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FRACTAL DIMENSIONS IN NON-LINEAR SYSTEM DYNAMICS OF REALITY

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Abstract

This paper treats a) the s.c. 'capacity' and 'alternate' fractal dimensions (fr.dim.), b) together with numerous illustrating examples of geometry, nature and modern arts, c) basin boundaries being often fr.dim. d) finally recent 'control algorithms' for reducing chaotic motions into periodic ones.

Keywords: capacity, pointwise, correlation, information, Ljapunov fr.dim.; fr.basin boundaries, control algorithms.

1. Preliminary Remarks

1.1 Since approx. 3 centuries, the study of a *dynamical system* (dS) given by the *differential equation* (DE), mainly *linear* (lin.) ones and initial date

$$m\ddot{x} = f(x, \dot{x}, t)\ ; \quad x(t_0) = x_0\ , \quad \dot{x}(t_0) = v_0$$

had performed the classical task: *to predict the motion* (as 'history') of S far into the future, using some counting device. In our century, the (electric, later electronic) computers had brought greater possibilities for such far prediction.

However, in the last 1–2 decades, certain exact sciences (e.g. fluid, then solid mechanics, later electric, electronic, physical-mathematical-technical etc. branches, too) had discovered special, s.c. *chaotic motions* (Ch-m) *in non-linear* (nlin.) dS, which *cannot* be predicted generally into the far future and exiges also new concepts, ideas, theories and methods. It became obvious till now, that α) the Ch-m can appear in all nlin.dS, β) it opened a new age in the dynamics and γ) brought a type of revolution into the exact sciences [2], [6].

1.2 Be characterized shortly the **class of Ch-m** in (deterministic) *nlin.dS*! – a) The motion of nlin.dS α) – e.g. over a value δ of control parameter (e.g. the frictional one $\delta = c/\omega$) – can be *regular* (vibration with period T, tendig at $t \to \infty$ e.g. to a stable limit cycle (LC) $\hat{G}_T(\hat{\varrho}_0)$; then β) – under the values of a certain sequence $\delta_1 > \delta_2 > \dots \delta_n > \delta_\infty$ – the sequential *bifurcations*

$\hat{\varrho} \to \hat{\varrho}_i \to \hat{\varrho}_j \to \hat{\varrho}_q$ (1; $i = 1,2$; $j = i+2, \ldots$; $q = 2^{n-2}+2$) and stable *period-duplications* $\hat{G}_T \to \hat{G}_{2T} \to \hat{G}_{2^2T} \to \hat{G}_{2^nT}$ happen; finally γ) – under a heaping value $\delta_\infty > \delta$ – the asymptotic motion on the LC $\hat{G}_{2^nT}$ becomes an *irregular* (aperiodical), s.c. **chaotic** one; its trajectories are contracted to a funny (strange) attractor, on which the points jump irregularly; consequ. the *prediction* of this Ch-m appears totally *impossible* (practically, already for $n > N$). (This is the very frequent FEINGENBAUM way toward the Ch., but also other ways exist, too (see in [3]). In other words, the approaching way α)-β) can be qualified as a *deterministic input* of Ch-m (without random or unpredictable inputs and parameters), over δ_1 with T periodic, then under $\delta_1 > \ldots > \delta_n$ with $2T, \ldots, 2^nT$ periodic vibration, which **transits** on the final way γ) under δ_∞ into a *stochastic output* of Ch-m, under δ_∞ with an aperiodic, irregular jumping on a funny attractor. [Obviously, the Ch-m is not a random motion (as e.g. the BROWNIAN one) with only statistically measured parameters and truly without input data]. – b) A Ch-m is very *sensitive to* the *initial conditions* (IC), that is small differences in the IC can produce very great (enormous) divergencies in the final phenomena. – c) It bears a *loss of information* about IC, when the uncertainty $dA_0 = dx_0^2$ at time $t_0 = 0$ (in regular S) grows during t exponentially to $dA_t = dA_0 e^{ht}$ (in ch.S). – d) Its consequence $h = \frac{1}{t} \ln \frac{dA_t}{dA_0}$ is related (through the entropy) to the s.c. LJAPUNOV *exponent* (see in [2], [3]) measuring the divergency of trajectories in the phase plane $(x, \dot{x}, t)$. – e) Searching for the geometry of the (irregular become, s.c.) Ch-m, the s.c. '*strange attractor*' (Str-att) appears, as unusual (maze-like, multisheeted) structure in the phase space. – f) It is often measured by *fractal dimension* (fr.dim.). – g) A cross section of Str-att produced by the s.c. POINCARÉ *map* (Pc-map) a thread-like set of points shows also fr. properties. – h) The transition between *basin* of ch. and periodic motions in IC or parameter space is often qualified as *fr. basin boundary*.

1.3 Such and other properties of Ch-m were treated in detail in our papers [6]-[7] and mainly in our series of papers [3] (recommended also for postgraduate students and doctorands, too), therefore it is unnecessary to repeat them now. Obviously, it will be here sufficient to recall shortly the basic facts, notions, methods, etc., which are in a relation near enough with the fractal lines, dimensions, basin boundaries, etc. So they can help to fit – in this long 'fr. chapter' – into the mentioned series, (which has given till now only short information about the HAUSDORFF's definition).

