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ON HYBRID DYNAMICAL MODELS OF THE VEHICLE TRACK SYSTEM[1]

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Abstract

In the applications of vehicle system dynamics it is very important to create effective simulation models to predict the motion and loading processes of the components in the complex 'vehicle - track' system. The paper deals with the exact mathematical description of a four-axle bogie-vehicle in the framework of an in-plane *hybrid dynamical system model*. In the latter the railway track is modelled by a homogeneous beam on damped linear foundation, while the vehicle is modelled by a lumped parameter linear dynamical system. The interaction between the track and the vehicle models in vertical plane is described by a Hertzian spring and damper, belonging to the linearized vertical contact force transfer. Formulation of the mathematical models, as well as the closed-form solutions for the excitation-free system are presented.

Keywords: vehicle - track system, continuous beam model, hybrid systems of linear differential equations.

1. Introduction

The system model is shown in *Fig. 1.* This in-plane dynamical model is a typical *hybrid* one, as it consists of a continuous subsystem (the track) treated as a Bernoulli–Euler beam on damped Winkler foundation, and the lumped parameter vehicle subsystem describing a four-axle railway vehicle. The connection of the two subsystems mentioned above is realized by contact springs/dampers.

The track model has the following parameters: rail density ρ, cross section area A of the two rails, moment of inertia I of the two rails, Young modulus E of the rail, foundation stiffness s and foundation damping k.

The vehicle parameters are as follows: wheelset masses m_1, m_2, m_3 and m_4, bogie masses M_1 and M_2, carbody mass M_3, bogie moments of inertia Θ_1 and Θ_2, carbody moment of inertia Θ_3, vertical wheelset suspension stiffnesses s_i and vertical wheelset suspension dampings k_i for $i = 1, \dots, 4$,

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axlebases $l_1 + l_2$ and $l_3 + l_4$, bogiebase $l_5 + l_6$, coefficients a_i of the velocity-square dependent air drag and vertical distances h_i between the action lines of the air drag and mass centres of the bogies for $i = 1, 2$ and that of the carbody for $i = 3$. There are ten free coordinates describing the positions of the masses in the vehicle subsystem: vertical displacements of the bogies z_{01}, z_{02} and the carbody z_{03}, angular displacements of the bogies ψ_1, ψ_2 and the carbody ψ_3, and vertical displacements of the wheelsets Z_i, $i = 1, \ldots, 4$. Six further displacements on the vehicle are important to determine the motion-state dependent vertical forces transmitted through the suspension springs and dampers. The points on the bogies located over the wheelsets and the point on the carbody located over the bogies are indicated in *Fig. 1*, and their displacements can be expressed by using z_{0j} and ψ_j in the way

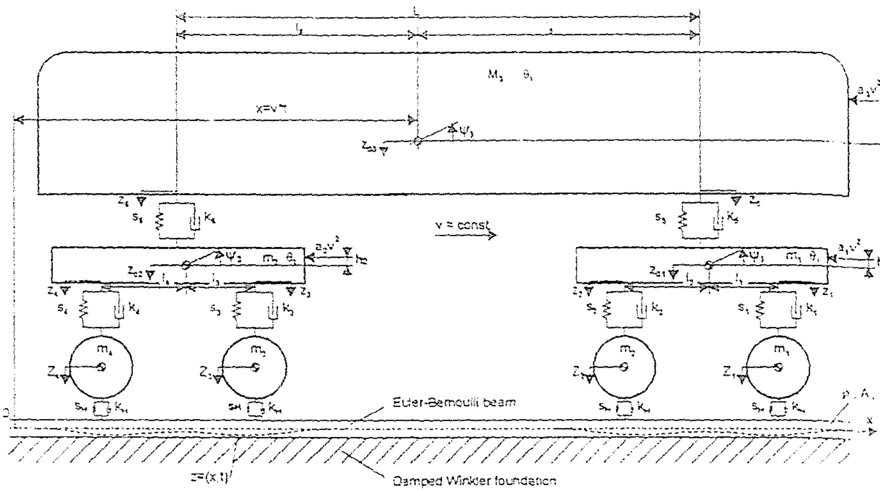

Fig. 1. The track–vehicle system

$$z_{2j-1} = z_{0j} - l_{2j-1}\psi_j, \quad z_{2j} = z_{0j} + l_{2j}\psi_j, \quad j = 1,\ 2,\ 3. \tag{1}$$

