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THEORETICAL INVESTIGATIONS INTO THE DYNAMICAL PROPERTIES OF RAILWAY TRACKS USING A CONTINUOUS BEAM MODEL ON ELASTIC FOUNDATION

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Abstract

The development in railway vehicle technology requires adequate dynamical analyses to ensure accurate and reliable information on the expected loading conditions of the vehicle components in the period of design. This paper takes an attempt to formulate an exact mechanical and mathematical description of the wheel track system. The wheel is considered as a rigid disk, while the track is modelled by an Euler-Bernoulli beam on damped elastic foundation. The connection of the wheel and the track is realized by the linear Hertzian spring and damper. Stability considerations, critical speed determination and solution of the boundary value problem will be carried out. Also the complex eigenfrequencies will be pointed out.

Keywords: wheel-track system, continuous beam model, linear partial differential equations.

1. Mechanical Model and Mathematical Description

The in-plane model introduced in [9] consists of a moving loaded wheel on elastic foundation where the contact between the wheel and the rail is modelled through a paralelly connected linear Hertzian spring and damping. The model is shown in *Fig 1.*

The partial differential equation of the Bernoulli-Euler beam on an elastic Winkler foundation has the form

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = (T_0 - m\ddot{Z})\delta(x - vt)\,, \tag{1}$$

where $z(x,t)$ denotes the vertical displacement of the rail, and $Z(t)$ stands for the vertical displacement of the wheel. Here the positive real parameters I, E, A, ρ and s mean the moment of inertia, Young's modulus,

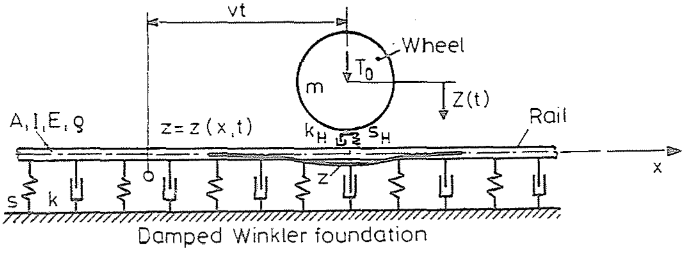

Fig. 1.

cross-section area, mass density and distributed stiffness, respectively. The nonnegative real parameter k stands for distributed damping, while the real parameter v denotes steady longitudinal speed of the wheel.

The solutions $z(x, t)$ of the partial differential *Eq.* (1) must satisfy the boundary condition

$$\lim_{x \to \pm\infty} z(x, t) = 0 . \tag{2}$$

Eq. (1) is coupled with the ordinary differential equation

$$T_0 - m\ddot{Z} = k_H \left(\dot{Z} - \frac{\mathrm{d}}{\mathrm{d}t} z(vt, t) \right) + s_H \left(Z - z(vt, t) \right) \tag{3}$$

with initial conditions

$$Z(0) = Z \qquad \text{and} \qquad \dot{Z}(0) = V_0 . \tag{4}$$

Here the constant positive parameters T_0, m, s_H and k_H stand for the wheel load, the mass of the wheel, the stiffness and damping of the Hertzian spring, respectively.

Remark 1.1. Partial differential *Eq.* (1) is meant in the distribution or generalized function sense, i. e. z is in fact a linear functional on the vector space $C_0^\infty(R^2)$ of smooth functions vanishing outside a bounded closed set in R^2, and δ stands for the unit impulse or Dirac's δ-distribution, see e. g. [8].

We are looking for a solution of the system (1–4) in the form

$$z(x, t) = B_0(\xi) + \sum_{i=1}^{2} B_i(\xi) e^{w_i t} , \qquad Z(t) = \beta_0 + \sum_{i=1}^{2} \beta_i e^{w_i t} , \tag{5}$$

where $\xi := x - vt$ is the relative longitudinal displacement, $B_j(\xi)$ is a complex valued function and β_j is a complex constant for $j = 0, 1, 2$.

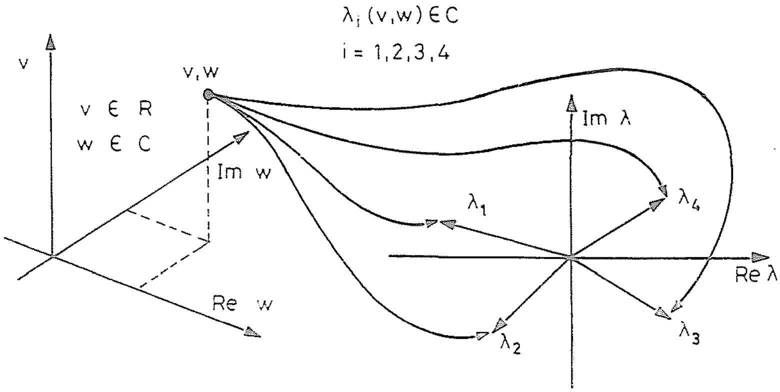

Fig. 2.

Definition 1.2. A complex number w_i satisfying (5) is called *a complex frequency* of the system (1–4).

Our first goal is to obtain a solution for the partial differential equation

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = e^{wt}\delta(x - vt)\,, \tag{6}$$

where w is a complex number, i. e. a possible complex frequency, under boundary condition (2). Such problems have been solved by KENNEY [6], MATHEWS [7] and FILIPPOV [3] for the classical case $w = i\omega$, $\omega - R$. In this paper we use another method, which is similar to that of DE PATER [2] in the classical case.