2. Definition of the 'Capacity' as Fractal Dimension (Fr.Dim.)

2.1 A very intuitive (geometric) measure for the dimension of a set of points has been introduced by HAUSDORFF (7/4, [3], [5]). This is a general definition, which can furnish – occasionally – a fr. number, as the dimension of the examined set, so it is suitable to classify the POINCARÉ map of numerous nlin. systems giving quantitative measure for the fr. properties of their Str-att. – We describe now the HAUSDORFF's *definition of the s.c. 'capacity'*, but later we will mention some other definition given e.g. by MANDELBROT, FARMER, etc.

2.2 Let us observe now a *set* S_d of points in the (integer) n-dimensional *space* S^n ($\supset S_d$), e.g. a uniform distribution of N_0 points a) along some $d = 1$ dim. (plane or space) curve G_1 in the space S^3, or b) N_0 uniformly distributed points on some $d = 2$ dim. surface $F_2 \subset S^3$. Then we try to *cover* this set of points with small n (= 3) dim. cubes of side $\varepsilon > 0$ (or spheres of radius $\varepsilon > 0$), namely using such covering cubes in minimal number $N(\varepsilon) < N_0$. If N_0 is large enough, then $N(\varepsilon)$ will scale for $d = 1, 2$ and for arbitrary d ($\leq n$) dim. – intuitively and approximately – as

$$N(\varepsilon) \approx 1/\varepsilon , \tag{1}$$

$$N(\varepsilon) \approx 1/\varepsilon^2 , \tag{2}$$

$$N(\varepsilon) \approx 1/\varepsilon^d = (1/\varepsilon)^d \quad (\varepsilon, d > 0) . \tag{3}$$

There is expected a limit behaviour

$$N(\varepsilon) \approx (1/\varepsilon)^d \to +\infty \quad \text{at} \quad \varepsilon \to +0 , \tag{4}$$

namely faster at larger $d > 0$ (connected with the information on G_1, F_2 and S_d's spatial placing, at increased accuracy for $\varepsilon \to +0$). The *Eqs.* (3)[1]–(4) show a natural way to the approaching value d got explicitly by logarithm of both sides:

$$\ln N(\varepsilon) \approx d \cdot \ln(1/\varepsilon) , \tag{5}$$

$$d \approx \ln N(\varepsilon) / \ln(1/\varepsilon) , \tag{6}$$

then to the exact value d_c (referring with a subscript to the name 'capacity') *defined* by the limit formula:

$$d_c = \lim_{\varepsilon \to +0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} , \quad \text{with implicit requirement} \quad N_0 > N(\varepsilon) \to +\infty . \tag{7}$$

It gives in simple cases the usual *integer* dim. d (= $1, 2, 3, \ldots$) (see the examples 1a–1c); but it furnishes in numerous chaotic cases non-integer = fraction result, sc. *fractal* dim. (see the examples 2–3).

2.3 Look at some simple, then complicated examples to calculate exactly the integer or fractal dim. of a set of points on a curve or surface.

1/a) Linear distribution points:

$$d = \ln N(\varepsilon) / \ln(1/\varepsilon) = \ln 10 / \ln(1/0.1) = 1 \ldots\ldots \quad \text{(int.dim.)} .$$

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[1]One can write more fully: $N(\varepsilon) \approx C(1/\varepsilon)^d$, but the limit $\varepsilon \to +0$ on $d \approx [\ln N(\varepsilon) + \ln C] / \ln(1/\varepsilon)$ makes disappear the term of C.

1/b) Linear distribution on a curve:

$$d = \ln N(\varepsilon)/\ln(1/\varepsilon) = \ln 10/\ln 10 = 1\ldots \quad \text{(int.dim.)}\ .$$

Fig. 1.

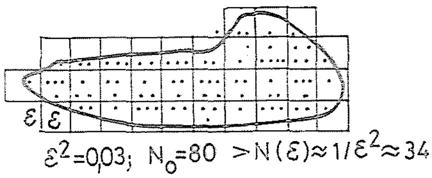

Fig. 2.

1/c) Planar distribution of points:

$$d = \ln N(\varepsilon)/\ln(1\varepsilon) = \ln 34/\ln 33^{\frac{1}{2}} =$$

$$= 2 \cdot \ln 34/\ln 33 \approx 2\ldots \quad \text{(int.dim)}.$$

There was a sole step of the covering with a unique ε^2; it can be continued (with finer ε^2), expecting a better approach to 2 (*Fig. 3*).

