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QUASI-RECURRENT WEYL SPACES

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Abstract

In this work we define Quasi-Recurrent Weyl spaces and examine the hypersurfaces of them.

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1. Introduction

An n-dimensional manifold W_n is said to be a Weyl space if it has a conformal metric tensor g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 , \tag{1}$$

where T_k denotes a covariant vector field and $\nabla_k g_{ij}$ denotes the usual covariant derivative.

Under renormalization of the fundamental tensor of the form

$$\breve{g}_{ij} = \lambda^2 g_{ij} \tag{2}$$

the complementary vector T_k is transformed by the law

$$\breve{T}_i = T_i + \partial_i \ln \lambda , \tag{3}$$

where λ is a function.

Writing (1) out in full, we have

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2T_k g_{ij} = 0 ,$$

where Γ^i_{kl} are the connection coefficients of the form

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}(g_{mk}T_l + g_{ml}T_k - g_{kl}T_m) . \tag{4}$$

A quantity A is called a satellite with weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\check{A} = \lambda^p A$$

under the renormalization (2) of the metric tensor g_{ij}.

The prolonged covariant derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by [1]

$$\dot{\nabla}_k A = \nabla_k A - pT_k A \ . \tag{5}$$

REMARK 1 The prolonged covariant derivative preserves the weight.

The curvature tensor R^i_{jkl} of the Weyl connection is defined by

$$R^i_{jkl} = \frac{\partial}{\partial x^k}\Gamma^i_{jl} - \frac{\partial}{\partial x^l}\Gamma^i_{jk} + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk} \tag{6}$$

and the Ricci tensor R_{ij} of the Weyl connection is

$$R_{ij} = R^m_{ijm} \ . \tag{7}$$

Since the Weyl connection is not metric, the Ricci tensor R_{ij} is not necessarily symmetric. In fact, $R_{[ij]} = n\nabla_{[i}T_{j]}$. We remark that if T_j is a gradient, then the space is Riemannian.

It is easy to see that the covariant curvature tensor R_{lijk} is a satellite of g_{ij} with weight $\{2\}$.

The Bianchi identity for the Weyl space is, by [2]

$$\dot{\nabla}_l R^h_{ijk} + \dot{\nabla}_k R^h_{ilj} + \dot{\nabla}_j R^h_{ikl} = 0 \ . \tag{8}$$

2. Quasi-Recurrent Weyl Spaces

A non-flat Weyl space $W_n(g_{ij}, T_k)$ will be called quasi-recurrent ($(QRW)_n$ in short) if the curvature tensor satisfies the following condition for some non-zero covariant vector field ϕ_k ($\neq T_k$)

$$\dot{\nabla}_s R_{ijkl} = 2\phi_s R_{ijkl} + \phi_i R_{sjkl} + \phi_j R_{iskl} + \phi_k R_{ijsl} + \phi_l R_{ijks} \ . \tag{9}$$

A $(QRW)_n$ manifold can be Weyl recurrent, i.e. it can, beside (9), satisfy

$$\dot{\nabla}_s R_{ijhk} = \varphi'_s R_{ijhk} \ . \tag{10}$$

We examine the spaces satisfying (9) but not satisfying (10). Recurrent Weyl spaces have been examined in [2]. We note that ϕ_k is a satellite of g_{ij}

with weight $\{0\}$. By multiplying (9) by g^{il} and summing up with respect to i and l we get

$$\dot{\nabla}_s R_{jk} = 2\phi_s R_{jk} + \phi_j R_{sk} + \phi_k R_{js} + \phi_i (R^i_{sjk} + R^i_{jks}) \,. \tag{11}$$

Similarly transvecting (11) by g^{jk} we obtain

$$\dot{\nabla}_s R = 2\phi_s R + \phi_k (R^k_{s.} + R^k_{.s} + R^{ik}_{si..} + R^{ki}_{..is}) \,, \tag{12}$$

where $R^k_{s.} = R_{sj} g^{kj}$ and $R^k_{.s} = R_{js} g^{jk}$.

Hence the scalar curvature of the $(QRW)_n$ satisfies (12).

By changing the indices j and k in (11) and subtracting the resulting equation from the one obtained therefrom we obtain

$$\dot{\nabla}_s F_{jk} = \frac{2(n+1)}{n} \phi_s F_{jk} + \phi_k F_{js} + \phi_j F_{sk} \,, \tag{13}$$

where $F_{jk} = R_{[jk]}$.

In fact this is the relation between the complementary vector T_k and the recurrent vector ϕ_k.

3. Hypersurfaces of Quasi-Recurrent Weyl Spaces

Let $W_n(g_{ij}, T_k)$ be a hypersurface, with coordinates u^i $(i = 1, 2, \cdots, n)$ of a Weyl space $W_{n+1}(g_{ab}, T_c)$ with coordinates x^a $(a = 1, 2, \cdots, n+1)$. The metrics of W_n and W_{n+1} are connected by the relations

$$g_{ij} = g_{ab} x^a_i x^b_j \qquad (i, j = 1, 2, \cdots, n; \;\; a, b = 1, 2, \cdots, n+1) \,, \tag{14}$$

where x^a_i denotes the covariant derivative of x^a with respect to u^i.

