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ELASTIC WAVES IN ROTATING RAILWAY WHEEL[1]

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Abstract

The study is devoted to the analysis of the influence of angular velocity of the railway wheel on the velocity of the elastic wave propagation. The wheel is modelled as an elastic curved Thimoshenko beam lying on continuous, inertial elastic foundation, which creates a wheel plate. Radial and circumferential vibrations are also taken into account (two-dimensional model). The mathematical description of the model consists of the system of two partial, conjugated, differential equations. The solution of the system of equations is determined by the Fourier method. The solutions in wave form are obtained taking advantage of the dependencies resulting from the characteristic determinant. Each monochromatic wave has four angular phase velocities. Two of them have a sense consistent with a vector of angular velocity of a rotating wheel, while the two other ones have opposite sense.

Keywords: wave propagation, Timoshenko beam, railway wheel.

1. Introduction

The increase of rail vehicles speed in passenger traffic as well as carrying capacity of freight traffic, observed now, is connected with the increase in dynamic load of the wheel–rail system. A tendency is observed to use the optimum parameters of this system. Resonance states and critical ones are connected with great probability of overloadings. A phenomenon of this kind is, among others, the self-excited escalation of disturbances, which is connected with the problem of instability of rolling motion of the wheelsets. Such phenomena should be eliminated in railway traffic.

The propagation of the elastic waves in railway rails, modelled as a beam on elastic foundation, is a relatively well-known problem and broadly discussed in papers [1, 2, 3]. However, the typical analysis of the dynamic properties of railway wheels resolves itself into the analysis of its free vibrations or forced ones. This consists of searching functions and eigenvalues of boundary problem, or determining amplitude-frequency characteristics, which then are used to examine such phenomena like noise emission by

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railway wheels, stability of rolling motion, variability of contact force and so on. An exception is paper [4], in which an attempt was made to explain the phenomenon of wheel corrugation on the basis of ultrasonic surface waves. The papers dealing with elastic waves, the propagation of which is connected with the shifts of the whole cross-section of wheel tyre, are not known to the author.

This paper is devoted to the analysis of the effect of rotational speed of the railway wheel as taken on the phase velocities of harmonic waves. The considerations were restricted only to transversal (bending) waves which are propagating in the plane of wheel and longitudinal (circumferential) waves. The excitation of waves was assumed as the initial condition resulting from the wheel deformation by radial force. To solve the equation of a wheel motion in the form of travelling waves, the Fourier method was used. This method is usually applied to continuous systems analysis with solutions in the form of standing waves. In the paper, using the relationships resulting from the characteristic determinant, the solutions in the form of travelling waves are estimated.

2. Physical and Mathematical Model of Wheel

The railway wheel tyre is modelled as an elastic curved Timoshenko beam connected with the axle by inertial continuous elastic foundation of Winkler type. The elastic foundation forms the wheel disc. The use of the theory of curved beams in the model construction causes that the cross-section of tyre can preserve its real shape. The detailed description of the two-dimensional model of a wheel and derivation of motion equations are presented in [5, 6]. In *Fig. 1,* only the systems of coordinates used for the mathematical description of the model are presented:

- polar system of coordinates φ, R, the pole being in the wheel centre, stiffly connected with rotating wheel. In this system, the geometrical axis of wheel tyre is described,
- polar system of coordinates φ_1, R, the pole lying in the wheel centre, used to the description of the wheel rotational motion,
- rectangular system ξ, η, the beginning of which O_1 lying on the geometrical axis of tyre and being determined by spatial gap φ or φ_1; axes ξ, η, are locally tangent and normal to tyre axis. This system is used to describe displacements, internal forces and cross-section of tyre.

The geometrical axis of a tyre is meant to be the geometric locus of centres of gravity of cross-sections of not deformed wheel-tyres. If the angular velocity of wheel $\dot{\varphi}_0 = \text{const}$, thus between the coordinates in the polar system φ, R and φ_1, R, the following relationship is met:

$$\varphi_1 = \varphi + \dot{\varphi}_0 t\,. \tag{1}$$

The system of homogeneous conjugated differential equations describing, in the coordinate system φ, R, free vibrations of the tyre of railway wheel rotating with an angular velocity $\dot{\varphi}_0$, has the following shape:

$$\begin{aligned}
&-m_1\frac{\partial^3 v}{\partial\varphi\partial t^2} + m_2\frac{\partial^2 u}{\partial t^2} - \frac{EA}{R}\frac{\partial^2 u}{\partial\varphi^2} + \frac{EAR-k_u h r_t^2}{R^2}\frac{\partial v}{\partial\varphi} + \frac{k_u r_t^2}{R^2}u- \\
&\quad - 2m_3\dot{\varphi}_0\frac{\partial v}{\partial t} = 0\,, \\
&m_4\frac{\partial^4 v}{\partial\varphi^2\partial t^2} - \frac{EJ}{R^3}\frac{\partial^4 v}{\partial\varphi^4} - m_5\frac{\partial^3 u}{\partial\varphi\partial t^2} - m_6\frac{\partial^2 v}{\partial t^2} + \frac{k_u h^2 r_t R - 2EJ}{R^3}\frac{\partial^2 v}{\partial\varphi^2}+ \\
&\quad + \frac{EAR - k_u h r_t^2}{R^2}\frac{\partial u}{\partial\varphi} - \frac{EJ + EAR^2 + k_v r_t R^3}{R^3}v + 2m_7\dot{\varphi}_0\frac{\partial^2 v}{\partial\varphi\partial t}- \\
&\quad - 2m_8\dot{\varphi}_0\frac{\partial u}{\partial t} = 0\,,
\end{aligned} \tag{2}$$

where:

u, v	—	are the displacements of point O_1 along axis ξ, η,
m_i	—	are the reduced masses of wheel tyre and disc,
k_u, k_v,	—	are the coefficients of the elastic foundation stiffness in the circumferential and radial directions, respectively,
E	—	is the modulus of elasticity of the tyre material,
r_t, R	—	are the radii of wheel disc and geometrical axis of tyre,
A	—	is the area of the tyre cross-section.

The system of *Eqs.* (2) constitutes the mathematical two-dimensional model of the railway wheel rotating with angular velocity $\dot{\varphi}_0$. The first of *Eqs.* (2) describes the motion of a wheel tyre in the circumferential direction, whereas the second one — in the radial direction. If the displacements of point O_1 are $u(\varphi, t)$ and $v(\varphi, t)$, thus the displacements of an arbitrary point B, lying on the transversal cross-section of the wheel tyre may be calculated from relationship:

$$\begin{aligned}
u_B &= u - \frac{\eta}{R}\left(u + \frac{\partial v}{\partial\varphi}\right)\,, \\
v_B &= v\,.
\end{aligned} \tag{3}$$

Eqs. (2) describe the vibrations of wheel tyre in its plane. In fact, these vibrations are conjugated by inertia forces with the vibrations from the wheel plane. In the present paper, this conjugation is neglected, as an assumption was made that vibrations in the plane and out of the plane of the wheel are mutually independent. In reality, this conjugation is small.

