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TRACK–VEHICLE IN-PLANE DYNAMICAL MODEL CONSISTING OF A BEAM AND LUMPED PARAMETER COMPONENTS

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Abstract

This paper deals with the exact mathematical description of a simple in-plane track–vehicle dynamical system model. The railway track is modelled by a beam on damped linear foundation, while the two-axle railway vehicle is modelled by a lumped parameter linear dynamical system. The interaction between the track and the vehicle in vertical plane is described by the Hertzian spring and damper, belonging to the linearized vertical contact force transfer. Formulation of the mathematical models, as well as the closed form solutions for the excitation-free system are presented.

Keywords: track/vehicle dynamical system, hybrid systems of differential equations.

1. The Track–Vehicle System Model

The system model is shown in *Fig. 1.* The in-plane dynamical model is a typical *hybrid* one, as it consists of a continuum subsystem, i.e. the track, treated as an Euler–Bernoulli beam on damped Winkler foundation, and a lumped parameter vehicle subsystem describing the two-axle railway vehicle. The connection of the two subsystems mentioned is realized by the contact springs/dampers.

The track model parameters are the following: rail density ρ, cross section area of the two rails A, moment of inertia of the two rails I, Young modulus of the rail E, foundation stiffness s and foundation damping k. The vertical position of the rails is described by bivariate function $z(x,t)$, the so called rail deflection function. Here x stands for the longitudinal coordinate of the track.

The vehicle parameters are as follows: wheelset masses m_1 and m_2, carbody mass m, carbody moment of inertia Θ, vertical wheelset suspension stiffnesses s_1 and s_2, vertical wheelset suspension dampings k_1 and k_2, axlebase $L = l_1 + l_2$, coefficient a of the velocity-square dependent air drag and the vertical distance h between the action line of the air drag and the mass centre of the carbody. There are four free coordinates describing the positions of the masses in the vehicle subsystem: vertical displacement of

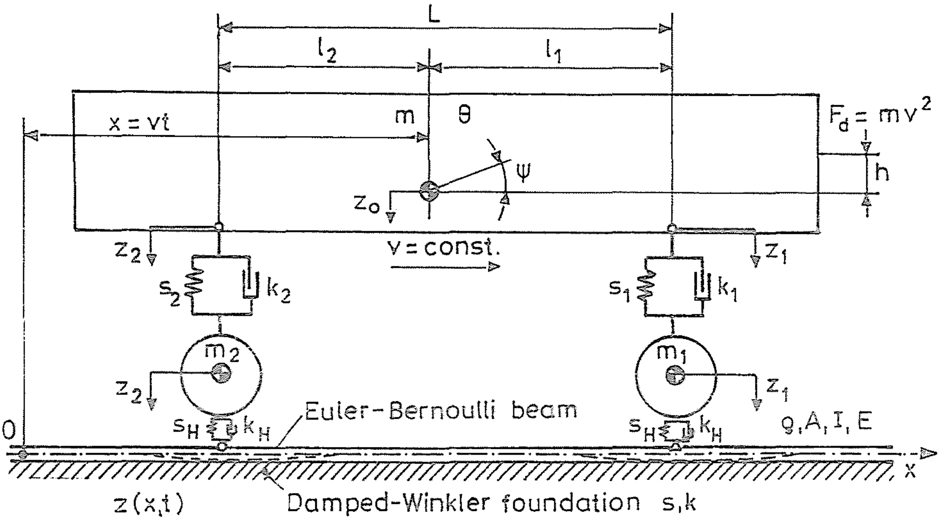

Fig. 1. Vehicle–track system model (continuum rail model)

the carbody z_0, angular displacement of the carbody ψ, and vertical displacements of the wheelsets Z_1 and Z_2. Two further vertical displacements are important on the carbody to determine the motion-state dependent vertical forces transmitted through the suspension springs and dampers. The points on the carbody located over the wheelsets are indicated in *Fig. 1* and their displacements can be expressed by using z_0 and ψ in the following way: $z_1 = z_0 - l_1\psi$ and $z_2 = z_0 + l_2\psi$.

The interaction of the track and the vehicle is realized through the Hertzian springs and dampers of linearized stiffness s_{H} and damping factor k_{H}. The actual operation condition of the vehicle is reflected in the constant velocity v of the carbody mass centre. The longitudinal position of the latter under this condition is given by product vt. So, the longitudinal coordinates of the wheelset/track contact points are $x_1 = vt + l_1$ and $x_2 = vt - l_2$.

