MEASUREMENT OF TENSOR OF INERTIA WITH TETRAHEDRON METHOD

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Abstract

Several sphere of technical life requires the tensor of inertia of a machine components. The calculation is only an approaching one, therefore the application of any measuring method seems to be powerful way. This paper introduces a new method for the experimental determination of the center of mass and the tensor of inertia of machine components with complex shape and nonhomogeneous mass distribution. The swinging experiment was carried out on model engine made of wood, so the measured results confirm the applicability of the elaborated method.

Keywords: tensor of inertia, tetrahedron method, center of mass, pendulum, swinging time, eigenvalues, eigenvectors.

1. Introduction

In the latest years, the requirements of comfort of vehicles have gradually increased. A modern vehicle cabin is exempt from noise and vibration. The service lifetime of structural units of vehicle is very important, especially in the case of trucks and buses. It may decrease if the building of the engine in the vehicle body is not tuned dynamically well. These are the reasons why the optimal engine suspension has always been an actual task for engineers for decades.

The engine is fastened to the chassis or bodyframe with elastic elements. Actually the vehicle bodies have elasticity, however, they are more rigid than the applied engine bedding.

The purposes, which can be realized with elastic suspensions, are as follows:

- isolating low frequency components from road excitation. These components increase comfort and can decrease the dynamic stresses.
- isolating high frequency components from engine excitation,
- engine is defended from stochastic road excitation,
- bodywork is manufactured with errors equalizing,

- setting eigenfrequency of engine (as vibrating system) so that the parameters of dynamic system are optimal,
- determination of stiffness and damping of rubber bedding: at low frequency large, at high frequency small damping is advantageous.

2. The Equation of Moving of 'Engine on Ground'

The motion of engine, as a rigid body is described in coordinate system x, y, z with fixed axis in space. The engine with elastic suspension is a six-degree of freedom swinging system, which can move along and can turn round the above three axes. The following matrix equation describes a motion of rigid body with elastic suspension (supposing that the displacements are small and the nonlinearities are negligible):

$$\mathbf{M}\ddot{q} + \mathbf{K}\dot{q} + \mathbf{S}\mathbf{q} = \mathbf{F},$$

where:

- $\mathbf{M}$ mass matrix,
- $\mathbf{K}$ damping matrix and
- $\mathbf{S}$ stiffness matrix

are quadratic, real, symmetrical, positive matrices.

Matrix $\mathbf{M}$ contains primary and secondary moments of inertia of engine and mass of engine. (The examination was performed in a coordinate system where origin is the center of mass of the engine, consequently the primary moments are zeros.)

Matrices $\mathbf{K}$ and $\mathbf{S}$ contain damping and stiffness factors, primary and second-order moments of elastic rubber bedding in directions x, y and z.

Vector $\mathbf{q}$ contains moving components and turning components in three directions.

Vector $\mathbf{F}$ contains exciting forces and moments.

3. Determination of Matrix of Inertia of Engine Gearbox Unit

3.1. Possible Measuring Principles

The authors who had studied this theme earlier used diagonal matrix form. They suppose that principal axes of inertia of engine are almost equal or parallel to the typical geometric axes. It may result in a rather great error. This supposition is only an approach if the engine is vertical, but for horizontal engines (bus) or tilted engines (passenger car) is absolutely wrong.

Other authors mention deviations of principal axes from geometric axis in space, but they do not recommend any method for determination of the correct principal axis. The only exception is a paper [1]; it recommends a concrete measuring method.

We need to determine the magnitudes of principal moment of inertia and position of principal axis, so we should determine tensor of inertia of engine gearbox unit. The engines have very various shape, volume and mass. Therefore the elaborated method must be applicable for all kinds of engines.

Determination of moment of inertia and center of mass of engine gearbox unit by calculation is a difficult problem, because structural elements consist of nonhomogeneous and very complicated spare parts. Applying CAD we can get wrong estimation. So the solution of problem in practice can only be accomplished by a measuring method. The most known methods are as follows:

The *torsional pendulum* method as a swinging experiment round a fixed axis is unsuitable in this problem. Firstly, we should know the direction of the principal axis. Suspension point of torsional pendulum must be positioned at the vertical centroidal axis. Secondly, the engine has a complex shape and irregular mass distribution. The engine can move irregularly because of centroidal products of inertia, so this measurement is unusable.