If we are looking for a solution of *Eq.* (6) in the form

$$z(x,t) = B(\xi)e^{wt}\,, \tag{7}$$

then by substitution we obtain the ordinary differential equation

$$EIB^{\mathrm{IV}} + \rho Av^2B'' - v(k + 2\rho Aw)B' + (s + kw + \rho Aw^2)B = \delta \tag{8}$$

with characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho Av^2\lambda^2 - v(k + 2\rho Aw)\lambda + (s + kw + \rho Aw^2)\,. \tag{9}$$

It is to be mentioned that the roots of (9) will be considered as functions of the two essential parameters, namely v and w. In *Fig 2* the allocation of the roots $\lambda_i(v, w)$; $i = 1, 2, 3, 4$ is visualized as lying in the complex plane positioned at a height v.

In case of our dynamical problem it is enough to restrict ourselves to the following, so-called moderately damped case, however, computations of the following chapters can be carried out in overdamped cases, too.

Definition 1.3. We call our system *moderately damped* if the relation

$$4\rho As - k^2 > 0$$

is satisfied.

2. Stability Analysis of the Characteristic Polynomial

We are interested in the question concerning the number of roots of the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho Av^2\lambda^2 - v(k + 2\rho Aw)\lambda + (s + kw + \rho Aw^2)$$

with $\mathrm{Re}(\lambda) > 0$, where $4\rho As - k^2 > 0$ is satisfied, i. e. the system is moderately damped.

For technical reasons we shall introduce the following nondimensional variables:

instead of λ the complex variable

$$\mu := \lambda \sqrt[4]{\frac{4EI\rho A}{4\rho As - k^2}},$$

instead of v the nonnegative real parameter

$$c := v \sqrt[4]{\frac{4\rho^3 A^3}{EI(4\rho As - k^2)}},$$

and instead of w the complex parameter

$$\nu := \frac{k + 2\rho Aw}{\sqrt{4\rho As - k^2}}.$$

Substituting the above new variables, our characteristic polynomial will have the relatively simple form

$$p(\mu) = \mu^4 + (c\mu - \nu)^2 + 1. \tag{10}$$

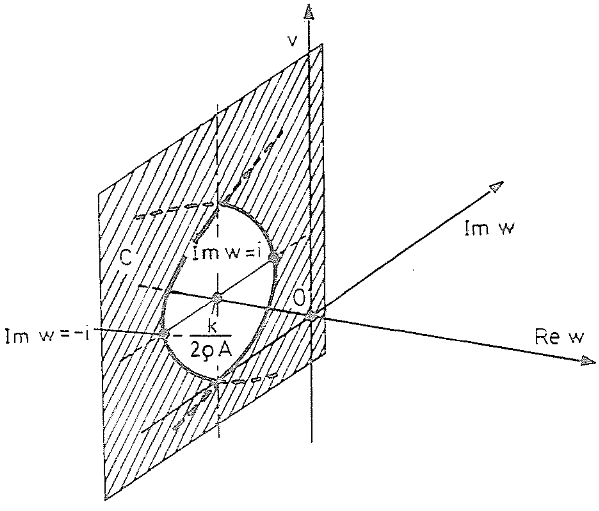

Fig. 3.

Theorem 2.1. If $\mathrm{Re}(w) \neq \dfrac{-k}{2\rho A}$, then the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho A v^2 \lambda^2 - v(k + 2\rho A w)\lambda + (s + kw + \rho A w^2)$$

of a moderately damped system has two roots in the left-hand halfplane and two roots in the right-hand halfplane.

(The assertion can be illustrated by *Fig 3,* in which, on the one hand, the 'parameter space' of pairs (v, w) can be seen with the critical plane located at $\mathrm{Re}(w) = -\dfrac{k}{2\rho A}$, perpendicular to axis $\mathrm{Re}(w)$, while the allocation of roots $\lambda_i(v, w)$; $i = 1, 2, 3, 4$, on the other.)

Proof. We utilize the generalized Routh–Hurwitz theorem in the form of GANTMACHER [5].

Let a and b denote the real and the imaginary parts of ν, respectively. Then the characteristic polynomial (10) has the form

$$p(\mu) = \mu^4 + (c\mu - a - bi)^2 + 1\,. \tag{11}$$

Instead of (11) we shall use the polynomial

$$ip(i\mu) = i(\mu^4 - c^2\mu^2 + 2bc\mu + a^2 - b^2 + 1) + 2a(c\mu - b)\,. \tag{12}$$

We compute the resultant

$$D_8 = \begin{vmatrix} 1 & 0 & -c^2 & 2bc & a^2-b^2+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2ac & -2ab & 0 & 0 & 0 \\ 0 & 1 & 0 & -c^2 & 2bc & a^2-b^2+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2ac & -2ab & 0 & 0 \\ 0 & 0 & 1 & 0 & -c^2 & 2bc & a^2-b^2+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2ac & -2ab & 0 \\ 0 & 0 & 0 & 1 & 0 & -c^2 & 2bc & a^2-b^2+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2ac & -2ab \end{vmatrix} =$$

$$= 16a^4\left(b^4 + c^4(a^2+1)\right)$$

of the polynomial (12).

Suppose $D_8 \neq 0$. Then (11) has no imaginary roots.