3. Solution to the Boundary Problem

As it was formerly mentioned the solution of system of *Eqs.* (2) is searched by using the Fourier method in the area of complex variable in the form:

$$\begin{aligned} u(\varphi,t) &= U(\varphi)e^{rt}\,, \\ v(\varphi,t) &= V(\varphi)e^{rt}\,. \end{aligned} \tag{4}$$

When substituting (4) into (2), the following is obtained:

$$\begin{aligned} &EARU'' - \left(r^2R^2m_2 + k_ur_t^3\right)U + \left(r^2R^2m_1 + k_uhr_t^2 - EAR\right)V' + \\ &\quad + 2rR^2\dot{\varphi}_0m_3V = 0\,, \\ &EJV^{\mathrm{IV}} - \left(r^2R^3m_4 - 2EJ + k_uh^2r_tR\right)V'' + \\ &\quad + \left(EJ + EAR^2 + k_vr_tR^3 + r^2R^3m_6\right)V - 2rR^3\dot{\varphi}_0m_7V' + \\ &\quad + R\left(r^2R^2m_5 + k_uhr_t^2 - EAR\right)U' + 2rR^2\dot{\varphi}_0m_BU = 0\,. \end{aligned} \tag{5}$$

Solution of equation system (5), which meets the boundary condition, being in the case of wheel the continuity conditions, will be the solution to the boundary problem of the system. The solution of *Eqs.* (5) is searched in the form of the following trigonometric series:

$$\begin{aligned} U(\varphi) &= \frac{A_0}{2\pi} + \sum_{n=1}^{\infty}\left(A_n\cos n\varphi + B_n\sin n\varphi\right)\,, \\ V(\varphi) &= \frac{C_0}{2\pi} + \sum_{n=1}^{\infty}\left(C_n\cos n\varphi + D_n\sin n\varphi\right)\,, \end{aligned} \tag{6}$$

where: A_n, B_n, C_n, D_n — are the Fourier's coefficients, whereas each term of series (6) is required to fulfil simultaneously *Eqs.* (5) and the boundary conditions. Thus each term of series (6) will be the eigenfunction of the boundary problem.

The integration of *Eqs.* (5) within the limits $0 \div 2\pi$ leads to the system of algebraic equations:

$$\begin{aligned}&\left(r_0^2R^2m_2+k_ur_t^3\right)A_0-2r_0\dot{\varphi}_0R^2m_3C_0=0\,,\\&2r_0\dot{\varphi}_0R^3m_8A_0+\left(EJ+EAR^2+k_vr_tR^3+r_0^2R^3m_6\right)C_0=0\,,\end{aligned}\tag{7}$$

where: $r = r_0$ — is the eigenvalue of 'zero' mode of vibrations; it is the mode the mode characterized by null number of nodal diameters.

Condition for the existence of a non-trivial solution of *Eqs.* (7) is that the characteristic determinant should be equal to zero. From this condition, a biquadratic characteristic equation is obtained, the roots of which are imaginary values:

$$\begin{aligned}r_{01,2}&=\pm j\omega_{01}\,,\\r_{03,4}&=\pm j\omega_{02}\,,\end{aligned}\tag{8}$$

where: ω_{0k} — are frequencies of 'zero' mode of wheel vibrations.

It results from (7) that for angular velocity of wheel, $\dot{\varphi}_0=0$ modes of vibration have the following shape:

$$\begin{aligned}&U_{01}(\varphi)=A_{01}\quad\text{and}\quad V_{01}(\varphi)=0\quad\text{for}\quad\omega_{01}\,,\\&U_{02}(\varphi)=0\quad\text{and}\quad V_{02}(\varphi)=C_{02}\quad\text{for}\quad\omega_{02}\,.\end{aligned}\tag{9}$$

For $\dot{\varphi}_0\neq 0$, the following relationship is fulfilled:

$$C_{0i}=\overline{H}_{0i}A_{0i}\,,\qquad i=1,\,2,\,3,\,4\tag{10}$$

and taking into account (8):

$$\begin{aligned}\overline{H}_{01}&=-\overline{H}_{02}=jH_{01}\,,\\\overline{H}_{03}&=-\overline{H}_{04}=jH_{02}\,,\end{aligned}\tag{11}$$

where:

$$H_{0k}=\frac{R^2m_2\omega_{0k}^2-k_ur_t^3}{2Rm_3\dot{\varphi}_0\omega_{0k}}\qquad\text{for}\qquad k=1,\,2\,,\tag{12}$$

which corresponds to the following modes of vibration:

$$\begin{aligned}&U_{01}(\varphi)=A_{01}\quad\text{and}\quad V_{01}(\varphi)=H_{01}A_{01}\quad\text{for}\quad\omega_{01}\,,\\&U_{02}(\varphi)=A_{02}\quad\text{and}\quad V_{02}(\varphi)=H_{02}A_{02}\quad\text{for}\quad\omega_{02}\,.\end{aligned}\tag{13}$$

It follows from the previous considerations hitherto, that for $\dot{\varphi}_0 = 0$ and $\dot{\varphi}_0 = \mathrm{const}$, the 'zero' mode of vibrations is characterized by two frequencies. There is, however, a principal difference between these two cases. For $\dot{\varphi}_0 = 0$, the circumferential vibrations occur independently from radial vibrations, whereas for $\dot{\varphi}_0 = \mathrm{const}$, the conjugation of motion in circumferential direction with the motion in radial direction exists. The source of conjugation is inertia forces resulting from Coriolis acceleration.

The remaining eigenvalues and eigenfunctions are determined as a result of finite sin- and cosin-transformation of system (5), which results in a system of four algebraic equations:

$$
\begin{aligned}
&\left(r_n^2 + \omega_{pn}^2\right) B_n - \left(a_{1n} r_n^2 + a_{2n}\right) C_n - 2 r_n \dot{\varphi}_0 \frac{m_3}{m_2} D_n = 0\,, \\
&\left(r_n^2 + \omega_{pn}^2\right) A_n - 2 r_n \dot{\varphi}_0 \frac{m_3}{m_2} C_n + \left(a_{1n} r_n^2 + a_{2n}\right) D_n = 0\,, \\
&\left(a_{3n} r_n^2 + a_{4n}\right) A_n + 2 r_n \dot{\varphi}_0 \frac{m_8}{n^2 m_4 + m_6} B_n + 2 n r_n \dot{\varphi}_0 \frac{m_7}{n^2 m_4 + m_6} C_n + \\
&\qquad + \left(r_n^2 + \omega_{gn}^2\right) D_n = 0\,, \\
&2 r_n \dot{\varphi}_0 \frac{m_8}{n^2 m_4 + m_6} A_n - \left(a_{3n} r_n^2 + a_{4n}\right) B_n + \left(r_n^2 + \omega_{gn}^2\right) C_n - \\
&\qquad - 2 n r_n \dot{\varphi}_0 \frac{m_7}{n^2 m_4 + m_6} D_n = 0\,,
\end{aligned}
\tag{14}
$$

where:

$r = r_n$ — eigenvalue of the n^{th} mode of vibrations,

$\omega_{pn}^2 = \frac{n^2 EAR + k_u r_t^3}{R^2 m_2}$ — the square of the n^{th} frequency that is not conjugated with the defection of longitudinal vibrations of the tyre,

$\omega_{gn}^2 = \frac{(n^2-1)EJ + EAR + r_t R(n^2 k_u h^2 + k_v R^2)}{R^3(n^2 m_4 + m_6)}$ — the square of the n^{th} frequency that is not conjugated with the tangential displacement of transverse vibrations of the tyre,

$$
a_{1n} = -\frac{n m_1}{m_2}\,, \qquad a_{2n} = \frac{n\left(EAR - k_u h r_t^2\right)}{R^2 m_2}\,,
$$

$$
a_{3n} = -\frac{n m_5}{n^2 m_5 + m_6}\,, \qquad a_{4n} = \frac{n\left(EAR - k_u h r_t^2\right)}{R^2\left(n^2 m_4 + m_6\right)}\,.
$$

In the area of the stable rotating motion of a wheel, the roots of characteristic equation of system (14) are imaginary values:

$$\begin{aligned} r_{n1,2} &= \pm j\omega_{n1}\,, \qquad & r_{n3,4} &= \pm j\omega_{n2}\,, \\ r_{n5,6} &= \pm j\omega_{n3}\,, \qquad & r_{n7,8} &= \pm j\omega_{n4}\,, \end{aligned} \tag{15}$$

where: ω_{nk} — frequency of n^{th} mode of free vibrations.

If $\dot{\varphi}_0 = 0$, thus $\omega_{n1} = \omega_{n2}$ and $\omega_{n3} = \omega_{n4}$. It follows from those above that the rotating motion of a wheel causes a reduplication of frequency of free vibrations for each mode of vibrations, except the 'zero' mode. This is the so-called bifurcation of vibration frequency. It should be also mentioned that the domain of unstable rotating motion of wheel is situated within the range of very high velocities [7], not in testing from practical point of view.

The conjugation of circumferential vibrations with radial ones causes, that the constant multipliers of the given mode of vibrations are mutually dependent. These relationships are obtained from (14)

$$\begin{aligned} B_{ni} &= \overline{H1}_{ni} A_{ni}\,, \\ C_{ni} &= \overline{H}_{ni} A_{ni}\,, \\ D_{ni} &= \overline{H1}_{ni} A_{ni}\,, \qquad i = 1,\, 2,\, 3,\, \ldots,\, 8. \end{aligned} \tag{16}$$

By taking into account (15), it is obtained:

$$\begin{aligned} \overline{H1}_{n,2i-1} &= -\overline{H1}_{n,2i} = j h_{ni}\,, \\ \overline{H}_{n,2i-1} &= -\overline{H}_{n,2i} = j H_{ni}\,, \qquad i = 1,\, 2,\, 3,\, 4\,, \end{aligned} \tag{17}$$

where:

$$h_{ni} = 2\dot{\varphi}_0 \omega_{ni} \times$$

$$\times \frac{n m_2 m_7 \left(\omega_{pn}^2 - \omega_{ni}^2\right) + m_2 m_8 \left(a_{2n} - a_{1n}\omega_{ni}^2\right) + m_3 \left(n^2 m_4 + m_6\right) \left(a_{4n} - a_{3n}\omega_{ni}^2\right)}{m_2 \left(n^2 m_4 + m_6\right) \left[\left(\omega_{pn}^2 - \omega_{ni}^2\right) \left(\omega_{gn}^2 - \omega_{ni}^2\right) - \left(a_{2n} - a_{1n}\omega_{ni}^2\right) \left(a_{4n} - a_{3n}\omega_{ni}^2\right)\right] - 4 m_3 m_8 \dot{\varphi}_0^2 \omega_{ni}^2}\,,$$

$$H_{ni} = \frac{\omega_{pn}^2 - \omega_{ni}^2}{h_{ni}\left(a_{2n} - a_{1n}\omega_{ni}^2\right) - 2\dot{\varphi}_0 \omega_{ni} \frac{m_3}{m_2}}\,.$$

As the modes of vibrations are determined with an accuracy to the constant multiplier, it is possible to take assumption that for $n = 1,\, 2,\, 3,\, \ldots$, the system of eigenfunctions has the shape:

$$\begin{aligned} U_{ni}(\varphi) &= A_{ni}\left(\cos n\varphi + \overline{H1}_{ni} \sin n\varphi\right)\,, \\ V_{ni}(\varphi) &= \overline{H}_{ni} A_{ni}\left(\cos n\varphi + \overline{H1}_{ni} \sin n\varphi\right)\,, \end{aligned} \tag{18}$$

for $i = 1,\, 2,\, 3,\, \ldots,\, 8$.

4. Solution to Initial Problem in Vibrational Form

The general solution of equation system (2) may be written in the form of infinite sum of linearly independent particular solutions:

$$\begin{aligned} u(\varphi,t) &= \sum_{i=1}^{4} A_{0i}\exp(r_{0i}t) + \sum_{n=1}^{\infty}\sum_{i=1}^{8} A_{ni}\left(\cos n\varphi + \overline{H1}_{ni}\sin n\varphi\right)\exp(r_{ni}t)\,,\\ v(\varphi,t) &= \sum_{i=1}^{4}\overline{H}_{0i}A_{0i}\exp(r_{0i}t)+\\ &\quad+\sum_{n=1}^{\infty}\sum_{i=1}^{8}\overline{H}_{ni}A_{ni}\left(\cos n\varphi+\overline{H1}_{ni}\sin n\varphi\right)\exp(r_{ni}t)\,. \end{aligned} \tag{19}$$

Solution (19) may be written in real form by using (8), (11), (15), (17) as well as when introducing new constants with the help of relationships:

$$\begin{gathered} A_{0,2i-1} = \frac{1}{2}\left(K_{0i} - jL_{0i}\right)\,, \qquad A_{0,2i} = \frac{1}{2}\left(K_{0i} + jL_{0i}\right)\\ \text{for} \qquad i = 1,\, 2\,,\\ A_{n,2i-1} = \frac{1}{2}\left(K_{ni} - jL_{ni}\right)\,, \qquad A_{n,2i} = \frac{1}{2}\left(K_{ni} + jL_{ni}\right)\\ \text{for} \qquad i = 1,\, 2,\, 3,\, 4\,. \end{gathered} \tag{20}$$

Substituting (20) into (19), it is obtained:

$$\begin{aligned} u(\varphi,t) &= \sum_{i=1}^{2}\left(K_{0i}\cos\omega_{0i}t + L_{0i}\sin\omega_{0i}t\right)+\\ &\quad+\sum_{n=1}^{\infty}\sum_{i=1}^{4}\left[\left(K_{ni}\cos\omega_{ni}t + L_{ni}\sin\omega_{ni}t\right)\cos n\varphi+\right.\\ &\quad\left.+h_{ni}\left(L_{ni}\cos\omega_{ni}t - K_{ni}\sin\omega_{ni}t\right)\sin n\varphi\right]\,,\\ v(\varphi,t) &= \sum_{i=1}^{2} H_{0i}\left(K_{0i}\cos\omega_{0i}t - L_{0i}\sin\omega_{0i}t\right)+\\ &\quad+\sum_{n=1}^{\infty}\sum_{i=1}^{4} H_{ni}\left[\left(L_{ni}\cos\omega_{ni}t - K_{ni}\sin\omega_{ni}t\right)\cos n\varphi-\right.\\ &\quad\left.-h_{ni}\left(K_{ni}\cos\omega_{ni}t - L_{ni}\sin\omega_{ni}t\right)\sin n\varphi\right]\,. \end{aligned} \tag{21}$$