Applying *bifilar pendulum* the body must be turned around the center of mass, while hanging on two parallel cords. The centroidal products of inertia distort swinging and deflect from ideal axis.

The *trifilar pendulum* differs from the previous one in the circumstance that the body should hang on three cords. Disadvantages are the same as with bifilar pendulum.

The *elastic system* is more correct, since the engine is rotated about a fixed horizontal axis and is supported by a spring, which is positioned in the vertical, centroidal plane of the engine being perpendicular to the axis. Therefore the body of the engine cannot deviate from its axle. For measuring we should build equipment and the spring stiffness should be known, however, it is very complicated and expensive in practice.

Applying a *physical pendulum* offers the simplest and easiest solution. If a rigid body is deflected from its equilibrium position, it can turn round about a fix axis. Because of disturbing factors (friction, air resistance), the deflection must be small (5°). To this measurement the engine should be hanged in two points (I, K) and the examined axis will be parallel to the line IK. The paper [1] presents an engine hanged in two cords, with horizontal axis IK. This is actually a double pendulum. It can be treated as a physical pendulum, if the center of mass of the engine remains in the plane determined by two cords, which move parallel with each other during the swinging motion. The parallel motion of cords is not always perfect, because of the inappropriate initial displacement. However, if the distance between the center of mass and swinging axis is large, the measure data can be inaccurate since the parallel axis theorem results large additional members.

The principle of measuring and its properties are shown in *Table 1*:

Table 1.

Measuring principles	Properties	Usable in theory
Torsional pendulum	Directions of principal axes must be known; requires calibrated elastic elements; correct in the case of rigid body with regular shape	– –
Bifilar and trifilar pendulum	The position of center of mass must be known; correct in the case of rigid body with regular shape	–
Elastic system	Requires equipment of platform and calibrated elastic elements	+
Physical pendulum	Accurate when an arbitrary superficial point is attached to a fix pin	+

Having compared various measurement methods, for this problem it has been found that the most suitable one was the physical pendulum with a bit correction.

Correct measure with physical pendulum is only possible if suspension elements are on the surface of the body. After averaging, if the time of one complete swinging is denoted by T, then the moment of inertia on the axis IK is

$$J_{i,k} = \frac{T^2 mgs}{4\pi^2} \tag{1}$$

where:

m mass of engine;

s distance between center of mass (examined axis) and axis IK;

g gravity.

The moment of inertia of the examined axis computed with parallel axis theorem:

$$J_{s,i,k} = J_{i,k} - ms^2. \tag{2}$$

Consequently, the moment of inertia can be determined with physical pendulum method. For the determination of matrix M we have to measure by the use of six different axes.

According to study [1], this method is the most suitable one. The author suggests that the measurement should be carried out using six axes intersecting center of mass. Three axes must be mutually perpendicular to each other and the remaining three can be optional. The study [3] measures with axes at right angles. It can be realized with clamping equipment, but it depends on the object to be measured.

The other problem with the physical pendulum method is the determination of the position of center of mass. Position of center of mass affects the accuracy of the measured data. Our purpose is to develop a measuring principle and method:

- that can be realized everywhere in practice and gives correct results with low costs,
- for which there is no geometrical restriction and which does not need any clamping equipment to the measurement,
- by which the six elements of the tensor of inertia and three center of mass coordinates could simply be computed.
  The six axes do not have to be perpendicular to each other, but they have to fulfil the requirements as follows:
- three or more axes must not be parallel to each other,
- maximum five axes can intersect one another, the sixth one must be deviating,
- more than three axes do not have to cross each other in one point,
- more than three axes do not lie in the same plane and the axes in one plane need not be parallel and they do not have to cross each other in the same point.

3.2. Theoretical Fundamentals of the Tetrahedron Method

The six edges of tetrahedron satisfy the requirements, and these edges do not intersect the center of mass. Further advantage is that only four points are necessary for the measurement. Choosing two points at each vertex of tetrahedron, six axes are established, so the measurement is executable.

