PERIODICA POLYTECHNICA SER. TRANSP. ENG. VOL. 23, NO. 1–2, PP. 45–52 (1995)

COMPARISON OF DISCRETE AND CONTINUOUS RAIL MODELS

Zoltán ZÁBORI

Department of Railway Vehicles
Technical University of Budapest
H–1521 Budapest, Hungary

Received: November 8, 1994

Abstract

This paper shows a comparison between the continuous and discrete rail models. The discrete rail model consists of rigid bodies which are connected with each other by springs and dampers. In the discrete track model the rails are connected with the sleepers by springs and dampers, modelling the pads and fastenings. The continuous rail model is a flexible beam connected with the sleeper masses in discrete points by springs and dampers. The paper introduces a comparative analysis of the two models from the point of view of the shape function of the rail models in case of a moving vertical force. The results give a possibility to identify the parameters of the discrete rail model with the knowledge of the dynamical processes of the continuous rail model.

Keywords: discrete rail model, continuous rail model, dynamical simulation.

1. Introduction

The investigation into the dynamical processes of the railway track – vehicle system has a great importance recently. The structure of the railway track can be modelled as a system of elastically supported continuous beam and sleeper masses connected with the beam and the basic plane elastically and dissipatively. There are moving vertical forces on the beam modelling the vertical wheel loads of a railway vehicle. Due to the vertical forces also vertical displacements of the beam and sleeper masses should be reckoned with.

The goal of this paper is to analyse and compare the dynamical processes of different railway track models.

The railway track is modelled on the one hand as a continuous beam supported elastically in discrete points and as an elastic chain consisting of discrete masses connected with each other elastically and connected with the sleepers in discrete points, on the other. A longitudinally moving vertical force acts on the rail models. The solutions of the equations of motion of the two dynamical track models can be compared from that point of view, if a the good approximation property of the results yielded by the discretized model is ensured.

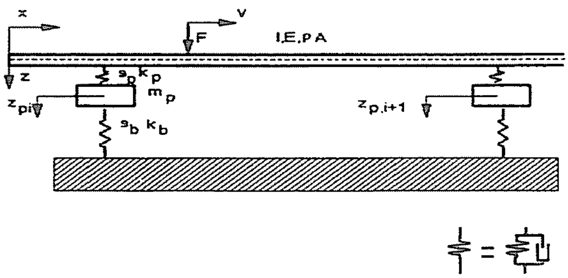

Fig. 1. Continuous rail model

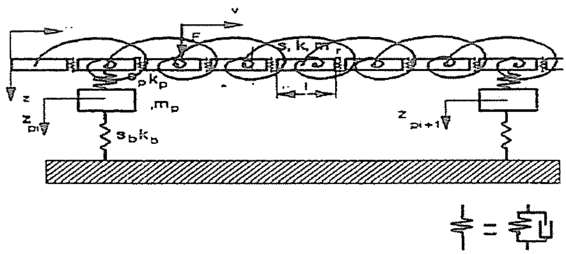

Fig. 2. Discretized rail model

2. Track Modelling

The continuous rail model of the railway track is shown in *Fig. 1.* The in-plane dynamical model consists of an elastic beam supported elastically in discrete points and the model of the sleeper masses is connected with the beam and the stationary basic plane by parallelly connected springs and dampers modelling the rail pads and the ballast. The parameters of the beam are: moment of inertia I, Young modulus E, density ρ and cross-section area A in the model. s_p is the stiffness of the spring, k_p is the damping between the beam and the sleeper mass, while s_b stands for the stiffness and k_b for the damping of the ballast between the sleeper mass and the stationary basic plane. The effect of the wheelset is represented by vertical force F, moving at a constant longitudinal velocity v.

The second model of the railway track is shown in *Fig. 2.* The discretized model of the rail consists of brick-form masses connected with each other by vertical and bending springs (s_p) and vertical and bending dampers (k_p). The length of one mass element is designated by l. Moving vertical force F acts on the discrete track model representing the wheelset load of the vehicle.

3. Motion Equations

a. The equations of motion of the continuous rail model are determined by the known equation of the Euler–Bernoulli beam and by using Newton's 2nd law for the motion of the sleeper masses of the model.