The integration constants occurring in relationships (21) are determined from the following initial conditions:

$$
\begin{aligned}
u(\varphi,0) &= u_0\,, &\qquad \frac{\partial u(\varphi,t)}{\partial t}\Big|_{t=0} &= \dot{u}_0\,,\\
v(\varphi,0) &= v_0\,, &\qquad \frac{\partial v(\varphi,t)}{\partial t}\Big|_{t=0} &= \dot{v}_0\,,
\end{aligned}
\tag{22}
$$

When substituting (22) into (21) and using the orthogonality of the system of eigenfunctions, constants K_{0i} and L_{0i} are calculated from relationship:

$$
\begin{aligned}
\sum_{i=1}^{2} K_{0i} &= \frac{1}{2\pi}\int\limits_0^{2\pi} u_0 \mathrm{d}\varphi\,, &\qquad \sum_{i=1}^{2} H_{0i}L_{0i} &= \frac{1}{2\pi}\int\limits_0^{2\pi} v_0 \mathrm{d}\varphi\,,\\
\sum_{i=1}^{2} H_{0i}\omega_{0i}K_{0i} &= -\frac{1}{2\pi}\int\limits_0^{2\pi} \dot{v}_0 \mathrm{d}\varphi\,, &\qquad \sum_{i=1}^{2} \omega_{0i}L_{0i} &= \frac{1}{2\pi}\int\limits_0^{2\pi} \dot{u}_0 \mathrm{d}\varphi\,,
\end{aligned}
\tag{23}
$$

whereas constants K_{ni} and L_{ni} are obtained from system of equations:

$$
\sum_{i=1}^{4} K_{ni} = \frac{1}{\pi}\int\limits_0^{2\pi} u_0 \cos n\varphi \mathrm{d}\varphi\,,
$$

$$
\sum_{i=1}^{4} h_{ni}L_{ni} = \frac{1}{\pi}\int\limits_0^{2\pi} u_0 \sin n\varphi \mathrm{d}\varphi\,,
$$

$$
\sum_{i=1}^{4} h_{ni}H_{ni}K_{ni} = -\frac{1}{\pi}\int\limits_0^{2\pi} v_0 \sin n\varphi \mathrm{d}\varphi\,,
$$

$$
\sum_{i=1}^{4} H_{ni}L_{ni} = \frac{1}{\pi}\int\limits_0^{2\pi} v_0 \cos n\varphi \mathrm{d}\varphi\,,
$$

$$
\sum_{i=1}^{4} \omega_{ni}h_{ni}K_{ni} = -\frac{1}{\pi}\int\limits_0^{2\pi} \dot{u}_0 \sin n\varphi \mathrm{d}\varphi\,,
$$

$$
\sum_{i=1}^{4} \omega_{ni}L_{ni} = \frac{1}{\pi}\int\limits_0^{2\pi} \dot{u}_0 \cos n\varphi \mathrm{d}\varphi\,,
$$

$$\sum_{i=1}^{4} \omega_{ni} H_{ni} K_{ni} = -\frac{1}{\pi} \int_0^{2\pi} \dot{v}_0 \cos n\varphi \mathrm{d}\varphi \,,$$

$$\sum_{i=1}^{4} \omega_{ni} h_{ni} H_{ni} L_{ni} = -\frac{1}{\pi} \int_0^{2\pi} \dot{v}_0 \sin n\varphi \mathrm{d}\varphi \,. \tag{24}$$

By using the relationships:

$$\begin{gathered} AU_{ni} = \sqrt{K_{ni}^2 + L_{ni}^2} \,, \\ \sin \psi_{ni} = \frac{K_{ni}}{AU_{ni}} \,, \qquad \cos \psi_{ni} = \frac{L_{ni}}{AU_{ni}} \,, \\ AV_{ni} = \sqrt{(H_{ni} L_{ni})^2 + (H_{ni} L_{ni})^2} \,, \\ \sin \varphi_{ni} = \frac{H_{ni} K_{ni}}{AV_{ni}} \,, \quad \cos \varphi_{ni} = \frac{H_{ni} L_{ni}}{AV_{ni}} \,, \end{gathered} \tag{25}$$

the free vibrations of wheel tyre rotating with constant velocity and expressed by formulae (21) may be written in the following form:

$$\begin{aligned} u(\varphi, t) &= \sum_{i=1}^{2} AU_{0i} \sin (\omega_{0i} t + \psi_{0i}) + \\ &+ \sum_{n=1}^{\infty} \sum_{i=1}^{4} AU_{ni} \left[\cos n\varphi \sin (\omega_{ni} t + \psi_{ni}) + h_{ni} \sin n\varphi \cos (\omega_{ni} t + \psi_{ni}) \right] \,, \\ v(\varphi, t) &= \sum_{i=1}^{2} AV_{0i} \cos (\omega_{0i} t + \varphi_{0i}) + \\ &+ \sum_{n=1}^{\infty} \sum_{i=1}^{4} AV_{ni} \left[\cos n\varphi \cos (\omega_{ni} t + \varphi_{ni}) - h_{ni} \sin n\varphi \sin (\omega_{ni} t + \varphi_{ni}) \right] \,. \end{aligned} \tag{26}$$

Formulae (26) constitute a vibrational form of solution of equation system (2) and describe free vibrations of railway wheel tyre at $\dot{\varphi}_0 = \text{const}$, which are excited by arbitrary initial conditions (22).

5. Solution in Wave Form

By taking into account the fact that the values of frequency ω_{ni} are determined from the condition that the characteristic determinant of the system of *Eqs.* (14) is equal to zero, it may be proved that the dimensionless coefficient h_{ni} may have only two values: $+1$, or -1. If additionally the values are ordered $\omega_{n1} < \omega_{n2} < \omega_{n3} < \omega_{n4}$, then:

$$\begin{aligned} &h_{n1} = h_{n4} = -1\,, \\ &h_{n2} = h_{n3} = 1 \qquad \text{for} \qquad n = 1,\, 2,\, 3, \ldots\,. \end{aligned} \tag{27}$$

As h_{ni} does not depend on the excitation manner, then relationships (27) are valid for arbitrary initial conditions. When substituting (27) into (26), the solution of *Eq.* (2) is obtained in the form of waves:

$$\begin{aligned} u(\varphi, t) = &\sum_{i=1}^{2} AU_{0i} \sin\left(\omega_{0i} t + \psi_{0i}\right) + \\ &+ \sum_{n=1}^{\infty} \left[\sum_{i=1,4} AU_{ni} \sin\left(\omega_{ni} t - n\varphi + \psi_{ni}\right) + \right. \\ &\left. + \sum_{i=2,3} AU_{ni} \sin\left(\omega_{ni} t + n\varphi + \psi_{ni}\right) \right] , \\ v(\varphi, t) = &\sum_{i=1}^{2} AV_{0i} \cos\left(\omega_{0i} t + \varphi_{0i}\right) + \\ &+ \sum_{n=1}^{\infty} \left[\sum_{i=1,4} AV_{ni} \cos\left(\omega_{ni} t - n\varphi + \varphi_{ni}\right) + \right. \\ &\left. + \sum_{i=2,3} AV_{ni} \cos\left(\omega_{ni} t + n\varphi + \varphi_{ni}\right) \right] , \end{aligned} \tag{28}$$

where:

AU_{ni}, AV_{ni} — amplitudes of harmonic waves: both longitudinal and transversal ones,

ψ_{ni}, φ_{ni} — initial phases.