For measuring we only need a timer and a plummet (for determination of the position of mass center). (*Fig. 1.*)

The steps of process:

a) After measuring swinging time of the physical pendulum, we calculate the moments of inertia of axis ($J_{i,k}$, where, $k = 1 \ldots 4$ and $i \neq k$).

b) Determination of the center of mass as intersection of planes. By two suspension points of the body and a third point to be determined by plummet we can describe the equation of plane. This plane contains center of mass. The applied method leads to the six considered planes intersecting the center of gravity, therefore in principle three planes determine it. We must carry out the measurements with respect to six axes, contained by the above mentioned six planes. Consequently, the help of the axes we can determine the equations of these planes. We

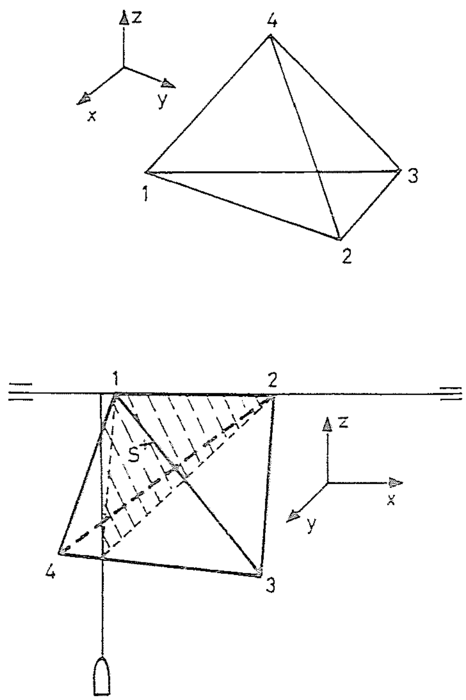

Fig. 1.

get twenty points $\left(\binom{6}{3} = 20\right)$, for center of mass and these points are close to each other. However, because of the errors of measurements, these points are not in the same position. For example, taking the average value of the coordinates the position of center of mass can be determined with acceptable error.

c) Determination of the matrix of moments of inertia in the center of mass $(J_{s(x,y,z)})$.

d) Determination of the principal values and principal directions.

4. Experimental Control of the Tetrahedron Method

4.1. Introducing the Applied Model

As a demonstration of the measuring method we made an experiment by a rigid body with known tensor of inertia. The model consists of three billets of wood (*Figs. 2* and *3*). It can be regarded as a serial engine gearbox object.

Supposing that the used beech-tree blocks have homogeneous mass distribution, we computed the position of the center of mass of the body and the centroidal moments of inertia. Four rings were fixed to billets in four points as shown in *Fig. 3.* The billet's hanging on these rings can also be seen in *Fig. 3.* The origin of coordinate system x, y, z is point 1. The axes of the coordinate system can be seen in *Figs. 2, 3.*

These four points in space determine a tetrahedron. The computed swinging time and moment of inertia (J_{ik}) about six axes are contained in *Table 2.* The centroidal moments of inertia ($J_{S,ik}$) are also being shown in *Table 2.* Two points positioned in the considered axis denote the applied axis.

Table 2.

Axis	Computed swinging time (s)	Computed J_{ik} value ($kgcm^2$)	Computed J_{Sik} value ($kgcm^2$)
12	0.7344	427	143
13	0.7344	422	144
14	0.9832	277	240
23	0.9815	1616	339
24	0.8148	669	208
34	0.8158	668	211

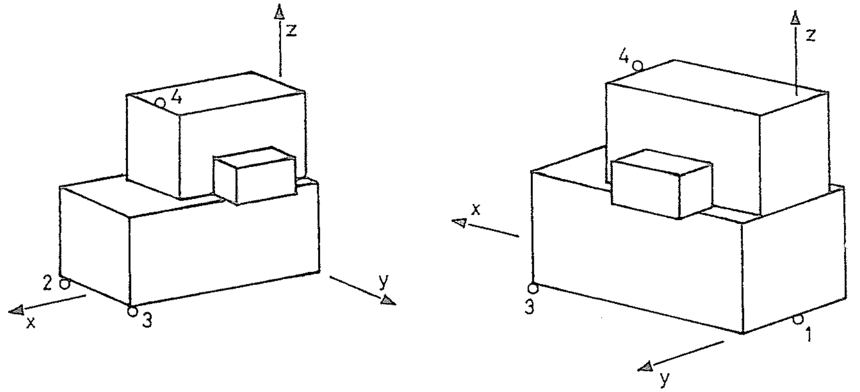

Fig. 2.