Thus, the equation of motion for the beam is a fourth order linear partial differential equation

$$IE\frac{\partial^4 z(x,t)}{\partial x^4} + \rho A\frac{\partial^2 z(x,t)}{\partial t^2} = -\sum_{(i)} k_i\left(\frac{\partial z(x,t)}{\partial t} - z'_{pi}(t)\right)\delta(x-x_i) - $$
$$-\sum_{(i)} s_i(z(x,t) - z_{pi}(t))\delta(x-x_i) + F(t)\delta(x - vt)\,, \qquad (1)$$

where $z(x,t)$ is the vertical displacement of the beam and x_i is the sequence of the longitudinal position of the sleepers [1], [2], [3].

The equation of motion of the ith sleeper mass is the following second order ordinary linear differential equation:

$$m_i\ddot{z}_{pi} = s_i(z(x_i,t) - z_{pi}(t)) + k_i(z_t(x_i,t) - \dot{z}_{pi}(t))\,. \qquad (2)$$

Eqs. (1) and (2) determine an equation system consisting of one fourth order linear partial differential equation and number n second order linear ordinary differential equations.

b. The equation of motion for the discretized track model (shown in *Fig. 2*) can be written into the following form:

$$\mathbf{M}\ddot{z} + \mathbf{K}\dot{z} + \mathbf{S}z = b(t)\,, \qquad (3)$$

where $\mathbf{M}$ is the mass, $\mathbf{K}$ is the damping and $\mathbf{S}$ is the stiffness matrix, $z(t)$ is the vertical displacement and angular position vector and $b(t)$ is the excitation vector. Vector $b(t)$ can be written in the following form:

$$b_i(t) = \begin{cases} F & \text{if } \frac{(i-1)l}{v} < t < \frac{1}{v} \\ 0 & \text{otherwise} \end{cases} \quad (i = 1, 2, \ldots, 2n-1)\,, \qquad (4)$$

$$b_j(t) = \begin{cases} F[(i-1)l + 0.5l - vt] & \text{if } \frac{(i-1)l}{v} < t < \frac{1}{v} \\ 0 & \text{otherwise} \end{cases} \quad (j = 2, 4, \ldots, 2n)\,, \qquad (5)$$

The excitation function can be seen in the *Fig. 3*.
The structure of the stiffness matrix can be seen in the *Table 1*.

Table 1. The structure of the stiffness mat $\mathbf{S} =$

$+s$	$-s^*(1/2)$	$-s$	$-s^*(1/2)$							
$-s^*(1/2)$	$st + s(1/2)^\wedge 2$	$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$							
$-s$	$+s^*(1/2)$	$+2s + sp$		$-sp$		$-s$	$-s^*(1/2)$			
$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$		$2st + 2s(1/2)^\wedge 2$			$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$			
		$-sp$		$+sp + sb$						
					$2st + 2s(1/2)^\wedge 2$					
		$-s$	$+s * (1/2)$			$+2s$		$-s$	$-s * (1/2)$	
		$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$				$2st + 2s(1/2)^\wedge 2$	$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$	
						$-s$	$+s^*(1/2)$	$+2s$		$-s$
						$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$		$2st + 2s(1/2)^\wedge 2$	$+s^*(1/2)$
								$-s$	$+s^*(1/2)$	$+2s$
								$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$	
										$-s$
										$-s^*(1/2)$

$-s^*(1/2)$										
$-st + s(1/2)^\wedge 2$										
	$-s$	$-s^*(1/2)$								
$2st + 2s(1/2)^\wedge 2$	$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$								
$+s^*(1/2)$	$+2s$		$-s$	$-s^*(1/2)$						
$-st + s(1/2)^\wedge 2$		$2st + 2s(1/2)^\wedge 2$	$+s * (1/2)$	$-st - s(1/2)^\wedge 2$						
	$-s$	$+s^*(1/2)$	$+2s$		$-s$	$-s^*(1/2)$				
	$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$		$2st - 2s(1/2)^\wedge 2$	$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$				
			$-s$	$-t * (1/2)$	$+2s + sp$		$-sp$		$-s$	$-s^*(1/2)$
			$-s^*(1/2)$	$-st - s(1/2)^\wedge 2$		$2st + 2s(1/2)^\wedge 2$			$+s^*(1/2)$	$-st + s(1/2)^\wedge 2$
					$-sp$		$+sp + sb$			
								$2st + 2s(1/2)^\wedge 2$		
					$-s$	$+s^*(1/2)$			$+s$	$+s^*(1/2)$
					$-s^*(1/2)$	$-st + s(1/2)^\wedge 2$				$+s^*(1/2)\ st + s(1/2)^\wedge 2$