The interaction of the track and the vehicle is realized through *Hertzian springs and dampers* of linearized stiffness s_H and damping factor k_H. The actual operation condition of the vehicle is reflected in the constant velocity v of the carbody mass centre. The longitudinal position of the latter one under this condition is given by product vt. The longitudinal coordinates of the wheelset/track contact points are

$$x_1 = vt + l_5 + \frac{l_1 + l_2}{2}, \quad x_2 = vt + l_5 - \frac{l_1 + l_2}{2},$$

$$x_3 = vt - l_6 + \frac{l_3 + l_4}{2}, \quad x_4 = vt - l_6 - \frac{l_3 + l_4}{2}.$$

Thus, the track-vehicle dynamical system can be characterized by the parameter vector $\mathbf{p}$ of dimension 43. It has the form

$$\mathbf{p} = [\rho, A, I, E, s, k; l_1, l_2, l_3, l_4, l_5, l_6, h_1, h_2, h_3, m_1, m_2, m_3, m_4, M_1, M_2, M_3,$$

$$\Theta_1, \Theta_2, \Theta_3, s_1, s_2, s_3, s_4, s_5, s_6, k_1, k_2, k_3, k_4, k_5, k_6, a_1, a_2, a_3; s_H, k_H; v]^T .$$

The motion conditions can be studied by looking for function $z(x,t)$ of track deflection and free coordinates $Z_i(t)$, $i = 1, \ldots, 4$; $z_{0j}(t)$, $\psi_j(t)$, $j = 1, 2, 3$ characterizing the vehicle subsystem. The governing set of motion equations is established in the next chapter, where we decompose the original system into three submodules in order to obtain a reduced system of simpler character in the mathematical sense.

In the following chapters we solve our reduced system with the help of methods from [2] based on the pioneering work of DE PATER [1], and this way, generalizing the results of [3], we give a closed-form solution for the original problem.

2. Governing Equations of the System

2.1. Equations of Motion

Track deflection is described by the known Bernoulli–Euler beam equation

$$EI\frac{\partial^4 z}{\partial x^4}+\rho A\frac{\partial^2 z}{\partial t^2}+k\frac{\partial z}{\partial t}+sz = \sum_{i=1}^{4}\left(g_i(t) + m_i(g - \ddot{Z}_i)\right)\delta\left(x - (vt + L_i)\right) \quad (2)$$

on elastic foundation in the presence of forces describing the vertical interaction between xthe track and the wheels at contact points with

$$L_1 := l_5 + \frac{l_1 + l_2}{2}, \quad L_2 := l_5 - \frac{l_1 + l_2}{2},$$

$$L_3 := -l_6 + \frac{l_3 + l_4}{2}, \quad L_4 := -l_6 - \frac{l_3 + l_4}{2}.$$

The equations of motion for the vehicle subsystem are determined by using Newton's 2nd law for the rigid body components. On the wheelsets and the bogies they have form

$$g_i(t) := k_i\left(\dot{z}_i - \dot{Z}_i\right) + s_i\left(z_i - Z_i\right) =$$

$$k_H\left(\dot{Z}_i - \frac{\mathrm{d}}{\mathrm{d}t}z(vt + L_i, t)\right) + s_H\left(Z_i - z(vt + L_i, t)\right) - m_i\left(g - \ddot{Z}_i\right),$$

$$i = 1, \ldots, 4, \quad (3)$$

$$g_5(t) := k_5\left(\dot{z}_5 - \frac{\dot{z}_1 + \dot{z}_2}{2}\right) + s_5\left(z_5 - \frac{z_1 + z_2}{2}\right), \tag{4}$$

$$g_6(t) = k_6\left(\dot{z}_6 - \frac{\dot{z}_3 + \dot{z}_4}{2}\right) + s_6\left(z_6 - \frac{z_3 + z_4}{2}\right). \tag{5}$$