2) KOCH *curve* (1904) treated in MANDELBROT's book (1977). The **increasing** geometric procedure α) sets out from an interval G_0 of length $L_0 = \varepsilon_0 = 1$. β) divides it into 3 segments of length $\varepsilon_1 = 1/3$ and γ) replaces the middle one by 2 segments of similar length $1/3$; the new curve G_1 of $N_1 = 4$ sides has obviously the total length $L_1 = N_1\varepsilon_1 = 4/3$. The continuation happens by repeating of the former triple-step S_3 ($\alpha - \gamma$) for all the 4 sides, namely the n^{th} S_3 results $N_n = 4^n$ segments of length $\varepsilon_n = (1/3)^n$ with the total length $L_n \triangleq N_n\varepsilon_n = (4/3)^n$. Tending $n \to +\infty$, so $G_n \to G$, $\varepsilon_n \to +0$, $N_n \to +\infty$ and $L_n = N_n\varepsilon_n \to +\infty$, then

$$d_c = \lim_{n\to\infty} \frac{\ln N_n}{\ln(1/\varepsilon_n)} = \lim_{n\to\infty} \frac{\ln 4}{\ln 3} = 1.26185\ldots \quad \text{(fr.dim.)}$$

and the fractional line G_n of N_n segments – looking fuzzy – becomes a continuous, but nowhere differentiable limit curve G (*Fig. 4*). This set of points $G_n \to G$ of dim. $d_c \approx 1.26$ appears as trying to cover more than a line, but reaching to fulfill less than an area only, having nevertheless some properties of area, as a young boy's scribbling with coloured crayons on a piece of sidewalk. – We will find such fractal – like structures for basin boundaries of periodic attractors (see e.g. [5] p. 244) and for boundaries between periodic and Ch-m (see e.g. here. p.12) therefore this KOCH curve is very important for the nlin. dynamics.

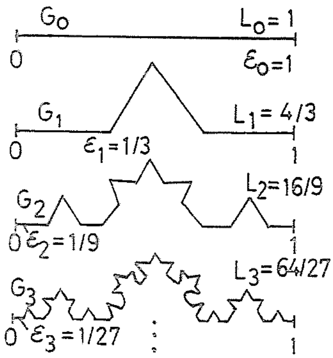

Fig. 3.

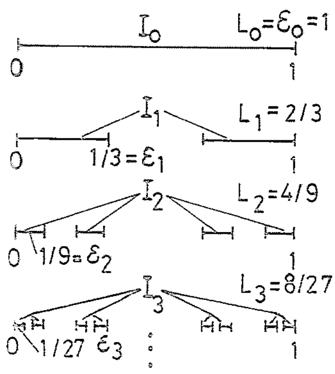

Fig. 4.

3) CANTOR *set* (discovered in 1883) can be produced by a **decreasing** geometric process. Namely, this also very significant concept for nlin. systems can be origined by repeated removing finer and finer pieces from the initial line (counter KOCH curve, by repeated complementing smaller and smaller segments to the initial interval). – The construction's procedure begins with a triple-step S_3: α) to take an interval I_0 of length $L_0 = \varepsilon_0 = 1$, β) to divide it into 3 parts of length $\varepsilon_1 = 1/3$ and γ) to omit the middle one and to keep the remaining $N_1 = 2$ parts as union I_1 with total length $L_1 = N_1 \varepsilon_1 = 2/3$. – Continuation by repeating of S_3: after the n^{th} S_3, there is the remaining I_n with $N_n = 2^n$ segments of length $\varepsilon_n = (1/3)^n$ and total length

$L_n \overset{\wedge}{=} N_n \varepsilon_n = (2/3)^n$ (*Fig. 5*). At $n \to +\infty$, these limits appear: $I_n \to I$, $\varepsilon \to +0$, $N_n \to +\infty$, $L_n \to +0$, then

$$d_c = \lim_{n\to\infty} \frac{\ln N_n}{\ln(1/\varepsilon_n)} = \lim_{n\to\infty} \frac{n \ln 2}{\ln 2} = \frac{\ln 2}{\ln 3} = 0.63092\ldots \qquad \text{(fr.dim.)}$$

Consequently, the infinite point-series I of dim $d_c = 0.63\ldots$ shows itself more, than a point (of dim. 0), but less than a line (of dim.1). On this discontinuous fr. set, one can generate a continuous fr. function, namely by integrating a distribution function of the total unit mass at the start on the total interval I_0, later on the remaining and decreasing CANTOR intervals $I_1, \ldots, I_n$, with increasing mass density. After the n^{th} step, when I_n consists of $N_n = 2^n$ parts of length $\varepsilon_n = (1/3)^n$, the density is $\varrho_n = (3/2)^n \overset{\wedge}{=} c_n$ for all the $N_n\varepsilon_n$-segments (obviously: $L_n\varrho_n \overset{\wedge}{=} N_n\varepsilon_n \cdot \varrho_n = 2^n(1/3)^n = 1$ total mass) and $\mathring{\varrho}_n = 0$ for all omitted (vacant) segments of $\mathring{I}_n$ $(\varepsilon_1, 2\varepsilon_2, \ldots, 2^n\varepsilon_n;$ $\mathring{L}_n = \frac{1}{3}\frac{1-(2/3)^n}{1-2/3} = 1-(2/3)^n$; $L_n + \mathring{L}_n = (2/3)^n + [1-(2/3)^n] = 1 \overset{\wedge}{=} L_0)$.