Now we compute the even order corner minors of determinant D_8:

$$D_0 = 1, \quad D_2 = 0, \quad D_4 = 0, \quad D_6 = (-2ac)^3,$$

so we have $D_{2h} \neq 0$, $D_{2h+2} = \ldots = D_{2h+2p} = 0$, $D_{2h+2p+2} \neq 0$ with $h = 0$, and with $p = \begin{cases} 2 & \text{if } c \neq 0, \\ 3 & \text{if } c = 0. \end{cases}$ Let n denote the number of roots μ of (11) with $\mathrm{Re}(\mu) > 0$.

If $ac > 0$, then $D_6 < 0$ and $D_8 > 0$ follows, and by the generalized Routh–Hurwitz theorem we have

$$n = \frac{1}{2}\left(p + 1 - (-1)^{\frac{p}{2}}\operatorname{sgn}\left(\frac{D_6}{D_0}\right)\right) + 1 = 2.$$

If $ac < 0$, then $D_6 > 0$ and $D_8 > 0$ follows, so we have

$$n = \frac{1}{2}\left(p + 1 - (-1)^{\frac{p}{2}}\operatorname{sgn}\left(\frac{D_6}{D_0}\right)\right) = 2.$$

If $c = 0$, then $n = \dfrac{p+1}{2} = 2$ follows.

Hence for $D_8 \neq 0$ the statement of the theorem is proved.

Condition $b^4 + c^4(a^2+1) = 0$ implies $b = c = 0$ and $\mu^2 = \pm i\sqrt{a^2+1}$. So in this case we also have two roots both in the left-hand and in the right-hand halfplane, just as in any case with $a \neq 0$. □

Theorem 2.2. Let us preserve the notations of the previous proof and suppose $\nu = bi$ is imaginary. Then characteristic polynomial (10) has two roots both in the left-hand and in the right-hand halfplane if and only if the relations

$$cr_0 - \sqrt{r_0^4 + 1} < b < -cr_0 + \sqrt{r_0^4 + 1}$$

are fulfilled, where

$$r_0 = \\ = c^{\frac{1}{3}} \sqrt{\frac{1}{12} \left(c^{\frac{4}{3}} + \sqrt[3]{c^4 + 216 + 12\sqrt{3(c^4 + 108)}} + \sqrt[3]{c^4 + 216 - 12\sqrt{3(c^4 + 108)}} \right)} .$$

(Such a b can exist only if $|c| < \sqrt{2}$ is satisfied.)

In the remaining cases p has at least two imaginary roots.

Proof. If p has an imaginary root $\mu = ri$, then the polynomial

$$q(r) := r^4 - (cr - b)^2 + 1$$

with real coefficients must have at least two real roots.

Polynomial q is a quadratic expression in b, so we have $b = cr \pm \sqrt{r^4 + 1}$, where r is a root of q.

The possible values of b that imply real roots r are in the range of the smooth real functions

$$b_\pm(r) = cr \pm \sqrt{r^4 + 1} .$$

Functions $b_\pm$ have zeroes if and only if $|c| \geq \sqrt{2}$, so in that case we always have real roots.

For the positive branch $b_+(r) = cr + \sqrt{r^4 + 1}$ we have

$$\lim_{r \to \pm\infty} b_+(r) = +\infty ,$$

while for the negative branch

$$\lim_{r \to \pm\infty} b_-(r) = -\infty$$

is satisfied.

Extrema can be found where the derivative

$$b'_\pm(r) = c \pm \frac{2r^3}{\sqrt{r^4 + 1}}$$

is vanishing, what implies a cubic equation in r^2: $\quad 4r^6 - c^2 r^4 - c^2 = 0$ with a nonnegative real solution

$$r_0^2=\frac{c^{\frac{2}{3}}}{12}\left(c^{\frac{4}{3}}+\sqrt[3]{c^4+216+12\sqrt{3(c^4+108)}}+\sqrt[3]{c^4+216-12\sqrt{3(c^4+108)}}\right).$$

Then $\min b_+ = b_+(-r_0)$ and $\max b_- = b_-(r_0)$, where r_0 is the positive square root of r_0^2.

Polynomial q can only have imaginary roots if $c=0$ and $|b|\geq 1$, but then b is in the range of one of the functions $b_\pm$.

If b is out of the range of functions $b_\pm$, then q has two conjugate non-imaginary root pairs, hence two of the roots $\mu = ri$ of p lay in the left-hand halfplane and two of them are in the right-hand side one. □

Remark 2.3. A complex frequency w can kill damping out if $\mathrm{Re}(w) = -\dfrac{k}{2\rho A}$. Then the situation is similar to that of the undamped classical case, see e. g. [1].

3. Critical Speeds

Definition 3.1. We call v a *critical speed* for the complex frequency w if the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho A v^2\lambda^2 - v(k+2\rho A w)\lambda + (s+kw+\rho A w^2)$$

has at least one multiple root. In this case w is called a *critical frequency* for v.

Remark 3.2. The analysis of the solutions for (6) over the critical speed in the classical case is given by BOGACZ, KRZYZINSKI and POPP in [1].

For the sake of simplicity instead of characteristic polynomial (9) we shall use our nondimensional characteristic polynomial

$$p(\mu) = \mu^4 + (c\mu - \nu)^2 + 1$$

of the moderately damped *Eq.* (6).

Theorem 3.3. Any nondimensional speed c appears as a critical speed for some nondimensional critical frequencies ν satisfying the equation

$$16\nu^6 + (c^4+48)\nu^4 + 4(5c^4+12)\nu^2 + (c^4-4)^2 = 0\,. \tag{13}$$

If neither $c = \pm\sqrt{2}$ nor $c = 0$, then there exist six distinct critical frequencies $\pm bi$, $\pm\nu$, $\pm\overline{\nu}$, where bi is imaginary and ν is neither real nor imaginary.