Thus, the track–vehicle dynamical system can be characterized by parameter vector $\underline{\mathbf{p}}$ of dimension 21. Its form is

$$\mathbf{p} = [\rho, A, I, E, s, k; l_1, l_2, h, m_1, m_2, m, \Theta, s_1, s_2, k_1, k_2, a; s_{\mathrm{H}}, k_{\mathrm{H}}; v]^T.$$

The motion conditions can be studied by seeking for the function $z(x,t)$ of the track deflection, and the free coordinates $z_0(t), \psi(t), Z_1(t)$ and $Z_2(t)$ characterizing the vehicle subsystem. The governing set of motion equations are established in the next chapter.

2. Mathematical Description of the System Model

The equations of motion are determined by using Newton's 2nd law for the rigid body components of the vehicle subsystem, and the known equation of the Euler–Bernoulli beam on elastic/damped foundation in the presence of forces describing the vertical interaction between the track and the wheels. The equations of motion of the wheelsets are the following:

$$F_i(\ddot{Z}_i, \dot{Z}_i, Z_i, \dot{z}_i, z_i) = s_i(z_i - Z_i) + k_i(\dot{z}_i - \dot{Z}_i) + m_i\mathrm{g} - m_i\,\ddot{Z}_i=$$

$$s_{\mathrm{H}}(Z_i - z(vt + L_i, t)) + k_{\mathrm{H}}(\dot{Z}_i - \frac{\mathrm{d}}{\mathrm{d}t}z(vt + L_i, t)), \quad i = 1, 2, \tag{1}$$

where $L_i = (-1)^{i+1}l_i$, $i = 1, 2$ stand for oriented lengths.

The vertical translatory motion of the carbody is governed by the following equation:

$$\sum_{i=1}^{2}[-s_i(z_i - Z_i) - k_i(z_i - Z_i)] + m\mathrm{g} - m\,\ddot{z}_0= 0. \tag{2}$$

The pitching motion equation has the following form:

$$\sum_{i=1}^{2}[s_i(z_i - Z_i) + k_i(\dot{z}_i - \dot{Z}_i)]L_i + hav^2 - \Theta\,\ddot{\psi}= 0. \tag{3}$$

The track deflection is described by the following fourth order linear partial differential equation:

$$IE\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = \sum_{i=1}^{2}\delta(x - (vt + L_i))F_i(\ddot{Z}_i, \dot{Z}_i, Z_i, \dot{z}_i, z_i). \tag{4}$$

Together with *Eqs.* (1–4) also relationships

$$z_0 = z_1 + l_1\psi = z_2 - l_2\psi \tag{5}$$

are in force.

We are able to eliminate z_0 and ψ from *Eqs.* (2–3) by expressing

$$\psi = \frac{z_2 - z_1}{L} \text{ and } z_0 = \frac{l_1 z_2 + l_2 z_1}{L}.$$

This way our original system can be simplified from the point of view of the mathematical treatment as follows.

Let us introduce functions

$$g_i(t) = F_i(\ddot{Z}_i\,(t), \dot{Z}_i\,(t), Z_i(t), \dot{z}_i\,(t), z_i(t)) + m_i(\ddot{Z}_i - \mathrm{g})$$

for $i = 1, 2$.

Then our differential equations can be written into the form

$$IE\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = \sum_{i=1}^{2} \delta(x - (vt + L_i))(g_i(t) - m_i(\ddot{Z}_i - \mathrm{g})), \quad (6)$$

$$g_i(t) = \sum_{j=1}^{2} a_{ij}\ \ddot{z}_j + b_i = s_i(z_i - Z_i) + k_i(\dot{z}_i - \dot{Z}_i) =$$

$$s_{\mathrm{H}}(Z_i - z(vt + L_i, t)) + k_{\mathrm{H}}(\dot{Z}_i - \frac{\mathrm{d}}{\mathrm{d}t}z(vt + L_i, t)) + m_i(\ddot{Z}_i - \mathrm{g}) \quad (7)$$

for $i = 1, 2$ with

$$a_{ij} = \frac{(-1)^{i+j+1}}{L^2}\left(\frac{mL_1^2L_2^2}{L_iL_j} + \Theta\right),\ b_i = \frac{(-1)^i}{L}\left(\frac{mL_1L_2\mathrm{g}}{L_1} + hav^2\right).$$

The solution has to satisfy boundary condition

$$\lim_{x\to\pm\infty} z(x, t) = 0 \quad (8)$$

and initial conditions

$$z_i(0) = z_{i0},\quad \dot{z}_i\ (0) = v_{i0},\ Z_i(0) = Z_{i0},\quad \dot{Z}_i\ (0) = V_{i0} \quad (9)$$

for $i = 1, 2$.