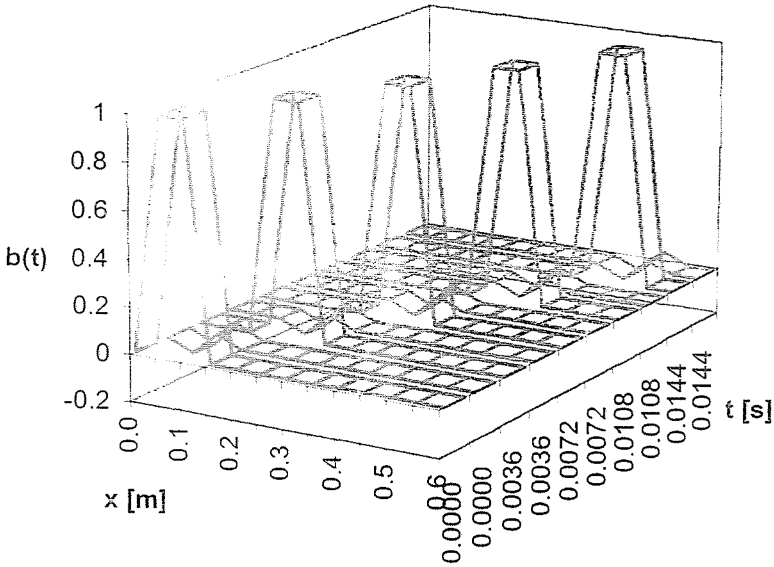

Fig. 3. Excitation function

4. Solution to the Equations of Motion

Equation system (1)-(2) can be solved by using Laplace-transform method. Define

$$Z(p,t) = L\{z(x,t)\} , \tag{6}$$

the Laplace-transform of function of $z(x,t)$ with respect to variable x.

The Laplace-transform of (1) can be written into the following form by considering $F(t)$ is constant:

$$(IEp^4 + \rho Av^2p^2 - \sum_{(i)} k_i vpe^{-px_i} + \sum_{(i)} s_i e^{-px_i})Z(p,t) =$$

$$= Fe^{-pvt} + \sum_{(i)}(k_i Z'_{tpi}(p,t) + s_i Z_{pi}(p,t))e^{-px_i} . \tag{7}$$

So the characteristic polynom of *Eq.* (7) is the following transcendent equation:

$$f(p) = IEp^4 + \rho Av^2p^2 - \sum_{(i)} k_i vpe^{-px_i} + \sum_{(i)} s_i e^{-px_i} = 0 , \tag{8}$$

Table 2. Data set of computation

I	=	$1.7415 \cdot 10^{-5}$ m^4	s_b	=	$3 \cdot 10^8$ N/m
E	=	$2.1 \cdot 10^{11}$ N/m^2	s_t	=	0
ρA	=	60 kg/m	k	=	0
m_r	=	6 kg	k_p	=	0
m_b	=	250 kg	k_b	=	0
s	=	$5 \cdot 10^{10}$ N/m	v	=	100 km/h
s_p	=	$3 \cdot 10^8$ N/m	F	=	1 N

where the unknown variable is p. The *Eq.* (8) can be solved graphically or numerically. The solution to (1) and (2) can be written into the following form:

$$z(x,t) = \sum_{(j)} C(p_j)e^{p_j(x-vt)}, \tag{9}$$

where C is the function of p_j, j is the index of the solution of *Eq.* (8), i is the index of the serial number of the support. The solution was approximated by substituting

$$\sum_{(i)} s_i e^{-px_i} = s' \sum_{(i)} s_i e^{-px_i} = s' \quad \text{and} \quad \sum_{(i)} k_i v p e^{-px_i} = k' v p, \tag{10}$$

where s' and k' are constants [1], and let $z_{pi}(x,t) = 0$ and $z'_{tpi}(x,t) = 0$ (the vertical motion and velocity of the ballast is approximately zero).

The differential equation system of the discrete rail model can be written into the following form by using the state space representation [2], [3], [4]:

$$\begin{bmatrix} \ddot{z}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1} & \mathbf{K} & -\mathbf{M}^{-1} & \mathbf{S} \\ \mathbf{E} & \mathbf{0} & & \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} b(t) \\ 0 \end{bmatrix}. \tag{11}$$

Set of *Eqs.* (10) can be solved in the time domain by using numerical method (e.g. Euler's method).