If $c = \pm\sqrt{2}$, then the critical frequencies are 0 and the square roots of $-\frac{1}{8}\left(13 \pm i\sqrt{343}\right)$.

If $c = 0$, then $\pm i$ appear as critical frequencies.

Proof. Characteristic polynomial (10) has multiple roots if and only if its discriminant D is vanishing. Evaluating D we have the result $\sqrt{D} = 16\left(c^8 + (\nu^4 + 20\nu^2 - 8)c^4 + 16(\nu^2+1)^3\right)$. From this we obtain *Eq.* (13), that is a cubic equation for ν^2 with real coefficients. The discriminant of this latter equation turns to be

$$d = \frac{2^{-16}}{27}c^8(c^4 + 108)^3 .$$

If $c = 0$, then $d = 0$ and we have $\nu^2 = -1$.

If c is nonzero, then $d > 0$ and we have one real solution and a conjugate pair for ν^2. We shall prove in 3.5 that this real solution is always nonpositive.

The complex solutions for ν can coincide if and only if $\nu = 0$, that implies $c = \pm\sqrt{2}$. □

The spatial allocation of the critical parameter pairs (c, ν) are visualized in *Fig 4.* As it is obvious from the Figure, we have a punched critical plane C' fitting on the imaginary axis of plane ν and four critical spatial curves γ_i, $i = 1, 2, 3, 4$ intersecting the imaginary axis of plane ν at $\nu = \pm i$.

Remark 3.4. If $c = \sqrt{2}$, then $v = \sqrt[4]{\dfrac{EI\left(4\rho As - k^2\right)}{\rho^3 A^3}}$, what is the generalization to the damped case of the well-known critical speed in the absence of damping, see e. g. [4].

Proposition 3.5. There are no critical speeds for $\nu = a \neq 0$ real.

If $\nu = bi$ imaginary, then there exists always a critical speed c satisfying

$$2c^4 = 8 + 20b^2 - b^4 + |b|(b^2+8)^{\frac{3}{2}} .$$

Proof. If we expand *Eq.* (13) by c, then we have a quadratic equation

$$c^8 + (\nu^4 + 20\nu^2 - 8)c^4 + 16(\nu^2+1)^3 = 0$$

in c^4, what can be solved as

$$2c^4 = 8 - 20\nu^2 - \nu^4 \pm \nu(\nu^2 - 8)^{\frac{3}{2}} . \qquad (14)$$

Formula (14) provides a critical speed if its right-hand side is nonnegative real.

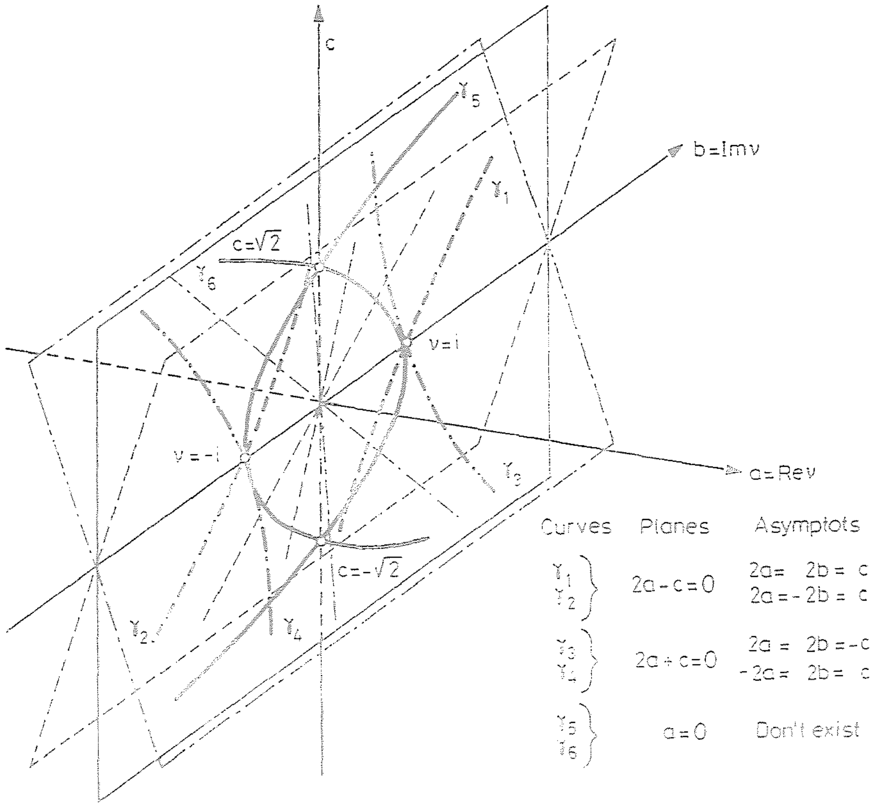

Fig. 4.

The right-hand side of (14) is vanishing if and only if $\nu = \pm i$.

If $\nu = a$, then $f(a) := 8 - 20a^2 - a^4 \pm a(a^2 - 8)^{\frac{3}{2}}$ is a smooth real function for $a^2 \geq 8$. f has no zeros and $f(\pm\sqrt{8}) < 0$, so f is always negative.