3. Solution to the Boundary Value Problem

We are looking for the solution $[z, z_1, z_2, Z_1, Z_2]^T$ of system (6–9) in the form

$$z(x, t) = \sum_{k=0}^{8} A_k(\xi)\mathrm{e}^{w_k t},$$

$$z_i(t) = \sum_{k=0}^{8} \xi_{ik}\mathrm{e}^{w_k t},\ Z_i(t) = \sum_{k=0}^{8} \zeta_{ik}\mathrm{e}^{w_k t},\ i = 1, 2,$$

where $\xi = x - vt$, $w_0 = 0$, while the w_k's for $k = 1, 2, ..., 8$ are the *complex frequencies* of the system and ξ_{ik}, ζ_{ik} are appropriate constants, all of them are to be determined later on.

Substituting our expected solution into the right-hand side of *Eq.* (6), the partial differential equation will have the form

$$IE\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = \sum_{i=1}^{2}\sum_{k=0}^{8} \delta(\xi - L_i)c_{ik}\mathrm{e}^{w_k t}$$

with $c_{i0} = b_i + m_i g$ and

$$c_{ik} = \left(\sum_{j=1}^{2} a_{ij}\xi_{jk} - m_i\zeta_{ik} \right) w_k^2 \text{ for } k = 1, 2, ..., 8. \tag{10}$$

Then applying the theory of such partial differential equations [1–5], we can use formulae of [7] to compute $A_k(\xi)$ as

$$A_k(\xi) = \sum_{i=1}^{2} c_{ik} B(\xi - L_i, w_k),$$

$$B(\eta, w) = \mathrm{H}(\eta) \left(\frac{e^{\lambda_1 \eta}}{P'(\lambda_1)} + \frac{e^{\lambda_2 \eta}}{P'(\lambda_2)} \right) - \mathrm{H}(-\eta) \left(\frac{e^{\lambda_3 \eta}}{P'(\lambda_3)} + \frac{e^{\lambda_4 \eta}}{P'(\lambda_4)} \right),$$

where the characteristic polynomial

$$P(\lambda) = IE\lambda^4 + \rho A v^2 \lambda^2 - v(k + 2\rho A w)\lambda + (s + kw + \rho A w^2)$$

has neither imaginary nor multiple roots (necessary and sufficent conditions are given in [7]), λ_1 and λ_2 are the roots of polynomial P with negative real parts, P' is the derivative of P, while H is Heaviside's unit jump function. (The formula for the multiple root case is given in [7].)

4. Determination of Complex Frequencies

Substituting the expected solutions into *Eqs.* (7), by comparing coefficients we obtain the system of equations

$$s_{\mathrm{H}}(\xi_{i0} - A_0(L_i)) - m_i g = s_i(\xi_{i0} - \zeta_{i0}) = b_i, \tag{11}$$

$$w_k^2 \sum_{j=1}^{2} a_{ij}\xi_{jk} = (s_i + k_i w_k)(\xi_{ik} - \zeta_{ik}) =$$

$$(s_{\mathrm{H}} + k_{\mathrm{H}} w_k)(\xi_{ik} - A_k(L_i)) + m_i w_k^2 \zeta_{ik} \tag{12}$$

for $i = 1, 2$ and $k = 1, 2, ..., 8$. System (12) contains unknowns w_k, ξ_{ik} and ζ_{ik}. In order to obtain the complex frequencies w_k, $k = 1, 2, ..., 8$ the latter unknowns can be eliminated. This procedure results in nonlinear equation

$$\det\left(\mathbf{C}(w)\right) = 0, \tag{13}$$

where $\mathbf{C}(w)$ is a w-dependent 2×2 matrix with entries

$$c_{ij}(w) = a_{ij} w^2 - (s_{\mathrm{H}} + k_{\mathrm{H}} w) \left\{ \delta_{ij} - \frac{w^2 a_{ij}}{s_i + k_i w} + \right.$$

$$\sum_{n=1}^{2}\left(-a_{nj}+m_n\delta_{nj}-\frac{m_n w^2 a_{nj}}{s_n+k_n w}\right) w^2 B(L_i-L_n,w)\Bigg\} - m_i w^2\delta_{ij}+\frac{m_i w^4 a_{ij}}{s_i+k_i w}.$$

In the above expression δ_{ij} stands for Kronecker's symbol. The solutions $w_1, w_2, \ldots, w_8$ are the complex frequencies of system (6–9). With the knowledge of these frequencies w_k one can easily determine constants ξ_{ik} and ζ_{ik} by solving linear equation system (11–12) together with initial conditions (9).

5. Conclusions

In this paper a new mathematical treatment has been elaborated for the solution of a set of equations describing the joint problem of the combined motions of the continuous track and the vehicle modelled as a lumped parameter system. The wheelsets of the vehicle are moving at a constant longitudinal velocity on the elastically/dissipatively supported beam at a constant longitudinal velocity. The two subsystems, connected with each other by the Hertzian springs/dampers, are completely characterized through the closed-form expressions based on the complex frequencies obtained from the solution of the auxiliary nonlinear equation.

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