The vertical translatory motion of the bogies and the carbody is governed by equations

$$\begin{aligned}
&-\sum_{i=1}^{2} g_i(t) + M_1(g - \ddot{z}_{01}) + g_5(t) = 0, \\
&-\sum_{i=3}^{4} g_i(t) + M_2(g - \ddot{z}_{02}) + g_6(t) = 0, \\
&-\sum_{i=5}^{6} g_i(t) + M_3(g - \ddot{z}_{03}) = 0.
\end{aligned} \tag{6}$$

The pitching motion equations are of the form

$$\begin{aligned}
&\sum_{i=1}^{2} g_i(t)(-1)^{i+1} l_i + h_1 a_1 v^2 - \Theta_1 \ddot{\psi}_1 + g_5(t)\frac{l_1 - l_2}{2} = 0, \\
&\sum_{i=3}^{4} g_i(t)(-1)^{i+1} l_i + h_2 a_2 v^2 - \Theta_2 \ddot{\psi}_2 + g_6(t)\frac{l_3 - l_4}{2} = 0, \\
&\sum_{i=5}^{6} g_i(t)(-1)^{i+1} l_i + h_3 a_3 v^2 - \Theta_3 \ddot{\psi}_3 = 0.
\end{aligned} \tag{7}$$

Together with *Eqs.* (2 – 7) also relationships

$$z_{0j} = z_{2j-1} + l_{2j-1}\psi_j = z_{2j} - l_{2j}\psi_j, \quad j = 1, 2, 3 \tag{8}$$

are in force. Utilizing formulae (1) variables z_{0j} and ψ_j, $j = 1, 2, 3$ can be eliminated.

System (2) – (8) has to satisfy boundary condition

$$\lim_{|x| \to \infty} z(x, t) = 0 \tag{9}$$

together with initial conditions

$$\begin{aligned}
&z_i(0) = z_{i0}, \quad \dot{z}_i(0) = v_{i0}, \quad i = 1, \ldots, 6, \\
&Z_i(0) = Z_{i0}, \quad \dot{Z}_i(0) = V_{i0}, \quad i = 1, \ldots, 4.
\end{aligned} \tag{10}$$

2.2. Submodules

A submodule, consisting of a box (standing for the carbody or one of the bogies) of mass m and moment of inertia Θ, is shown in *Fig. 2.* We take the actions of the weight mg, the air drag av^2 and three another exterior forces F_j, $j = 1, 2, 3$ into consideration. In this case the governing equations of the submodule are

$$\begin{aligned} &-F_1 - F_2 + m(g - \ddot{u}_0) + F_3 = 0, \\ &F_1 l_1 - F_2 l_2 + hav^2 - \Theta\ddot{\psi} + F_3 \frac{l_1 - l_2}{2} = 0, \\ &u_0 = u_1 + l_1\psi = u_2 - l_2\psi. \end{aligned} \tag{11}$$

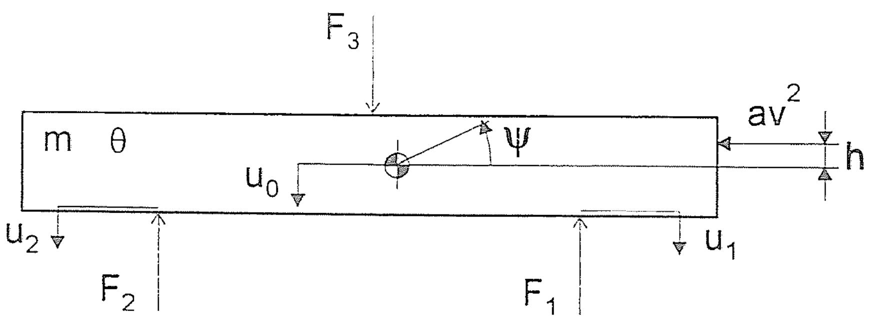

Fig. 2. A submodule

Here u_1 and u_2 are vertical displacements indicated in *Fig. 2,* while u_0 is the vertical displacement of the mass centre, and ψ stands for the angular displacement of the submodule.