If $\nu = bi$, then the positive branch $8 + 20b^2 - b^4 + b(b^2 + 8)^{\frac{3}{2}}$ is positive if $b > -1$, and the negative branch $8 + 20b^2 - b^4 - b(b^2 + 8)^{\frac{3}{2}}$ is positive if $b < 1$. □

Theorem 3.6. A frequency ν has more than one positive critical speed if and only if $\nu = bi$ is imaginary with $0 < |b| \leq 1$. In this case there are exactly two distinct positive critical speeds c satisfying

$$2c^4 = 8 + 20b^2 - b^4 \pm b(b^2 + 8)^{\frac{3}{2}} .$$

Proof. In 3.5 we have observed that in the case $\nu = bi$ both branches of $8 + 20b^2 - b^4 \pm b(b^2 - 8)^{\frac{3}{2}}$ are nonnegative if and only if $|b| \leq 1$. They coincide if and only if $b = 0$.

Suppose now that $\nu = a + bi$ is neither real nor imaginary, and both branches of $8 - 20\nu^2 - \nu^4 \pm \nu(\nu^2 - 8)^{\frac{3}{2}}$ are nonnegative real. This latter condition implies the reality of the square of both terms, that can be achieved only in the case $\nu^2 = -10 \pm i\sqrt{243}$. But in that case the negative branch would have an imaginary value. □

Later on we shall make use of the following statement.

Proposition 3.7. The polynomial

$$p(\mu) = \mu^4 + (c\mu - \nu)^2 + 1$$

has the quadruple root 0 if $c = 0$ and $\nu = \pm i$.

p has a pair of distinct double roots if $c = \pm\sqrt{2}$ and $\nu = 0$.

In all the other cases p can have at most one single double root.

Proof. Suppose p has two double roots μ_1 and μ_2. Then $\mu_2 = -\mu_1$ and $p(\mu) = (\mu^2 - \mu_1^2)^2$. By checking the coefficients either $\nu = 0$ follows, that implies $c = \pm\sqrt{2}$ by 3.5, and so we have $\mu_1 = \pm i$, or $c = \mu_1 = 0$ implies $\nu = \pm i$.

Let us suppose now that p has a triple root μ_1 and another root $\mu_2 = -3\mu_1$. Then $p(\mu) = (\mu - \mu_1)^3(\mu + 3\mu_1) = \mu^4 - 6\mu_1^2\mu^2 + 8\mu_1^3\mu - 3\mu_1^4$ follows, and we have either $c = \mu_1 = 0$, or $c^4 = -108$, where the latter condition contradicts to the reality of c. □

The following proposition illustrates the principle that critical speeds are critical in the sense that they separate regions with given numbers of imaginary roots.

Proposition 3.8. Let $\nu = bi$ be imaginary. Then the positive critical speed c with

$$2c^4 = 8 + 20b^2 - b^4 - |b|(b^2 + 8)^{\frac{3}{2}}$$

is the least positive speed where p has imaginary roots for a fixed $b \in [-1, 1]$.

Proof. Let us suppose that p has a double imaginary root $\mu = ri$. Then $p(\mu)$ and the derivative $p'(\mu)$ must vanish, and we have the equations

$$r^4 - (cr - b)^2 + 1 = 0$$

and

$$2r^3 - c(cr - b) = 0,$$

that lead to the equation

$$4r^6 - c^2r^4 - c^2 = 0.$$

This latter equation has been found in the proof of Thm. 2.2.

Its nonnegative real root r_0 determines the least frequency bi with imaginary roots for a given speed $c \leq \sqrt{2}$ by

$$|b| = -cr_0 + \sqrt{r_0^4 + 1} = -cr_0 + \frac{2r_0^3}{c}.$$

On the other hand we have

$$2c^4 = 8 + 20b^2 - b^4 - |b|\left(b^2 + 8\right)^{\frac{3}{2}}$$

for the least c for a given b by 3.6. □

Corollary 3.9. Formula $|b| = r_0\left(\frac{2r_0^2}{c} - c\right)$ with

$$r_0 = c^{\frac{1}{3}}\sqrt{\frac{1}{12}\left(c^{\frac{4}{3}} + \sqrt[3]{c^4 + 216 + 12\sqrt{3(c^4+108)}} + \sqrt[3]{c^4 + 216 - 12\sqrt{3(c^4+108)}}\right)}$$

is inverse to formula

$$2c^4 = 8 + 20b^2 - b^4 - |b|\left(b^2 + b\right)^{\frac{3}{2}}.$$

4. Solution of the Boundary Value Problem

In this chapter we construct a solution for the boundary value problem

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = e^{wt}\delta(x - vt),$$

$$\lim_{x \to \pm\infty} z(x,t) = 0,$$

where w is a complex number.

If we are looking for a solution of the form

$$z(x,t) = B(\xi)e^{wt},$$

where $\xi = x - vt$, then we obtain the inhomogeneous linear ODE

$$EIB^{\mathrm{IV}} + \rho Av^2B'' - v(k + 2\rho Aw)B' + (s + kw + \rho Aw^2)B = \delta \qquad (15)$$

with Dirac's δ-distribution as a right-hand side.

A particular solution B_p of (15), called the fundamental solution of the ordinary linear differential operator on the left-hand side, can be constructed as

$$B_p(\xi) = H(\xi)B_f(\xi),$$

where H is Heaviside's unit jump function and B_f is the solution of the homogeneous equation

$$EIB^{\mathrm{IV}} + \rho A v^2 B'' - v(k + 2\rho A w)B' + (s + kw + \rho A w^2)B = 0$$

under the initial conditions

$$B_f(0) = B_f'(0) = B_f''(0) \quad \text{and} \quad B_f'''(0) = \frac{1}{EI}, \tag{16}$$

see e. g. [8].