The mass on the interval $I_x = [0, x]$ at $x \in \mathring{I}_n$ will be calculated by integration

$$M_n(x) \overset{\wedge}{=} \int_0^x \varrho_n(\xi)d\xi = L'_n\varrho_n = N'\varepsilon_n\varrho_n = 2^\nu \cdot (1/2)^n = (1/2)^{n-\nu}$$

at $N' \overset{\wedge}{=} 2^\nu < 2^n \overset{\wedge}{=} N_n$, but at $\nu = n$ one has $M_n(1) = 2^n(1/2)^n = 1$ (*Fig. 6*).

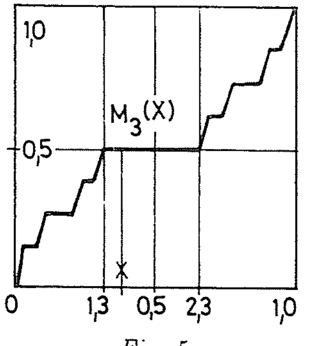

Fig. 5.

Its figure is a fractional, but continuous line consisting of oblique (increasing with $\tan\varphi = (3/2)^n$) and horizontal segments. – The limit curve at $n \to +\infty$ is the s.c. 'devil's staircase' $M(x)$ having $M'(x) = \varrho(x)\sum_{i=1}^{\infty}\delta(x - \xi_i)$.

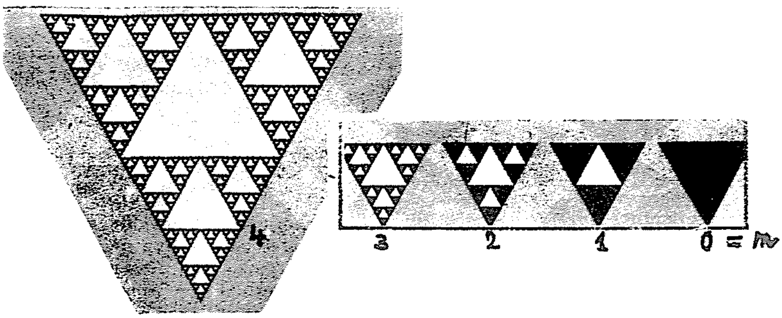

Fig. 6.

4) '**Decreasing**' triangular set: $T_0 = \frac{\sqrt{3}}{4}$, $T_1 = \frac{3}{4}T_0$, $T_2 = \frac{9}{16}T_0, \ldots$, $T_n = \left(\frac{3}{4}\right)^n$; $\varepsilon_n = (1/2)^n$, $N_n = 3^n$; $d_c = \ln 3/\ln 12 = 1.5737, \ldots$ (*Fig. 7*).

3. Alternate Definitions for the Fr.Dim.

3.1 The earlier introduced *capacity* d_c to measure the fr.dim. of Str-atts is a geometric metric (considering – without the frequency of orbit – the covering set of cubes or balls in phase space), but also a numeric one (counting the mentioned covering process often by computer). – The following alternate definitions – giving for many Str-atts roughly the same dim. – will be good controllers for the capacity d_c [5].

3.2 Pointwise dim. (Pw-dim.) On a long-time trajectory in phase space, we sign time-sampled points of motion in large number N_0. then place a sphere of measure r at some point ϱ_i of orbit and count the points in it: $N(r)$. The proportion $P(r, \varrho_i) = N(r, \varrho_i)/N_0$ gives us the (combinatorial) probability of finding a point in this sphere (from N_0 ones). – For a 1-dim. (closed periodic) orbit will be (at $r \to 0$, $N_0 \to \infty$): $P(r, \varrho_i) \approx br$; for a 2-dim. (toroidal, quasiperiodic) orbit: $P(r, \varrho_i) \approx br^2$; for a general case: $P(r, \varrho_i) \approx br^{d_p}$, consequ. [5]

$$\frac{\ln P}{\ln r} - \frac{\ln b}{\ln r} \approx d_p \,, \quad \text{finally} \quad d_p = \lim_{r \to 0} \frac{\ln P(r, \varrho_i)}{\ln r} \,. \tag{8}$$

For some attractor, d_p is independent of ϱ_i; but generally $d_p = d_p(\varrho_i)$, when it is suitable to count an *averaged Pw-dim.* on the randomly chosen set of points $\varrho_1, \ldots, \varrho_i, \ldots, \varrho_M$ at $M \ll N_0$ (e.g. distributed around the Str-att):

$$\frac{1}{M}\sum_{i=1}^{M} P(r, \varrho_i) \approx ar^{d_p} \,, \quad \ln \frac{1}{M}\sum_{i=1}^{M} P(r, \varrho_i) - \ln a \approx d_p \cdot \ln r \,,$$

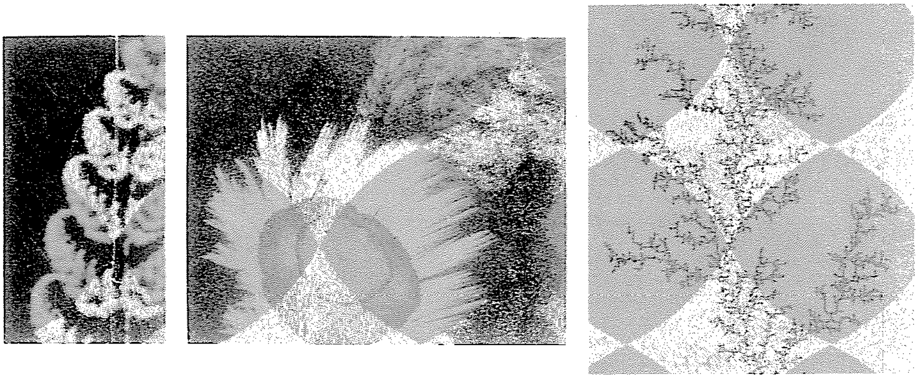

Fig. 7.