The solution of system (11) for variables F_1 and F_2 can be given by

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \mathbf{b} + \frac{F_3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{12}$$

where the entries of mass matrix $\mathbf{A}$ and force vector $\mathbf{b}$ can be determined by formulae

$$(\mathbf{A})_{ij} := \frac{1}{(l_1 + l_2)^2} \left((-1)^{i+j+1}\Theta - \frac{m l_1^2 l_2^2}{l_i l_j} \right)$$

and

$$(\mathbf{b})_i := \frac{1}{l_1 + l_2} \left((-1)^i hav^2 - \frac{m l_1 l_2 g}{l_i} \right)$$

for $i, j = 1, 2$, respectively.

2.3. New System Parameters

First we identify submodule parameters

$$u_1 := z_{2j-1}, \quad u_2 := z_{2j}, \quad m := M_j, \quad l_1 := l_{2j-1}, \quad l_2 := l_{2j},$$

$$h := h_j, \quad a := a_j, \quad \Theta := \Theta_j,$$

$$\mathbf{A} := \mathbf{A}_j, \quad \mathbf{b} := \mathbf{b}_j, \quad F_1 := g_{2j-1}, \quad F_2 := g_{2j} \quad \text{for} \quad j = 1, 2, 3,$$

and

$$F_3 := \begin{cases} g_5, & \text{if} \quad j = 1, \\ g_6, & \text{if} \quad j = 2, \\ 0, & \text{if} \quad j = 3, \end{cases}$$

in case of all the three of our submodules, separately.

The interactions between the three sumodules result in mass matrix

$$\mathbf{M} := \left[\begin{array}{cc|cc|cc} (\mathbf{A}_1)_{11} & (\mathbf{A}_1)_{12} & 0 & 0 & \frac{1}{2}(\mathbf{A}_3)_{11} & \frac{1}{2}(\mathbf{A}_3)_{12} \\ (\mathbf{A}_1)_{21} & (\mathbf{A}_1)_{22} & 0 & 0 & \frac{1}{2}(\mathbf{A}_3)_{21} & \frac{1}{2}(\mathbf{A}_3)_{12} \\ \hline 0 & 0 & (\mathbf{A}_2)_{11} & (\mathbf{A}_2)_{12} & \frac{1}{2}(\mathbf{A}_3)_{21} & \frac{1}{2}(\mathbf{A}_3)_{22} \\ 0 & 0 & (\mathbf{A}_2)_{21} & (\mathbf{A}_2)_{22} & \frac{1}{2}(\mathbf{A}_3)_{21} & \frac{1}{2}(\mathbf{A}_3)_{22} \\ \hline 0 & 0 & 0 & 0 & (\mathbf{A}_3)_{11} & (\mathbf{A}_3)_{12} \\ 0 & 0 & 0 & 0 & (\mathbf{A}_3)_{21} & (\mathbf{A}_3)_{22} \end{array} \right]$$

and force vector

$$\mathbf{f} := \left[\begin{array}{c} (\mathbf{b}_1)_1 + \frac{1}{2}(\mathbf{b}_3)_1 \\ (\mathbf{b}_1)_2 + \frac{1}{2}(\mathbf{b}_3)_1 \\ \hline (\mathbf{b}_2)_1 + \frac{1}{2}(\mathbf{b}_3)_2 \\ (\mathbf{b}_2)_2 + \frac{1}{2}(\mathbf{b}_3)_2 \\ \hline (\mathbf{b}_3)_1 \\ (\mathbf{b}_3)_2 \end{array} \right]$$

of the whole system.