Let us first suppose that our characteristic polynomial (9) has neither multiple nor imaginary roots. In this case we have two distinct characteristic roots λ_1 and λ_2 with negative real parts and two distinct characteristic roots λ_3 and λ_4 with positive real parts by Thms. 2.1–2.

A solution of the homogeneous differential equation (8) has the form

$$B(\xi) = \sum_{i=1}^{4} a_i e^{\lambda_i \xi}.$$

Initial conditions (16) result in the system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{EI} \end{bmatrix}$$

with solution

$$a_i = \frac{1}{EI} \prod_{\substack{j=1 \\ j \neq i}}^{4} \frac{1}{\lambda_i - \lambda_j} = \frac{1}{P'(\lambda_i)}, \tag{17}$$

where P is the derivative of (9):

$$P'(\lambda) = 4EI\lambda^3 + 2\rho A v^2 \lambda - v(k + 2\rho A w). \tag{18}$$

The general solution of the inhomogeneous *Eq.* (15) can be given as

$$B(\xi) = H(\xi)B_f(\xi) + B_h(\xi) = \sum_{i=1}^{4} (a_i H(\xi) + b_i) e^{\lambda_i \xi},$$

where $B_h(\xi) = \sum_{i=1}^{4} b_i e^{\lambda_i \xi}$ is the general solution of the homogeneous equation and $H(\xi)B_f(\xi) = \sum_{i=1}^{4} a_i H(\xi) e^{\lambda_i \xi}$ is the particular solution constructed above.

Our solution $B(\xi)$ must satisfy the boundary condition

$$\lim_{\xi \to \pm\infty} B(\xi) = 0\,,$$

that implies

$$b_i = \begin{cases} 0 & \text{if} \quad \mathrm{Re}(\lambda_i) < 0\,, \\ -a_i & \text{if} \quad \mathrm{Re}(\lambda_i) > 0\,. \end{cases}$$

Hence the solution of the original boundary value problem has the form

$$z(x,t) = \begin{cases} \left(a_1 e^{\lambda_1 \xi} + a_2 e^{\lambda_2 \xi}\right) e^{wt}, & \text{if} \quad \xi \geq 0\,, \\ -\left(a_3 e^{\lambda_3 \xi} + a_4 e^{\lambda_4 \xi}\right) e^{wt}, & \text{if} \quad \xi < 0\,, \end{cases}$$

where a_i can be computed from the characteristic roots by (17).

Theorem 4.1. If the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho A v^2 \lambda^2 - v(k + 2\rho A w)\lambda + (s + kw + \rho A w^2)$$

has no imaginary roots (necessary and sufficient conditions are given in Thms. 2.1–2), then the solution of the boundary problem

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A \frac{\partial^2 z}{\partial t^2} + k\frac{\partial z}{\partial t} + sz = e^{wt}\delta(x - vt)\,,$$

$$\lim_{x \to \pm\infty} z(x,t) = 0$$

is given as

$$z(x,t) = e^{wt}\left(H(\xi)\left(\frac{e^{\lambda_1 \xi}}{P'(\lambda_1)} + \frac{e^{\lambda_2 \xi}}{P'(\lambda_2)}\right) - H(-\xi)\left(\frac{e^{\lambda_3 \xi}}{P'(\lambda_3)} + \frac{e^{\lambda_4 \xi}}{P'(\lambda_4)}\right)\right), \tag{19}$$

where $\xi = x - vt$, $\lambda_1 \neq \lambda_2$ are the roots of the characteristic polynomial P with negative real parts, $\lambda_3 \neq \lambda_4$ are the characteristic roots with positive real parts, and P' is the derivative of P.

If $\lambda_1 = \lambda_2$, then instead of the first term in (19) we have

$$\frac{e^{\lambda_1 \xi + wt} H(\xi)}{EI(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}\left[\xi - \frac{4\lambda_1}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}\right].$$

If $\lambda_3 = \lambda_4$, then instead of the second term in (19) we get

$$\frac{-e^{\lambda_3\xi+wt}H(-\xi)}{EI(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left[\xi - \frac{4\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right].$$

Proof. The case of single roots has been discussed above.

By Proposition 3.7 we cannot have double roots on both sides together. The proof for the multiple root case can be given either in a similar way as it has been done for the single root case or by using L'Hôpital's rule.

Let us compute, for example, the following limit:

$$\lim_{\lambda_2\to\lambda_1}\frac{1}{\lambda_1-\lambda_2}\left[\frac{e^{\lambda_1\xi}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{e^{\lambda_2\xi}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right] =$$

$$= \frac{e^{\lambda_1\xi}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\left[\xi - \frac{2\lambda_1-\lambda_3-\lambda_4}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\right].$$

In characteristic polynomial (9) the cubic term is missing, so we have $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. Hence for $\lambda_1 = \lambda_2$ we get $2\lambda_1 - \lambda_3 - \lambda_4 = 4\lambda_1$.