The first of (28) relationships represents longitudinal (circumferential) waves, whereas the second — transversal (bending) ones. The functions described by formulae (28) are functions of two independent variables: the spatial variable φ and time variable t. For $n > 0$, each of the element series (28) represents an elastic harmonic (monochromatic) wave. The argument of each harmonic wave has characteristic property. For each variables φ

and moments t, which meet the conditions:

$$\begin{aligned} \varphi &= \frac{\omega_{ni}}{n}t + \text{const} \qquad \text{for} \qquad i = 1,\, 4\,, \\ \varphi &= -\frac{\omega_{ni}}{n}t + \text{const} \qquad \text{for} \qquad i = 2,\, 3\,, \end{aligned} \tag{29}$$

the value of displacement will not vary. It follows from this that in the system of coordinates φ, R the image of motion propagates at the angular phase velocity equal to ω_{n1}/n and ω_{n4}/n, with a sense compatible with the vector of angular velocity of wheel and with an opposite sense at the phase velocity ω_{n2}/n and ω_{n3}/n. Thus every monochromatic wave has four phase velocities.

In the system of coordinates φ_1, R (*Fig. 1*), relationships (29) have the form:

$$\begin{aligned} \varphi_1 &= \left(\dot\varphi_0 + \frac{\omega_{ni}}{n}\right)t + \text{const} \qquad \text{for} \qquad i = 1,\, 4\,, \\ \varphi_1 &= \left(\dot\varphi_0 - \frac{\omega_{ni}}{n}\right)t + \text{const} \qquad \text{for} \qquad i = 2,\, 3\,. \end{aligned} \tag{30}$$

Thus in this system for $i=1$, 4 and $i=2$, 3 the values of angular phase velocity are suitably increased and decreased by the value of angular velocity of wheel $\dot\varphi_0$. It follows also from the *Eq.* (30) that for every monochromatic wave, propagating in a sense opposite to rotation of wheel, two such velocities $\dot\varphi_0$ exist, at which for the observer connected with system φ_1, R, a stationary harmonic wave comes into being. The values of these velocities are determined by equation:

$$\dot\varphi_0 - \frac{1}{n}\omega_{ni}\left(\dot\varphi_0\right) = 0 \qquad \text{for} \qquad i = 2,\, 3\,. \tag{31}$$

This would mean that the angular velocity $\dot\varphi_0$ becomes equal to angular phase velocity ω_{ni}/n. The wheel has 'catch up' the image of deformations, propagating in a direction opposite to the sense of velocity. From practical point of view, this phenomenon is meaningless as it occurs at very high velocities of wheel.

6. Results of Calculations

In *Tables 1, 2, 3, 4, 5* and *6* results of calculations concerning the propagation of elastic harmonic waves in the rotating railway wheel are given. The waves are excited by the following initial conditions:

$$u(\varphi,0) = u_0\,, \qquad v(\varphi,0) = v_0\,, \qquad \dot u_0(\varphi,0) = 0\,, \qquad \dot v_0(\varphi,0) = 0\,, \tag{32}$$

displacements u, v corresponding to deformations of wheel by the action of concentrated radial force $P = 1$ N.

The calculations were made for a wheel of nominal diameter 0.95 m rotating at angular velocity 0, 116.9, 292.4 1/s, the tyre thickness being 0.075 m (*Tables 1, 2, 3*) and 0.03 m (*Tables 4, 5, 6*). The angular velocities 0, 116.9, 292.4 1/s correspond to linear velocities 0, 200 and 500 km/h in rolling motion of wheel of nominal diameter. In the first type page, the lengths of waves are given in radians, in the second one in phase velocity of waves. Signs '+' or '−' show the sense of angular phase velocity of harmonic waves in accordance with or opposite, respectively, to the sense of angular velocity of a wheel. The amplitudes of waves are given in dB, the reference level being 10^{-11} m. In *Figs. 3, 4, 5* the propagation of elastic harmonic wave packets is shown, the packet consisting of sum of wave numbers from 1 to 100. Time t after which the actual position of wave packet is shown in the following Figures, was calculated from the condition that at wheel velocity $\dot{\varphi}_0 = 0$, the harmonic wave of maximum phase velocity has run the way 'alfa'. The dotted axes of symmetry show the rolling of wheel after time t.