2.4. Reduced System

With the help of the above procedure we are now able to write down the governing equations of our original system in a more compact form as

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A \frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz =$$

$$\sum_{i=1}^{4} \left(\sum_{j=1}^{6} m_{ij}\ddot{z}_j + f_i + m_i(g - \ddot{Z}_i) \right) \delta\left(x - (vt + L_i)\right), \tag{13}$$

$$\sum_{j=1}^{6} m_{ij}\ddot{z}_j + f_i = k_i(\dot{z}_i - \dot{Z}_i) + s_i(z_i - Z_i) =$$

$$k_H\left(\dot{Z}_i - \frac{\mathrm{d}}{\mathrm{d}t}z(vt - L_i, t)\right) + s_H\left(Z_i - z(vt - L_i, t)\right) - m_i(g - \ddot{Z}_i), \quad i = 1, \ldots, 4, \tag{14}$$

$$\sum_{j=1}^{6} m_{ij}\ddot{z}_j + f_i = k_i\left(\dot{z}_i - \frac{\dot{z}_{2i-9} + \dot{z}_{2i-8}}{2}\right) + s_i\left(z_i - \frac{z_{2i-9} + z_{2i-8}}{2}\right), \quad i = 5, 6, \tag{15}$$

where m_{ij} and f_i stand for the entries of the mass matrix and the force vector defined in the previous chapter, respectively.

System (13) – (15) has to satisfy boundary condition

$$\lim_{|x|\to\infty} z(x, t) = 0, \tag{16}$$

together with initial conditions

$$\begin{aligned} z_i(0) &= z_{i0}, \quad \dot{z}_i(0) = v_{i0}, \quad i = 1, \ldots, 6, \\ Z_i(0) &= Z_{i0}, \quad \dot{Z}_i(0) = V_{i0}, \quad i = 1, \ldots, 4. \end{aligned} \tag{17}$$

3. Solution to the Boundary Problem

Let us look for the solution of problem (13) – (17) in the form

$$z(x, t) := \sum_{k=0}^{20} A_k(x - vt)e^{w_k t},$$

$$z_i(t) := \sum_{k=0}^{20} \xi_{ik} e^{w_k t}, \quad i = 1, \ldots, 6,$$

$$Z_i(t) := \sum_{k=0}^{20} \zeta_{ik} e^{w_k t}, \quad i = 1, \ldots, 4,$$

where constants w_k, ξ_{ik}, ζ_{ik} are subjects to be determined later.

Here $w_0 = 0$, function A_k is defined by formulae

$$A_k(\xi) = \sum_{i=1}^{4} \eta_{ki} B(\xi - L_i, w_k),$$

$$\eta_{ki} = \begin{cases} m_i g + f_i, & \text{if} \quad k = 0, \\ \left(\sum_{j=1}^{6} m_{ij}\xi_{jk} - m_i\xi_{ik}\right) w_k^2, & \text{if} \quad k = 1, \ldots, 20, \end{cases}$$

while $B(\xi,w)=\sum\limits_{j=1}^{4}\frac{e^{\lambda_j\xi}}{P'(\lambda_j)}\sigma_j H(\sigma_j\xi)$ is the solution of the underlying beam problem

$$EI\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}+k\frac{\partial u}{\partial t}+su=e^{wt}\delta(x-vt),\qquad \lim_{|x|\to\infty}u(x,t)=0$$

with $\xi=x-vt$, where the λ_j's are roots of characteristic polynomial

$$P(\lambda):=EI\lambda^4+\rho Av^2\lambda^2-v(k+2\rho Aw)\lambda+(s+kw+\rho Aw^2).$$

P' is the derivative of polynomial P, and $\sigma_j:=-\mathrm{sgn}(\mathrm{Re}\,\lambda_j)$ for $j=1,\dots,4$, see e.g. [2].