It can be easily checked that the solution obtained this way satisfies *Eq.* (6) and boundary condition (2). □

5. Determining Complex Frequencies

We introduce the complex function

$$g(w) := \frac{1}{P'_w(\lambda_1(w))} + \frac{1}{P'_w(\lambda_2(w))},$$

where

$$P_w(\lambda) = EI\lambda^4 + \rho Av^2\lambda^2 - v(k + 2\rho Aw)\lambda + (s + kw + \rho Aw^2),$$

$\lambda_1(w)$ and $\lambda_2(w)$ are the root branches of P_w with negative real parts, and $P'_w(\lambda) = 4EI\lambda^3 + 2\rho Av^2\lambda - v(k + 2\rho Aw)$ is the derivative of polynomial P_w.

$g(w)$ is well-defined if $\lambda_1(w) \neq \lambda_2(w)$.

If $\mathrm{Re}(w) \neq \frac{-k}{2\rho A}$ and $\lambda_1(w) = \lambda_2(w)$, then $g(w)$ can be defined as

$$g(w) := -\left(\frac{1}{P'_w(\lambda_3(w))} + \frac{1}{P'_w(\lambda_4(w))}\right)$$

by Proposition 3.7 and Thm. 4.1. Here λ_3 and λ_4 are the roots of P_w with positive real parts.

Hence $g(w)$ is correctly defined for any w with $\mathrm{Re}(w) \neq \dfrac{-k}{2\rho A}$.

Lemma 5.1. Function g preserves conjugation, i. e.

$$g(\overline{w}) = \overline{g}(w)\,.$$

Proof.

$$g(\overline{w}) = \frac{1}{P'_{\overline{w}}(\lambda_1(\overline{w}))} + \frac{1}{P'_{\overline{w}}(\lambda_2(\overline{w}))} =$$

$$\frac{1}{P'_{\overline{w}}\left(\overline{\lambda}_2(w)\right)} + \frac{1}{P'_{\overline{w}}\left(\overline{\lambda}_1(w)\right)} = \frac{1}{\overline{P}'_w(\lambda_2(w))} + \frac{1}{\overline{P}'_w(\lambda_1(w))} = \overline{g}(w)\,. \qquad \square$$

Now we return to the solution of our original system (1–4). We are looking for a solution of the form

$$z(x,t) = B_0(\xi) + \sum_{i=1}^{2} B_i(\xi)e^{w_i t}\,, \qquad Z(t) = \beta_0 + \sum_{i=1}^{2} \beta_i e^{w_i t}\,,$$

where w_1 and w_2 are distinct, at the moment unknown nonvanishing complex numbers, and $\xi = x - vt$. Note, that the mentioned w_i complex numbers will be reckoned with as known quantities, and later on they will be determined by solving an appropriate algebraic equation.

If we substitute these expected solutions into our PDE, then (1) splits into the following three ODEs:

$$EIB_0^{\mathrm{IV}} + \rho A v^2 B_0'' - vkB_0' + sB_0 = T_0\delta\,,$$

$$EIB_i^{\mathrm{IV}} + \rho A v^2 B_i'' - v(k + 2\rho A w_i)B_i' + (s + kw_i + \rho A w_i^2)B_i = -m\beta_i w_i^2 \delta\,,$$

$i = 1,\, 2$, and by Thm. 4.1 we have

$$B_0(0) = T_0 g(0) \qquad \text{and} \qquad B_i(0) = -m\beta_i w_i^2 g(w_i) \qquad \text{for} \qquad i = 1,\, 2\,.$$

If we substitute our solutions into (3), then we obtain equations

$$T_0 = s_H(\beta_0 - B_0(0))$$

and

$$(s_H + k_H w_i)(1 + m w_i^2 g(w_i)) = -m w_i^2\,, \qquad i = 1,\, 2\,.$$

From the first equation we can compute constant β_0:

$$\beta_0 = T_0 \left(\frac{1}{s_H} + g(0) \right) .$$

The second equation above gives the possibility to determine complex frequencies w_i.

Theorem 5.2. A complex number w is a complex frequency of the system (1–4) if and only if

$$\frac{1}{mw^2} + \frac{1}{s_H + k_H w} + g(w) = 0 \tag{20}$$

is satisfied. □

Proposition 5.3. Algebraic equation (20) can have only real solutions or conjugate pairs of solutions.

Proof. Lemma 5.1 shows that g preserves conjugation. Rational function $\frac{1}{mw^2} + \frac{1}{s_H + k_H w}$ clearly preserves conjugation. Hence if we have a solution w, then $\overline{w}$ is also a solution. □

Our original problem is correctly defined if we have exactly two complex frequencies. Numerical experiments support this consideration, so we are interested in the following two cases:

either we have two real frequencies,

or we have a conjugate pair of complex frequencies.

Theorem 5.4. If the algebraic equation

$$\frac{1}{mw^2} + \frac{1}{s_H + k_H w} + g(w) = 0$$

has two solutions w_1 and w_2, where both w_1 and w_2 are real, and damping k is nonvanishing, then the moderately damped system (1–4) always has the solution

$$z(x,t) = \sum_{j=0}^{2} B_j(\xi) e^{w_j t}, \quad Z(t) = \sum_{j=0}^{2} \beta_j e^{w_j t},$$

where $\xi = x - vt$, $w_0 = 0$ and

$$B_j(\xi) = 2\gamma_j \mathrm{Re} \left(H(\xi) \frac{e^{\lambda_{j1}\xi}}{P_j'(\lambda_{j1})} - H(-\xi) \frac{e^{\lambda_{j3}\xi}}{P_j'(\lambda_{j3})} \right)$$

with constants γ_j later to be determined. Here λ_{j1} and λ_{j3} are roots of the characteristic polynomial

$$P_j(\lambda) = EI\lambda^4 + \rho A v^2 \lambda^2 - v(k + 2\rho A w_j)\lambda + (s + kw_j + \rho A w_j^2)$$

with $\mathrm{Re}(\lambda_{j1}) < 0$ and $\mathrm{Re}(\lambda_{j3}) > 0$, and P_j' is the derivative of P_j for $j = 0, 1, 2$.