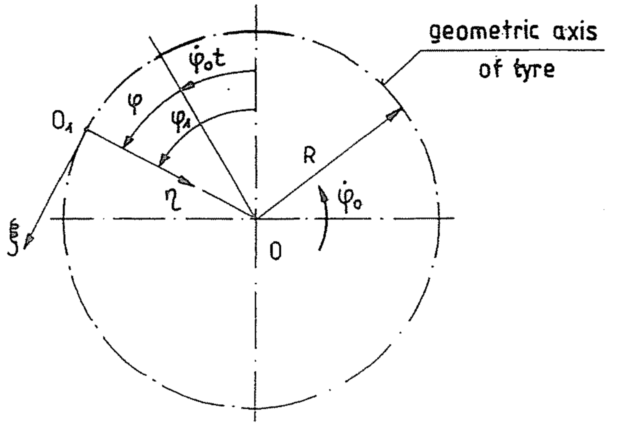

Fig. 1. Coordinate systems

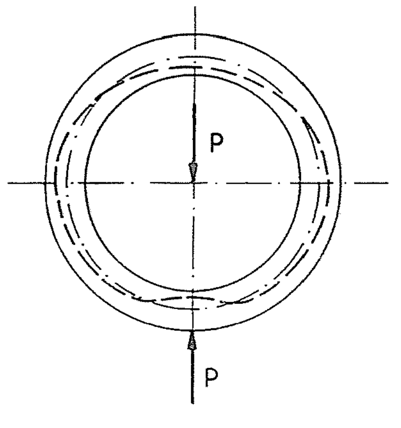

Fig. 2. Deformation of wheel by concentrated radial force

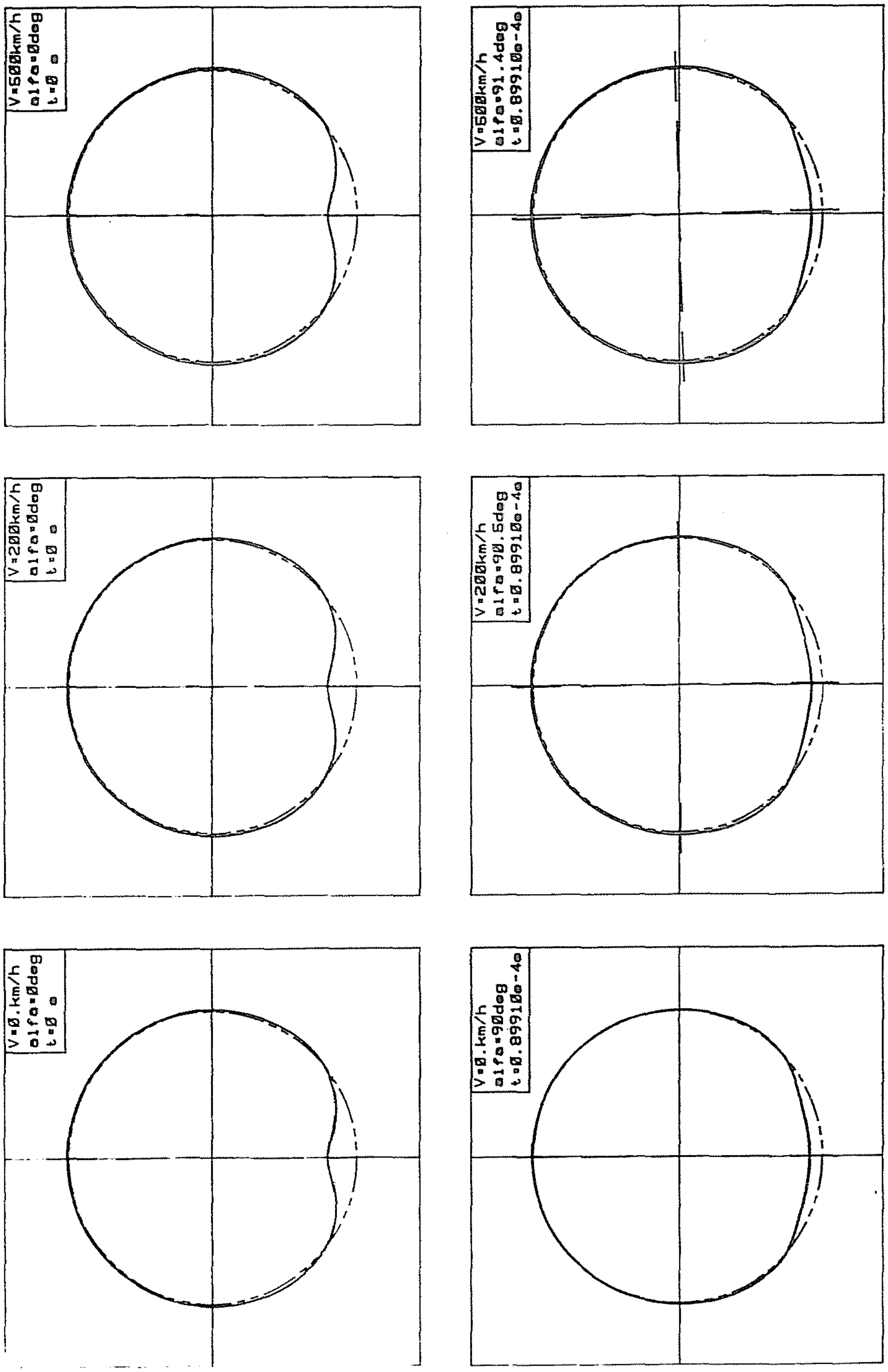

Fig. 3. Propagation of wave packets

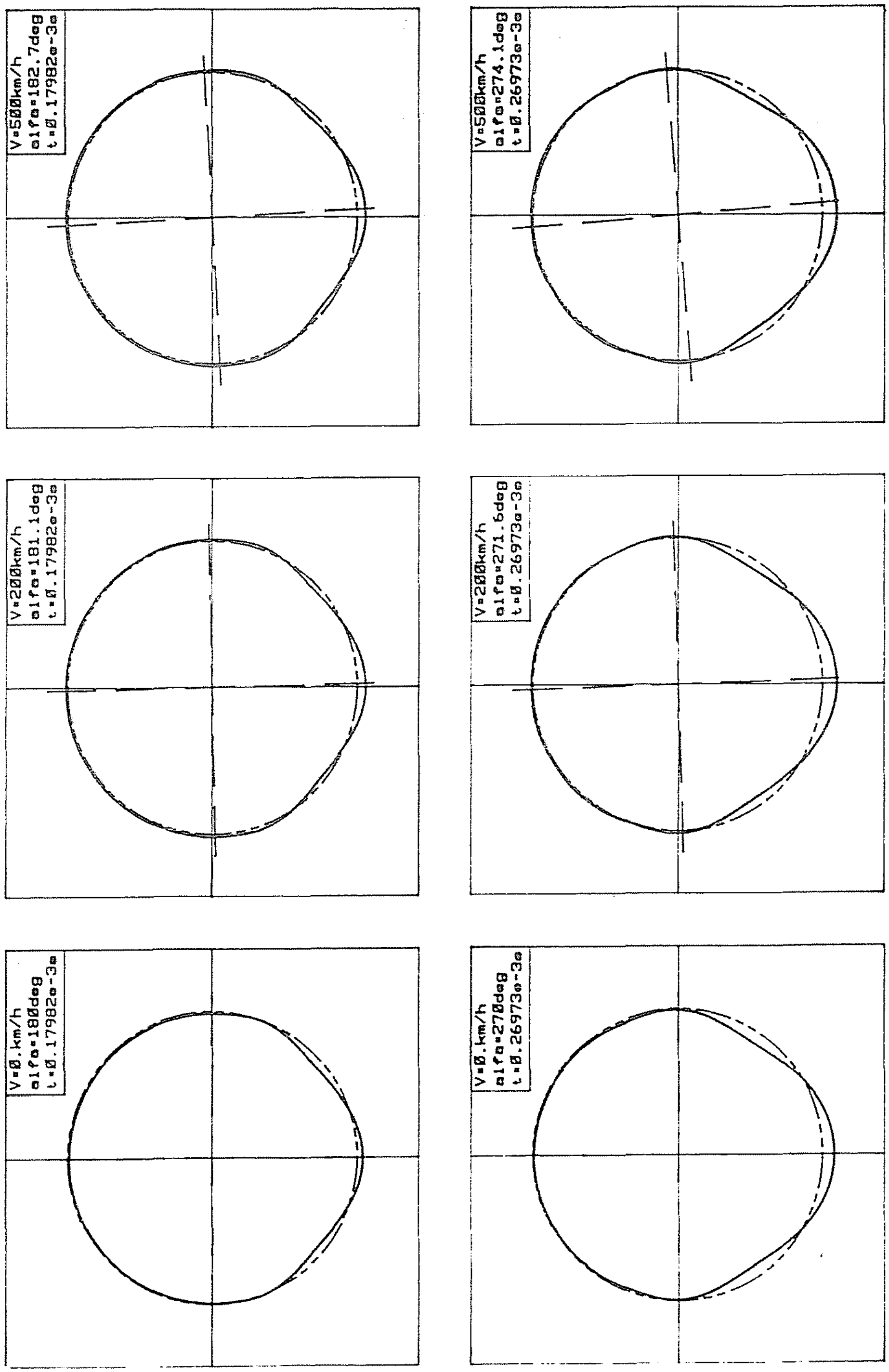

Fig. 4. Propagation of wave packets

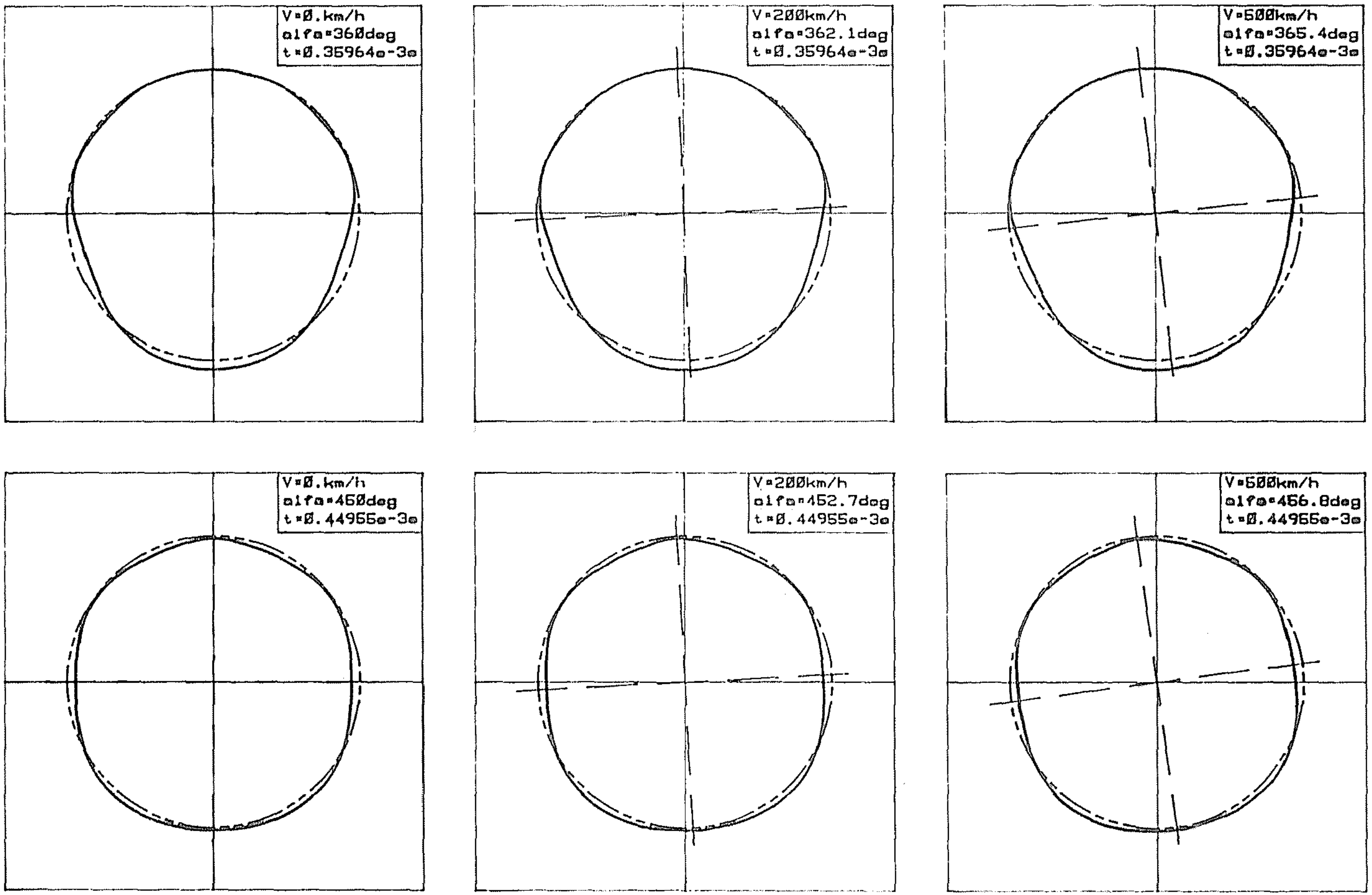

Fig. 5. Propagation of wave packets

Table 1

Wavelength [rd]	Wave phase velocity [1/s]	Wheel angular velocity 0 1/s (0 km/h) — Amplitude and phase of a wave: AU [dB]	ψ	AV [dB]	φ
	+6954	10.49	0	6.81	π
2π	−6954	10.49	π	6.81	π
	−17471	−5.42	0	−2.28	π
	+17471	−5.42	π	−2.28	π
	+4636	4.56	0	9.04	0
π	−4635	4.56	π	9.04	0
	−12729	−12.78	π	−17.06	0
	+12729	−12.78	0	−17.06	0
	+3509	−0.68	π	7.99	π
$\frac{2\pi}{3}$	−3509	−0.68	0	7.99	π
	−11882	−21.43	0	−28.86	π
	+11882	−21.43	π	−28.86	π
	+3152	−6.37	0	4.94	0
$\frac{\pi}{2}$	−3152	−6.37	π	4.94	0
	−11604	−28.28	π	−37.17	0
	+11604	−28.28	0	−37.17	0
	+3194	−12.61	π	0.57	π