$$d_p = \lim_{r \to 0} \frac{\ln \left[\sum_i P(r, \varrho_i)/M\right]}{\ln r} . \tag{9}$$

Practically, at $N_0 \approx 10^3 \sim 10^4$, one use $M \approx 10^2 \sim 10^3$.

3.3 Correlation dim. (Cr-dim.). It is used successfully since 1983, mainly by experimentalists, they find it often as related to the Pw-dim.

We discretize the (continuous) set to one of N points $\{\hat{\varrho}_i\}_N$ in the phase space, then count the distances $s_{ij} = |\hat{\varrho}_i - \hat{\varrho}_j| \triangleq \left[\sum_k (x_{ki} - x_{kj})^2\right]^{1/2}$ (or $s_{ij} = \sum_k |x_{ki} - x_{kj}|$) for the Cr-function [5]

$$C(r) = \lim_{N \to \infty} \frac{1}{N^2} \cdot \left(\begin{array}{c} \text{number of pairs } i, j \\ \text{with distances } s_{ij} < r \end{array} \right) = \lim_{N \to \infty} \frac{1}{N^2} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} 1(r - s_{ij}) .$$

that is the number of points ϱ_j in each sphere of centre ϱ_i and radius r (where the unit spring function $1(r - s_{ij}) = \begin{cases} 1 & \text{at} \quad r > s_{ij} \\ 0 & \text{at} \quad r < s_{ij} \end{cases}$; the sum is performed here about every point, but at the Pw-dim. about $M \ll N_0$ ones only). For many Str-att. one can find a power law (for $r \to 0$)

$$C(r) \approx a r^{d_G} , \quad \text{from which the Cr-dim. originates :} \quad d_G = \lim_{r \to 0} \frac{\ln C(r)}{\ln r} . \tag{10}$$

3.4 Information dim. (Inf. dim.) This definition is similar to one of d_c, but it tries to take into account the frequency of visits each covering cube by the trajectory (assumed: it is long enough to cover effectively the Str-att). Having again a set of points N_0 to discretize uniformly the (continuous) trajectory and covering it with a set of N cubes of size ε, one counts the number of points N_i in each of N cubes and the probability P_i of finding a point in the i^{th} cell:

$$P_i = N_i/N_0 \quad (N \ll N_0) \quad \sum_{i=1}^{N} P_i = 1\,. \tag{11}$$

Then the information entropy (approached for small ε, too) appears so:

$$I(\varepsilon) = -\sum_{i=1}^{N} P_i \ln P_i \approx \ln(1/\varepsilon)^{d_1} = -d_I \ln \varepsilon \tag{12}$$

and from this the definition of Inf. dim. origines [5]:

$$d_I = \lim_{\varepsilon\to 0} \frac{I(\varepsilon)}{\ln(1/\varepsilon)} \triangleq \lim_{\varepsilon\to 0} \frac{\sum_i P_i \ln P_i}{\ln \varepsilon}\,. \tag{13}$$

$I(\varepsilon)$ is a measure of the *unpredictability* in a system. – For uniform probability $P_i \triangleq N_i/N_0 = 1/N \triangleq P$, it has a *maximum*:

$$I(\varepsilon) \triangleq -\sum_{i=1}^{N} P_i \ln P_i = -N \cdot P \ln P = N \cdot \frac{1}{N} \ln N = \ln N(\varepsilon) = \hat{I}(\varepsilon)\,, \tag{14}$$

moreover

$$\hat{d}_I \triangleq \lim_{\varepsilon\to 0} \frac{\hat{I}(\varepsilon)}{\ln(1/\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \triangleq d_c\,; \tag{15}$$

as it is provable, $d_I \leq d_c$ in general.
– For a sole filled (and each other empty) cube $N_1 = N_0$, $P_1 = 1$ (so at $i \neq 1$; $N_i = P_i = 0$), there is $I(\varepsilon) = -P_1 \ln P_1 = -1 \cdot \ln 1 = 0$, consequ. $d_I = \lim_{\varepsilon\to 0} \frac{0}{\ln(1/\varepsilon)} = 0$; this is the case of *maximal predictability*.