The constants can be determined as

$$\beta_0 = T_0 \left(\frac{1}{s_H} + 2\mathrm{Re}\left(\frac{1}{P_0'(\lambda_{01})} \right) \right), \quad \beta_1 = \frac{w_2(\beta_0 - Z_0) + V_0}{w_1 - w_2},$$

$$\beta_2 = \frac{w_1(\beta_0 - Z_0) + V_0}{w_2 - w_1},$$

$\gamma_0 = T_0$ and $\gamma_1 = -m\beta_i w_i^2$ for $i = 1, 2$.

Proof. Real frequencies $w_i \neq \dfrac{-k}{2\rho A}$ cannot have a critical speed by Proposition 5.3. If $k > 0$ is satisfied, then 0 also cannot have a critical speed, so we have

$$B_j(\xi) = \left[H(\xi) \left(\frac{e^{\lambda_{j1}\xi}}{P_j'(\lambda_{j1})} + \frac{e^{\lambda_{j2}\xi}}{P_j'(\lambda_{j2})} \right) - H(-\xi) \left(\frac{e^{\lambda_{j3}\xi}}{P_j'(\lambda_{j3})} + \frac{e^{\lambda_{j4}\xi}}{P_j'(\lambda_{j4})} \right) \right] \gamma_j$$

by Thm. 4.1.

If w_j is real, then $\lambda_{j2} = \overline{\lambda}_{j1}$ and $\lambda_{j4} = \overline{\lambda}_{j3}$, and we obtain, e. g.

$$\frac{e^{\lambda_{j1}\xi}}{P_j'(\lambda_{j1})} + \frac{e^{\lambda_{j2}\xi}}{P_j'(\lambda_{j2})} = 2\mathrm{Re}\left(\frac{e^{\lambda_{j1}\xi}}{P_j'(\lambda_{j1})} \right).$$

Constants β_1 and β_2 can be determined by initial conditions (4). □

Theorem 5.5. If *Eq.* (20) has two nonreal solutions w and $\overline{w}$, and damping k is nonvanishing, then the moderately damped system (1–4) has the solution

$$z(x,t) = B_0(\xi) + \mathrm{Re}(B(\xi)e^{wt}), \qquad Z(t) = \beta_0 + \mathrm{Re}(\beta e^{wt})$$

with $\xi = x - vt$.

Here β_0 and B_0 are the same as in Thm. 5.4, while

$$\beta = \frac{\overline{w}(\beta_0 - Z_0) + V_0}{i\mathrm{Im}(w)}$$

and

$$B(\xi)=-m\beta w^2\left[H(\xi)\left(\frac{e^{\lambda_1\xi}}{P'(\lambda_1)}+\frac{e^{\lambda_2\xi}}{P'(\lambda_2)}\right)-H(-\xi)\left(\frac{e^{\lambda_3\xi}}{P'(\lambda_3)}+\frac{e^{\lambda_4\xi}}{P'(\lambda_4)}\right)\right],$$

where λ_1 and λ_2 are the roots of the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho Av^2\lambda^2 - v(k + 2\rho Aw)\lambda + (s + kw + \rho Aw^2)$$

with negative real parts, while λ_3 and λ_4 are the characteristic roots with positive real parts. If P has a multiple root, then formulae of Thm. 4.1 can be applied in computation of $B(\xi)$.

Proof. If $w_2 = \overline{w}_1$, then $\beta_2 = \overline{\beta}_1$ and $B_2(\xi) = \overline{B}_1(\xi)$, so we have $\beta_1 e^{w_1 t} + \beta_2 e^{w_2 t} = 2\text{Re}\left(\beta_1 e^{w_1 t}\right)$ and $B_1(\xi)e^{w_1 t} + B_2(\xi)e^{w_2 t} = 2\text{Re}\left(B_1(\xi)e^{w_1 t}\right)$. On the other hand $\beta_1 = \dfrac{\overline{w}_1(\beta_0 - Z_0)}{w_1 - \overline{w}_1} = \dfrac{\overline{w}_1(\beta_0 - Z_0)}{2iIm(w_1)} = \dfrac{\beta}{2}$ follows. □

Special Case 5.6. If we are looking for a real frequency w in the case $v = 0$, then *Eq.* (20) has the explicit form

$$\frac{1}{mw^2} + \frac{1}{s_H + k_H w} + \frac{1}{\sqrt[4]{64EI(s + kw + \rho Aw^2)^3}} = 0.$$

Limit Case 5.7. In the case $v \to \infty$ we have $\lim_{v\to\infty} g(w) = 0$, hence in this case the complex limit frequencies are

$$w_{1,2} = \frac{-k_H \pm i\sqrt{4ms_H - k_H^2}}{2m}.$$

6. Conclusions

In our paper a new mathematical treatment has been elaborated for the solution of a set of equations describing the joint problem of the combined motions of the continuous beam and the discrete wheel moving on the latter at a constant longitudinal speed. The two subsystems connected with each other by the Hertzian spring and damper are completely characterized through the closed-form expressions based on the complex frequencies obtained from the solution of the auxiliary algebraic equation.

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