After substitution we obtain for $k=0$ equations

$$\zeta_{i0}=\sum_{j=1}^{4}(f_j+m_jg)\left(\frac{\delta_{ij}}{s_H}+B(L_i-L_j,0)\right);\qquad \xi_{i0}=\frac{f_i}{s_i}+\zeta_{i0},\qquad i=1,\dots,4,$$

$$\xi_{i0}=\frac{f_i}{s_i}+\frac{\xi_{2i-9,0}+\xi_{2i-8,0}}{2},\qquad i=5,6, \tag{18}$$

while for $k=1,\dots,20$ we get the homogeneous system of linear equations

$$w_k^2\sum_{j=1}^{6}m_{ij}\xi_{jk}=(s_i+k_iw_k)(\xi_{ik}-\zeta_{ik})=$$

$$(s_H+k_Hw_k)\left(\zeta_{ik}-\sum_{l=1}^{4}w_k^2\left(\sum_{j=1}^{6}m_{lj}\xi_{jk}-m_l\zeta_{lk}\right)B(L_i-L_l,w_k)\right)+w_k^2m_i\zeta_{ik},$$

$$i=1,\dots,4,$$

$$w_k^2\sum_{j=1}^{6}m_{5j}\xi_{jk}=(s_5+k_5w_k)\left(\xi_{5k}-\frac{\xi_{1k}+\xi_{2k}}{2}\right),$$

$$w_k^2\sum_{j=1}^{6}m_{6j}\xi_{jk}=(s_6+k_6w_k)\left(\xi_{6k}-\frac{\xi_{3k}+\xi_{4k}}{2}\right). \tag{19}$$

From system (18) indeterminates

$$\zeta_{ik}=\sum_{j=1}^{4}\left(\delta_{ij}-\frac{w_k^2m_{ij}}{s_i+k_iw_k}\right)\xi_{jk},\qquad i=1,\dots,4$$

can be eliminated. This way we obtain a homogeneous system of linear equations in variables ξ_{ik} for any $k=1,\dots,20$, separately. In order to have nontrivial solutions, the determinant of the system in variables ξ_{ik}, $i=1,\dots,4$ must vanish, and we have the same nonlinear equation for determining complex frequency w_k for any $k>0$.

4. Determining Complex Frequencies

In order to compute the 20 complex eigenfrequencies w_k of our original system, we have to solve the unique nonlinear equation

$$\det \mathbf{C}(w) = 0, \tag{20}$$

where the entries of matrix function $\mathbf{C}(w)$ of type 6×6 are given by formulae

$$c_{ij} := \delta_{ij} - \frac{w^2 m_{ij}}{s_i + k_i w} -$$

$$- \begin{cases} w^2 \sum\limits_{l=1}^{4} \left(\frac{\delta_{ij}}{s_H + k_H w} + B(L_i - L_l, w) \right) \left(m_{lj} - m_j + \frac{w^2 m_l m_{lj}}{s_l + k_l w} \right) (1 - \delta_{5j} - \delta_{6j}), \\ \quad i = 1, \ldots, 4, \\ \frac{1}{2} (\delta_{2i-9,j} + \delta_{2i-8,j}), \\ \quad i = 5, 6 \end{cases}$$

$$j = 1, \ldots, 6.$$

If we can determine all the roots w_k, then constants ξ_{ik} and ζ_{ik} can be easily reckoned by the solution of the set of linear equations given by systems (17) – (18) and initial conditions (16).

Determination of all the roots of the implicit complex nonlinear *Eq.* (19) is not a simple task to solve. The authors are currently in the state of making experiments with the help of program MapleV.

5. Concluding Remarks

In this paper the module-based exact mathematical treatment was elaborated for the solution of the set of motion equations representing the combined dynamics of the continuous Euler–Bernoulli beam track model and the lumped parameter vehicle system model describing a traditional 4-axle bogie vehicle. The vehicle model is supposed to roll at a constant velocity on the stationary track model. The track sub-system and the vehicle sub-system are connected with each other through vertically defined linear Hertzian spring/damper systems. In accordance with the methods published in our previous papers [2], [3], closed-form expressions are derived for the solution to the set of equations of the combined hybrid dynamic model. The key of the solution is the correct treatment of the nonlinear algebraic equation, the roots of which supply the complex frequencies required by the closed-form solution.

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