Let still be mentioned *the q^{th} order Inf. entropy and dim.* (1984; useful in statistical mechanics and inf. theory):

$$I_q(\varepsilon) = \frac{1}{1-q} \ln \sum_{i=1}^{N} p_i^q\,, \quad d_q = \lim_{\varepsilon\to 0} \frac{I_q(\varepsilon)}{\ln(1/\varepsilon)}\,. \tag{16}$$

Its cases $q = 0, 1, 2$ (with $q = 1 + \Delta q \to 1$ at $q \to 0$) make connection with d_c, d_I and d_G so [5]:

$$I_0 = \ln \sum_{i=1}^{N} P_i^0 = \ln N \cdot 1 = \ln N \triangleq \hat{I}(\varepsilon)\,, \tag{17}$$

$$I_1 = \lim_{\Delta q\to 0} \frac{1}{\Delta q} \ln \sum_{i=1} P_i P_i^{\Delta q} = -\sum_{i=1}^{N} P_i \ln P_i \triangleq I(\varepsilon)\,. \tag{18}$$

$$I_2 = -\ln \sum_{i=1}^{N} P_i^2 = \lim_{N_0\to 0} \ln 2 \cdot N_0 C(\varepsilon)\,. \tag{19}$$

Finally, it was proved (1983), that $d_G \leq d_I$ are lower bounds of d_c, however, they are very close for many known Str-atts:

$$d_G \leq d_I \leq d_c \,. \tag{20}$$

3.5 Fr.dim. based on LJAPUNOV **(Lj.) numbers & exponents.** As memorable, there exponents $\lambda_i = \ln L_i$ measure the (rate of the) velocity of 2 trajectories (going out from $S_0(\varepsilon)$: $|q_0' - q_0| \leq \varepsilon$ and) **diverging** on the attractor with $|q_n - \hat{q}_i| \to \infty$ (at $n \to \infty$), or **converging** off the attractor toward another one with $q_n \to \hat{q}_i$, (at $n \to \infty$). During this dynamical process, the initial conditions' sphere $S_0(\varepsilon)$ is imagined to deform into an ellipsoid (in 3 dim.). – At a chaotic 2 dim. map $\hat{\varrho}_{n+1} = \mathbf{f}(\hat{\varrho}_n)$, the circle $C_0(\varepsilon)$ deforms into an ellipse having – after M_ε steps of iteration – the main axes $\overline{L}_1$ and $\overline{L}_2$, where $\overline{L}_i > 0$ at $(i = 1, 2)$ – as over the whole attractor averaged values – are the Lj. numbers, their logarithm $\overline{\lambda}_i = \ln \overline{L}_i$ the Lj. exponents. KAPLAN and YORK (1978) have proposed to calculate for a fr. attractor this Lj. *dim.*: [2]–[5]:

$$d_L = 1 + \frac{\ln \overline{L}_1}{\ln(1/\overline{L}_2)} = 1 - \frac{\overline{\lambda}_1}{\overline{\lambda}_2} \,. \tag{21}$$

A DE $\dot{\hat{\varrho}} = \mathbf{F}(\hat{\varrho}, t)$ of 4 dim. $(\hat{\varrho}, \dot{\hat{\varrho}} \in E_4)$ given for a dissipative system has a POINCARÉ map $\hat{\varrho}_{n+1} = \mathbf{f}(\hat{\varrho}_n)$ of **3 dim.** $(\hat{\varrho}_n; \hat{\varrho}_{n+1} \in E_3)$. For its Str-att. one can find

$$\overline{L}_1 > 1 \,, \quad \overline{L}_2 = 1 \,, \quad \overline{L}_3 < 1 \,, \tag{22}$$

that is the ellipsoid has tension, length-keeping, contraction in the 1^{st}, 2^{nd}, 3^{rd} main direction, resp. Because of dissipation, the ellipsoid's volume is less than the sphere's one, so that

$$\overline{L}_1 \overline{L}_2 \overline{L}_3 < 1 \,, \quad \text{but} \quad \overline{L}_1 \overline{L}_2 > 1 \,. \tag{23}$$

This circumstance leads us to use the K. and Y. *formula* (as the special case $k = 2$ of their general one) for Lj. *dim.*:

$$d_L = 2 + \frac{\ln(\overline{L}_1 \cdot 1)}{\ln(1/\overline{L}_3)} \triangleq 2 + \frac{\overline{\lambda}_1}{\overline{\lambda}_3} \,, \tag{24}$$

where it is difficult to measure the contraction's Lj. number $\overline{L}_3$.

For an N-**dim.** POINCARÉ **map** of such a system and at the order

$$\overline{L}_1 > \overline{L}_2 > \ldots > \overline{L}_k > \ldots \overline{L}_N \quad \text{with} \quad \overline{L}_1 \overline{L}_2 \ldots \overline{L}_k \geq 1 \,, \tag{25}$$

they have given for the Lj. *dim.* the following *general formula* [5]:

$$d_L = k + \frac{\ln(\overline{L}_1 \overline{L}_2 \ldots \overline{L}_k)}{\ln(1/\overline{L}_{k+1})} \triangleq k - \frac{\overline{\lambda}_1 + \overline{\lambda}_2 + \ldots + \overline{\lambda}_k}{\overline{\lambda}_{k+1}} \,, \tag{26}$$

which is also a lower bound for d_c, that is

$$d_L \leq d_c\,. \tag{27}$$

Remarkable that FARMER (1983) has given for the *Btr* the following connection (at $\overline{\lambda}_a = \overline{\lambda}_b = \overline{\lambda}$):

$$d_I = d_L = 1 + \frac{\ln(1/\alpha) + (1-\alpha)\ln[1/(1-\alpha)]}{\ln(1/\overline{\lambda}) + (1-\alpha)\ln(1/\overline{\lambda})} \stackrel{\wedge}{=} 1 + \frac{H(\alpha)}{\ln(1/\overline{\lambda})}\,; \tag{28}$$

moreover at $\alpha = 1 - \alpha = 1/2$ and $H(\alpha) = \ln 2$, one obtains:

$$d_I = d_L = d_c\,, \tag{29}$$

and the map is like the horseshoe, or CANTOR map. – Worth mentioning that the studied dynamical process can lead to a *nonuniform* POINCARÉ *map*, when the different fr.dims often yield different results.