It is easy to see that the prolonged covariant derivative of a satellite A, relative to W_n and W_{n+1}, are related by

$$\dot{\nabla}_k A = x^c_k \dot{\nabla}_c A \qquad (k = 1, 2, \cdots, n; \;\; c = 1, 2, \cdots, n+1) \,. \tag{15}$$

Let n^a be the contravariant components of the vector field of W_{n+1} normal to W_n which is normalized by the condition

$$g_{ab} n^a n^b = 1 \,. \tag{16}$$

The moving frame $\{x^i_a, n_a\}$ in W_n, reciprocal to the moving frame $\{x^a_i, n^a\}$ is defined by the relations

$$n_a x^a_i = 0 \,, \quad n^a x^i_a = 0 \,, \quad x^a_i x^j_a = \delta^j_i \,. \tag{17}$$

Remembering that the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a with respect to u^k is found as

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = \omega_{ik} n^a \; , \tag{18}$$

where ω_{ik} is the second fundamental form. It can be shown that ω_{ik} is a satellite of g_{ij} with weight $\{1\}$.

The following two relations, which are respectively the generalization of Gauss and Mainardi-Codazzi equations, are obtained in [2]

$$R_{pijk} = \Omega_{pijk} + \overline{R}_{dbce} x_p^d x_i^b x_j^c x_k^e \; , \tag{19}$$

$$\dot{\nabla}_k \omega_{ij} - \dot{\nabla}_j \omega_{ik} + \overline{R}_{dbce} x_i^b x_j^c x_k^e n^d = 0 \; , \tag{20}$$

where $\overline{R}_{dbce}$ is the covariant curvature tensor of W_{n+1} and Ω_{pijk} is the Sylvesterian of ω_{ij} defined by $\Omega_{pijk} = \omega_{pj}\omega_{ik} - \omega_{pk}\omega_{ij}$.

THEOREM 1 A hypersurface W_n of $(QRW)_{n+1}$ satisfies the following.

$$\begin{aligned} \dot{\nabla}_s R_{ijkl} - \dot{\nabla}_s \Omega_{ijkl} &= 2\phi_s(R_{ijkl} - \Omega_{ijkl}) + \phi_i(R_{sjkl} - \Omega_{sjkl}) + \\ &+ \phi_j(R_{iskl} - \Omega_{iskl}) + \phi_k(R_{ijsl} - \Omega_{ijsl}) + \\ &+ \phi_l(R_{ijks} - \Omega_{ijks}) + \overline{R}_{abcd} \dot{\nabla}_s (x_i^a x_j^b x_k^c x_l^d). \end{aligned} \tag{21}$$

Proof 1 By taking the prolonged covariant derivative of (19) we have

$$\dot{\nabla}_s R_{ijkl} = \dot{\nabla}_s \Omega_{ijkl} + \dot{\nabla}_s(\overline{R}_{abcd}) x_i^a x_j^b x_k^c x_l^d + \overline{R}_{abcd} \dot{\nabla}_s (x_i^a x_j^b x_k^c x_l^d) \; .$$

Moreover,

$$\dot{\nabla}_s R_{ijkl} = \dot{\nabla}_s \Omega_{ijkl} + \dot{\nabla}_e(\overline{R}_{abcd}) x_i^a x_j^b x_k^c x_l^d x_s^e + \overline{R}_{abcd} \dot{\nabla}_s (x_i^a x_j^b x_k^c x_l^d) \; .$$

By using (9), Mainardi-Codazzi equations and (15), we obtain the result.

THEOREM 2 A simply connected hypersurface of $(QRW)_{n+1}$ is Riemannian.

Proof 2 If we change the indices k, l, s cyclically in (21), we obtain two more equations. Namely

$$\begin{aligned} \dot{\nabla}_k R_{ijls} - \dot{\nabla}_k \Omega_{ijls} &= 2\phi_k(R_{ijls} - \Omega_{ijls}) + \phi_i(R_{kjls} - \Omega_{kjls}) + \\ &+ \phi_j(R_{ikls} - \Omega_{ikls}) + \phi_l(R_{ijks} - \Omega_{ijks}) + \\ &+ \phi_s(R_{ijlk} - \Omega_{ijlk}) + \overline{R}_{abcd} \dot{\nabla}_k (x_i^a x_j^b x_l^c x_s^d) \; , \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{\nabla}_l R_{ijsk} - \dot{\nabla}_l \Omega_{ijsk} &= 2\phi_l(R_{ijsk} - \Omega_{ijsk}) + \phi_i(R_{ljsk} - \Omega_{ljsk}) + \\ &+ \phi_j(R_{ilsk} - \Omega_{ilsk}) + \phi_s(R_{ijlk} - \Omega_{ijlk}) + \\ &+ \phi_k(R_{ijsl} - \Omega_{ijsl}) + \overline{R}_{abcd} \dot{\nabla}_l (x_i^a x_j^b x_s^c x_k^d). \end{aligned} \tag{23}$$

Adding these three equations and using Mainardi-Codazzi equations and Bianchi identities, we find

$$\phi_i(R_{ljsk} + R_{kjls} + R_{sjkl}) = 0 \,. \tag{24}$$

Since $\phi_i \neq 0$ we have

$$R_{ljsk} + R_{kjls} + R_{sjkl} = 0 \,. \tag{25}$$

From the first Bianchi identity we get

$$R_{(lj)sk} + R_{(kj)ls} + R_{(sj)kl} = 0 \,. \tag{26}$$

Hence, the result follows from the fact that $R_{(ij)sk} = 2g_{ij}T_{[s,k]}$.

A hypersurface W_n of W_{n+1} is called totally geodesic if $\omega_{ij} = 0$. Therefore we have the following theorem.

THEOREM 3 If the simply connected hypersurface of $(QRW)_{n+1}$ is totally geodesic, then the hypersurface is Riemannian quasi-recurrent.

Proof 3 Follows from (21) and the Theorem 2.

References

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