambda}$. The solution of it is

$$\left[\overline{w}_j^{01}, \overline{w}_j^{02}\right]\exp\left(\overline{\lambda}t\right). \tag{14}$$

Having all the eigenvalues and eigenvectors, a solution of the equation of motion can be given as a linear combination of functions (14), thus the stability requires negative real parts for all eigenvalues. From (13)

$$w_j^{02} = \lambda w_j^{01},$$

$$\frac{1}{\rho}\left(\widehat{K}_{jk}w_k^{01} + \widehat{L}_{jk}w_k^{02}\right) = \lambda w_j^{02},$$

then by substituting the first group of equations into the second one and using (10)

$$\left(\rho\lambda^2 v_j^0 - \widehat{K}_{jk}v_k^0 - \lambda\widehat{L}_{jk}v_k^0\right) = 0 \tag{15}$$

is obtained being a system of second order partial differential equations with boundary conditions. Thus the state of the material is stable, when for all values of λ, at which there exist nontrivial solutions of (15), $\Re\lambda < 0$. Case $\Re\lambda = 0$ is called the stability boundary. Then the state of the material under consideration may lose stability by either a static ($\Im\lambda = 0$) or dynamic ($\Im\lambda \neq 0$) bifurcation types of loss of stability. The relation between the classification above and the one used earlier into groups divergence and flutter can be studied by omitting the dissipation in (11) $\left(K_{ijklm}^2 = 0\right)$. Then after proper rearrangement *Eq.* (15) reads

$$\left(\widehat{K}_{jk} - \rho\lambda^2\delta_{jk}\right)v_k^0 = 0. \tag{16}$$

Let us substitute harmonic functions

$$v_j^0 = q_j \exp\left(i\ (n_m x_m)\right) \tag{17}$$

into (16) then equation

$$\left(\left(K_{jklm}\right) n_m n_k + \rho\lambda^2 \delta_{jl}\right) q_l = 0 \tag{18}$$

can be obtained. From *Eq.* (8) and (18) $c^2 = -\lambda^2$.

Then at divergence $c^2 = 0$ implies static bifurcation $\lambda = 0$, while for flutter c^2 complex we get $-\lambda^2$ complex, that is, $\lambda_{1,2,3,4} = \pm\alpha \pm i\beta$, where $\alpha,\ \beta > 0$, which indicates an unstable behavior, too.

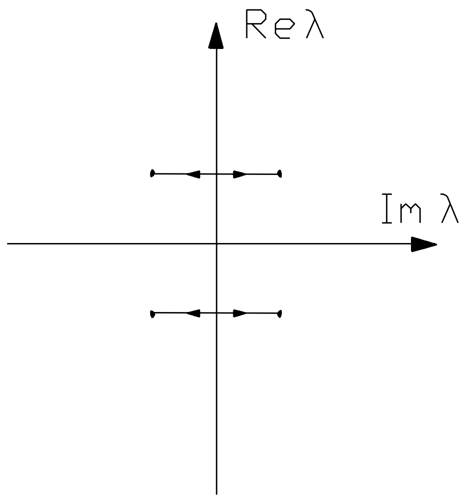

Fig. 1. Eigenvalues at flutter

At the end of this part we may state that divergence is identic to the loss of stability by losing also the uniqueness of the state of the material and a new localized solution can be detected. Let us now consider the onset of flutter, when an $0 < \alpha << 1$ appears and the critical eigenvalue moves off from the imaginary axis (*Fig. 1*) causing a special dynamic bifurcation.

Now (14) can be written as

$$\left[\overline{w}_j^{01}, \overline{w}_j^{02}\right] \exp\left(\alpha t\right) \exp\left(i\beta t\right),$$

thus (17) implies

$$v_j^0 = q_j \exp\left(\alpha t\right) \exp\left(i\ (n_m x_m + \beta t)\right).$$

Thus at flutter we obtain waves with exponentially increasing amplitudes as physical interpretation.

4. Summary

The loss of (Liapunov) stability of a dynamical system can be performed on two basic ways. It can be a static or a dynamic bifurcation. The classical setting of material instability uses divergence and flutter modes. Unfortunately, in the rate independent case the dynamical system defined by the basic equations of the solid body shows nongeneric behaviour and coexistent static and dynamic bifurcations occur. By introducing dissipative terms into the constitutive equation the stability investigation can be performed as a generic stability investigation. That is, we can now study the real parts of the eigenvalues of differential operators defined by the fundamental equations of the continuum.

The results show that divergence (or localization) means a static bifurcation instability. We also found interpretation for the flutter type of loss of stability in terms of dynamical systems theory. The appropriate physical phenomenon is the presence of a wave with increasing amplitude.

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