3.6A *The nature* exhibits fr. geometry in rich variety. Fr. curves (as fractioned lines consisting of straight pieces with free length and direction): a) border the coastal region of oceans, seas, lakes, b) similarly one of (pen)isle countries (as Island, England, Norway, etc.) (with longer frontier at finer measuring). Such (often randomlike) fr. formations appear (in the plane or space): c) on the ice of lake, as clefts: d) at the lightning, as trace lines of discharge; e) the contour line of mountain chains (looking from far); f) on the snowflake, as its contour and surface; g) on the frost – works of window, as strange figures; h) at the leaves' falling in windy autumn, as layered spread of foliage; i) fleecy clouds on the sky; j) the (randomly) ramifying of certain plants (e.g. cauliflower), bushes (e.g. blackberry); k) similar spread of weeds among the plants; l) sinking down sand grains during a sand storm; etc. [2], [5]. (*Fig. 8*).

3.6B *The artists'* sensitiveness to the fr. properties is remarkable. E.g. at the beginning of the century, a) the impressionists have used coloured points to make perceptible different effects in the space; b) in its 2$^{\mathrm{nd}}$ half, VASARELLY and others are using a rich world of colours and fitted geometric forms for various effects of space. c) Today, some textile designers create fr. figures for ladies' wear [2], [5].

4. Fractal Basin Boundaries

4.1 Attractors and their basins. In most *lin.* systems (given e.g. by a lDE), there is just one possible motion for certain input and one attractor:

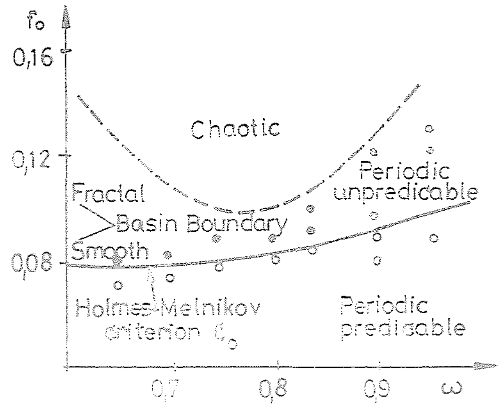

Fig. 8.

the equilibrium point. However, in *nlin.* systems (given e.g. by a nl.DE), more motions can occur depending on the input parameter and more attractors, too: equilibrium positions, periodic or limit cycle motions. These last ones are interesting now for us.

The range of values taken up by certain input or control parameter, for which the motion tends toward a given attractor, is called a *basin of attraction* in the parameter's space.

If there are two (or more) attractors, then their frontier-giving transition from one basin to an other – is called a *basin boundary*. – In *lin.* systems, it is expected, as a smooth, continuous line or surface and when its parameters are away from the input ones then their small uncertainties will not affect the outcome. – However, as the research has proved it, many *nlin.* systems exhibit nonsmooth, but fractal basin boundary: its existence is an essential part in the behaviour of nlin. systems. Small uncertainties in input, or other parameters may cause uncertainties in the outcome, so the predictability of motion can be impossible. – The (*Fig. 9*) shows certain smooth (continuously traced) and fractal (dotted lined) basin boundary, namely the fractal one for the HOLMES-MELNIKOV criterion ($f_0 > \frac{\gamma\sqrt{2}}{3\pi\omega}\cosh\frac{\pi\omega}{\sqrt{2}} \triangleq C_0$) at the two-well potential problem ($\ddot{x} = -\gamma\dot{x} + x(1 - x^2)/2 + f_0\cos\omega t$), the smooth one for the counter case ($f_0 \leq C_0$).

4.2 Let be mentioned the *complex map* $\overset{c}{L}$ to the series of regular complex function having a complex parameter ζ [2] [4]:

$$z_{n+1} \triangleq x_{n+1} + iy_{n+1} = (x_n^2 - y_n^2 + \xi) + i(2x_n y_n + \eta) \triangleq z_n^2 + \zeta\,.$$

The (black) domain of a) called as MANDELBROT set is the *fr. basin* of a parameter ζ, for which the long iteration (at $n \to \infty$) will remain bounded: $|z_{n+1}| < K$; the boundary of this domain shows fr. properties. – At