The computations were performed by using realistic data set for a 3 m long track section model shown in the *Table 2.*

The solutions obtained by the numerical computations are shown in *Fig. 4.* There are two curves in the Figure. The solid line represents the momentary deflection function of the continuous rail model. The strip of rectangles represents the momentary shape of the discretized rail. Each rectangle describes the displacement of the gravity point of the discrete brick element in the discretized model. The moving force for the instant of the representation is at position $x = 1$ m. In the Figure, the longitudinal distance x is taken on the horizontal axis.

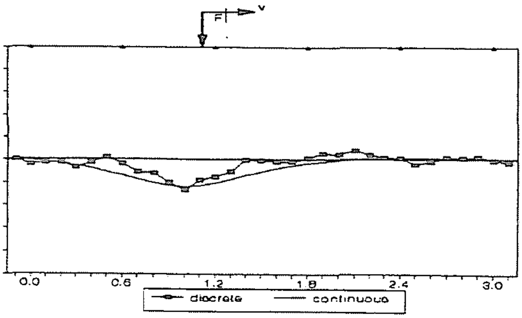

Fig. 4. Shape-function of 3 m long continuous and discrete rail model in case of moving load

It can be seen in the Figure, that the displacement function of the continuous rail model has a smooth shape with small curvature (two inflection points), while the discretized model has more changeable curvatures and more than two inflection points (the initial values were based on by the condition that the maximal vertical displacements of the two models should be equal).

5. Conclusions

It can be seen that the shapes of the displacement functions belonging to the two models are similar to each other but the degree of fitting is not satisfactory between them. Comparing the two computation methods, it is to be emphasized that the treatment of the continuous rail model is complicated because of the necessity of the solution of nonlinear *Eq.* (7), which can be solved only approximately.

The mathematical treatment of the discretized rail model is much more easy.

Further research is necessary to determine the optimum parameters of the discretization and the number of the elements used for representing a sleeper section of the rail.

References

[1] FORTIN, J-P.: Dynamic Track Deformation. *French Railway Review*, North Oxford Academic, 1983., Vol. 1. No. 1. pp. 3–12.

[2] ZOBORY, I.: Track-Vehicle System from the Point of View of a Vehicle Engineer. Transport Scientific Society, Vas County, Szombathely, 1991. pp. 19–42. (in Hungarian).

[3] TIMOSHENKO, S.: Method of Analysis of Statical and Dynamical Stresses in Rail, East-Pittsburgh, Pennsylvania, USA, 1978.

[4] FODOR, GY.: Laplace-Transformation. Technical Publisher, Budapest, 1967. (in Hungarian).

[5] RÉDEI, L.: Algebra, Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig. 1959. pp. 678–685.

[6] HIRATA, C.: An Experimental Study of the Rail Vibration Characteristics in Various Tracks. *Quarterly Reports JNR*, Vol. 21. No. 3.

[7] DESTEK, M.: Complex Investigation and Measuring Method of Track-Vehicle System. Transport Scientific Society, Vas County, Szombathely, 1991. pp. 43–51. (in Hungarian).

[8] TOYODA, M.: Test for Determining Optimum Spring Constant of Tie Pad. Quarterly Report of the Technical Railway Research Institute, 9, 1968/4.

[9] MEGYERI, J.: Track-Vehicle System from the Point of View of a Track Engineer. Transport Scientific Society, Vas County, Szombathely, 1991. pp. 10–18. (in Hungarian).

[10] ZOBORY, I. – ZOLLER, V. – ZÁBORI, Z.: Time Domain Analysis of a Railway Vehicle Running on a Discretely Supported Continuous Rail Model at a Constant Velocity *ICIAM95 Conference*, Hamburg, 1995 (to appear).

[11] ZÁBORI, Z.: The Lateral Dynamics of a Railway Wheelset Running on an Elastically Supported Track. *Periodica Polytechnica Transportation Engineering*, Budapest, Vol. 21. No. 3., 1994. pp. 281–287.

[12] ZOBORY, I. et al.: Theoretical Investigations into the Dynamical Properties of Railway Tracks Using a Continuous Beam Model on Elastic Foundation. *III. Mini Conference, Periodica Polytechnica*, Budapest Vol. 22., No. 1., 1994. pp. 35–54.