$\frac{2\pi}{5}$	−3194	−12.61	0	0.57	π
	−11481	−33.74	0	−43.33	π
	+11481	−33.74	π	−43.33	π
	+3418	−18.78	0	−4.20	0
$\frac{\pi}{3}$	−3418	−18.78	π	−4.20	0
	−11416	−38.24	π	−48.13	0
	+11416	−38.24	0	−48.13	0

Table 2

Wavelength	Wheel angular velocity 116.96 1/s (200 km/h)					
	Wave phase velocity	Amplitude and phase of a wave				
[rd]	[1/s]	AU [dB]	ψ	AV [dB]	φ
2π	+6847	10.64	0	7.01	π
	−7062	10.33	π	6.61	π
	−17368	−5.41	0	−2.16	π
	+17575	−5.42	π	−2.40	π
π	+4582	4.68	π	9.12	0
	−4690	4.44	0	8.95	0
	−12680	−12.79	π	−17.15	0
	+12799	−12.76	0	−16.97	0
$\frac{2\pi}{3}$	+3480	−0.58	0	8.05	π
	−3539	−0.79	π	7.93	π
	−11857	−21.52	0	−29.05	π
	+11908	−21.35	π	−28.68	π
$\frac{\pi}{2}$	+3132	−6.28	π	4.99	0
	−3172	−6.47	0	4.89	0
	−11588	−28.39	π	−37.38	0
	+11620	−28.17	0	−36.98	0
$\frac{2\pi}{5}$	+3179	−12.53	0	0.61	π
	−3209	−12.70	π	0.54	π
	−11469	−33.86	0	−43.52	π
	+11492	−33.62	π	−43.13	π
$\frac{\pi}{3}$	+3406	−18.71	π	−4.17	0
	−3430	−18.86	0	−4.22	0
	−11408	−38.36	π	−48.31	0
	+11425	−38.12	0	−47.95	0