Note that in the case of axis 14, the axis of suspension points and the parallel centroidal axis are too close to each other, therefore the calculated moment of inertia is not so accurate as the other ones.

The computed matrix $\mathbf{J}_{s(x,y,z)}$: (in $kgcm^2$)

$$\begin{bmatrix} 138 & 2 & 40 \\ 2 & 339 & -2 \\ 40 & -2 & 260 \end{bmatrix}.$$

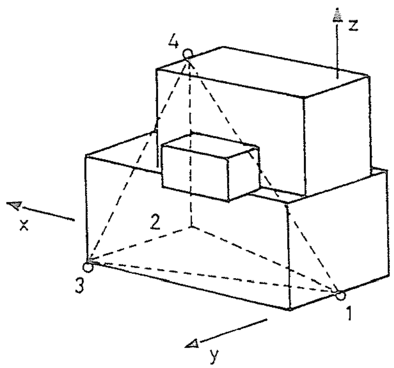

Fig. 3.

The three eigenvalues:

$$\begin{bmatrix} 339 & 0 & 0 \\ 0 & 272 & 0 \\ 0 & 0 & 126 \end{bmatrix}.$$

The principal directions:

$$\begin{aligned} v1 &= [\ 0.0056;\ 0.9996;\ -0.0280] \cong [0, 1, 0], \\ v2 &= [\ 0.2892;\ 0.0252;\ 0.9569], \\ v3 &= [-0.9573;\ 0.0135;\ 0.2889]. \end{aligned}$$

4.2. Experimental Results of the Tetrahedron Method

Arrangement of the measurement was very simple, since the joining rings were attached directly to the applied frame without cords. The billet of wood was deflected with small angle, and time of twenty complete swings was measured. Then averaged time was regarded as the real measured value. We computed the moments of inertia on the basis of *Eqs.* (1), (2), and the position of center of mass was calculated on condition that the mass density of block is homogeneous. The measured results are given in *Table 3*. Matrix $\mathbf{J}_{s(x,y,z)}$ computed from the measurement:

$$\begin{bmatrix} 139 & -5 & 20 \\ -5 & 344 & -9 \\ 20 & -9 & 255 \end{bmatrix}.$$

Eigenvalues:

$$\begin{bmatrix} 345 & 0 & 0 \\ 0 & 257 & 0 \\ 0 & 0 & 135 \end{bmatrix}.$$

Table 3.

Axis	Measured swinging time (s)	Deviation	J_{ik} value computed from measuring (kgcm^2)	J_{Sik} value computed from swinging time (kgcm^2)
12	0.7369	0.0785	429	146
13	0.7329	0.0590	420	143
14	0.9419	0.1288	254	217
23	0.9830	0.0884	1621	344
24	0.8214	0.0951	680	219
34	0.8239	0.0640	681	224

Eigenvectors:

$$\begin{aligned} v1 &= [-0.0326;\ 0.9940;\ -0.1048], \\ v2 &= [\ 0.1634;\ 0.1088;\ 0.9806], \\ v3 &= [-0.9860;\ 0.0148;\ 0.1659]. \end{aligned}$$

5. Conclusion

If we want to build in an engine in a vehicle we should know tensor of inertia of the engine, which is not given by engine factories. Several measuring methods are known for determination of this tensor, but because of several problems their applicability is restricted in engineering practice.

The advantages of the elaborated method are:

- very simple equipment is necessary with low cost,
- moreover, the processing of measured data is not difficult.

References

[1] GUTASSY, A.: Investigation of Engine Suspensions (in Hungarian).
[2] HERCZEG, I.: Experimental Method for Determination of Tensor of Inertia of Engine-Gearbox Unit (in Hungarian).
[3] THAMM, F. – SZABÓ, N.: Experimental Determination of the Complete Tensor of Inertia of a Specimen of Irregular Shape.