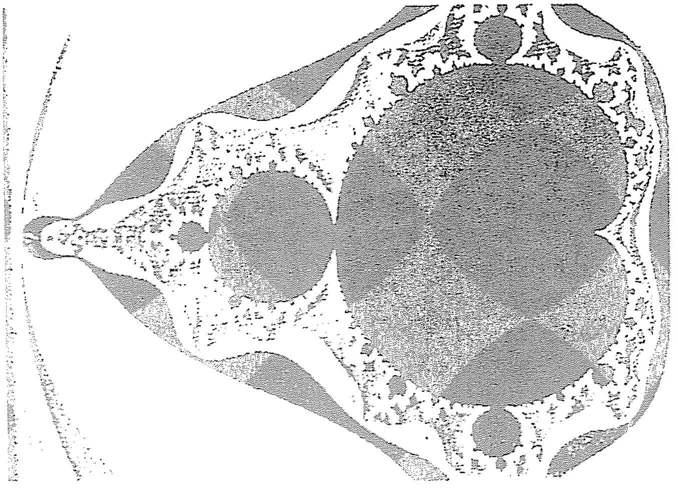

Fig. 9-a

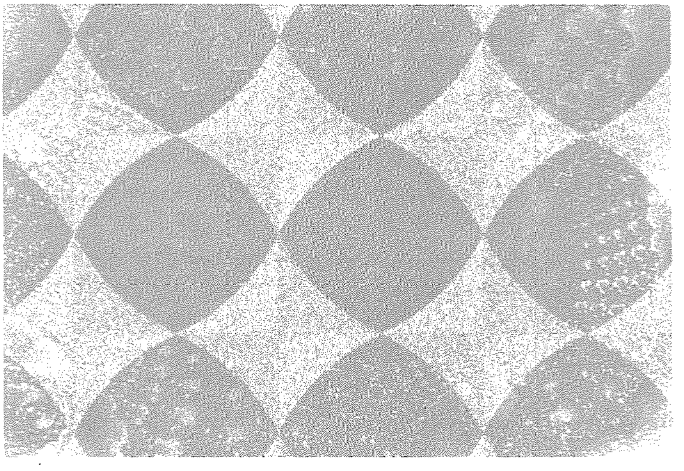

Fig. 9-b.

POINCARÉ map written by irregular complex functions

$$z_{n+1} = x_{n+1} + iy_{n+1} = u(x_n, y_n) + iv(x_n, y_n) \triangleq f(z_n) ,$$

one can meet a map b), which is different from a) in details, but related in central role of the fr. basin (here an oval one) and its fr. boundary (*Figs. 10.* and *11*).

5. Control of Chaotic Motions (Ch-m) into Periodic Ones

5.1 As it was stated (e.g. in 1.1), a Ch-m[2] cannot be predicted into future. Therefore the applied sciences (e.g. the appl. math.-phys.-biology-chemistry, etc.) intended recently to keep a firm hand on such a motion and reduce it possibly into a regular one. In the last 3–5 years, the researches have proved that the Ch. systems can be controlled really, that is their *Ch-m* can be moderated into a *periodic one*.

In research institutes of various applied sciences, mainly interdisciplinary teams have found several '**control algorithms**' (CA) for such purposes. Of course, these CA look very specific with strongly different details, but yet one can state some general steps St_k of common quality; such are e.g.: St_1 **diagnostic step**: one observes – with suitable feedback, or measuring tool – 'just where is walking the Ch. system S', that is in which direction and measure are deviating its control parameter (C_p) values from their normal ones; – St_2 **correcting step**: one betters the Ch-S's behaviour by small perturbations of the mentioned C_p, to drive its Ch-m towards a periodic one; $\mathrm{St}_{i>2}$, **repeating steps** of St_1 and St_2, too, for hindering S from reverting to the Ch [7].

5.2 Stay here some example!

1) It is obvious, that the *medical treatment* of an ill person can be considered, as a CA (it is suggested also by our naming of St_i). There is now the illness, as Ch; S_1 happens by a clinical thermometer, ECG, blood test, etc.; St_2 happens by prescribed medicines, dietary meal, gargling, inhaling, hydrotherapy, etc.: $\mathrm{St}_{i>2}$ are the repetition of St_1 and St_2; the restored normal state is the health.
2) Let be mentioned *some successful CA* from the last years! – a) OTT–GREBORI–YORKE (Maryland) CA (having St-type steps), which was the beginner of such experiences. – b) DITTO–RAUESEO–SPANO (Navy) C, which reduced the Ch-m of an elastic band in magnetic field into a regular one. – c) R. ROY and team (Georgia) increased the energy product of a solid laser – by slowing up its Ch – onto 10–15 times. – d) SHOWALTER and team with Hung. cooperation [7] examined resultsfully– a simple CA to regulate the chemical Ch etc.

5.3 Let us close this paper with the hope that the applied mathematics – in cooperation with other applied sciences – can promote surely *the quick development and the industrial propagation of this recent branch 'CA of Ch-m'*, namely by more fine and profound discovery of Ch-m (and C_p, sequential bifurcations, Str-att, fr.lines–dimensions–basin boundaries), by elaboration of optimal CA for various Ch systems, etc. The expected success of the 'controlled chaos' promises *a giant practical importance for the next decades.*

[2] which is signed e.g. just by fr. properties.

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