Table 3

Wavelength [rd]	Wave phase velocity [1/s]	Wheel angular velocity 292.39 1/s (500 km/h) Amplitude and phase of a wave			
		AU [dB]	ψ	AV [dB]	φ
2π	+6689	10.86	0	7.29	π
	−7227	10.10	π	6.30	π
	−17216	−5.41	0	−1.97	π
	+17733	−5.44	π	−2.58	π
π	+4503	4.85	π	9.24	0
	−4771	4.25	0	8.83	0
	−12606	−12.82	π	−17.30	0
	+12854	−12.74	0	−16.85	0
$\frac{2\pi}{3}$	+3436	−0.42	0	8.15	π
	−3584	−0.95	π	7.83	π
	−11819	−21.66	0	−29.34	π
	+11947	−21.23	π	−28.42	π
$\frac{\pi}{2}$	+3102	−6.15	π	5.06	0
	−3202	−6.61	0	4.82	0
	+11565	−28.57	π	−37.69	0
	+11645	−28.01	0	−36.69	0
$\frac{2\pi}{5}$	+3156	−12.41	0	0.66	π
	−3232	−12.82	π	0.48	π
	−11453	−34.05	0	−43.83	π
	+11510	−33.45	π	−42.85	π
$\frac{\pi}{3}$	+3387	−18.59	π	−4.13	0
	−3449	−18.98	0	−4.27	0
	−11394	−38.55	π	−48.59	0
	+11439	−37.95	0	−47.69	0

Table 4

Wavelength [rd]	Wave phase velocity [1/s]	Wheel angular velocity 0 1/s (0 km/h) — Amplitude and phase of a wave: AU [dB]	ψ	AV [dB]	φ
2π	+8739	10.56	π	3.69	π
	−8738	10.56	0	3.69	π
	−19102	−2.81	0	3.17	π
	+19102	−2.81	π	3.16	π
π	+6101	5.71	0	8.65	0
	−6101	5.71	π	8.65	0
	−12885	−6.90	π	−9.11	0
	+12885	−6.90	0	−9.11	0
$\frac{2\pi}{3}$	+4397	1.25	π	9.25	π
	−4397	1.25	0	9.25	π
	−11854	−15.22	0	−20.38	π
	+11854	−15.22	π	−20.38	π
$\frac{\pi}{2}$	+3464	−2.30	0	8.76	0
	−3463	−2.30	π	8.76	0
	−11550	−21.90	π	−27.93	0
	+11550	−21.90	0	−27.93	0
$\frac{2\pi}{5}$	+2968	−5.69	π	7.53	π
	−2968	−5.69	0	7.53	π
	−11432	−27.12	0	−33.24	π
	+11432	−27.12	π	−33.24	π
$\frac{\pi}{3}$	+2734	−9.28	0	5.58	0
	−2734	−9.28	π	5.58	0
	−11382	−31.32	π	−37.23	0
	+11382	−31.32	0	−37.23	0

Table 5

Wavelength [rd]	Wave phase velocity [1/s]	Wheel angular velocity 116.96 1/s (181 km/h)			
		Amplitude and phase of a wave			
		AU [dB]	ψ	AV [dB]	φ
2π	+8649	10.71	0	3.94	π
	−8827	10.40	π	3.43	π
	−19019	−2.93	0	3.28	π
	+19186	−2.71	π	3.05	π
π	+6043	5.81	π	8.72	0
	−6160	5.61	0	8.59	0
	−12832	−6.90	π	−9.14	0
	+12938	−6.91	0	−9.09	0
$\frac{2\pi}{3}$	+4362	1.35	0	9.30	π
	−4432	1.14	π	9.19	π
	−11825	−15.29	0	−20.51	π
	+11883	−15.15	π	−20.25	π
$\frac{\pi}{2}$	+3439	−2.20	π	8.82	0
	−3488	−2.40	0	8.71	0
	−11531	−22.00	π	−28.08	0
	+11570	−21.81	0	−27.79	0
$\frac{2\pi}{5}$	+2948	−5.59	0	7.58	π
	−2988	−5.78	π	7.48	π
	−11417	−27.23	0	−33.38	π
	+11447	−27.03	π	−33.11	π
$\frac{\pi}{3}$	+2717	−9.19	π	5.63	0
	−2752	−9.37	0	5.53	0
	−11370	−31.42	π	−37.35	0
	+11395	−31.23	0	−37.11	0

Table 6

Wavelength [rd]	Wave phase velocity [1/s]	Wheel angular velocity 292.39 1/s (453 km/h) — Amplitude and phase of a wave			
		AU [dB]	ψ	AV [dB]	φ
	+8516	10.93	0	4.30	π
2π	−8962	10.15	π	3.02	π
	−18899	−3.11	0	3.46	π
	+19316	−2.58	π	2.87	π
	+5956	5.96	π	8.82	0
π	−6249	5.46	0	8.48	0
	−12753	−6.90	π	−9.18	0
	+13018	−6.91	0	−9.06	0
	+4311	1.50	0	9.39	π
$\frac{2\pi}{3}$	−4484	0.98	π	9.11	π
	−11781	−15.40	0	−20.71	π
	+11928	−15.05	π	−20.08	π
	+3402	−2.05	π	8.90	0
$\frac{\pi}{2}$	−3526	−2.56	0	8.62	0
	−11502	−22.15	π	−28.30	0
	+11600	−21.67	0	−27.59	0
	+2918	−5.45	0	7.66	π
$\frac{2\pi}{5}$	−3019	−5.93	π	7.40	π
	−11395	−27.38	0	−33.59	π
	+11469	−26.88	π	−32.92	π
	+2691	−9.06	π	5.71	0
$\frac{\pi}{3}$	−2778	−9.51	0	5.45	0
	−11352	−31.57	π	−37.54	0
	+11413	−31.09	0	−36.94	0

7. Conclusion

It follows from *Tables 1, 2, 3, 4, 5* and *6* that for the two-dimensional model of railway wheel, every monochromatic wave has four phase velocities, two of them showing a sense consistent with a vector of angular velocity of wheel and the other two with sense opposite to that. This does mean that the propagation of waves occurs in the directions of opposite senses and for $\dot{\varphi}_0 = 0$, a full symmetry of values of phase velocities and amplitudes takes place. The rotating motion of wheel causes violation of propagation symmetry of elastic waves (in system φ, R) because inertia forces occur in conformity with Coriolis acceleration, thus the decrease of minor phase velocity and the increase of the higher one of waves running according to sense $\dot{\varphi}_0$ as well as the decrease of lower velocity of waves and the increase of higher velocity ones, running opposite to $\dot{\varphi}_0$. Waves running according to $\dot{\varphi}_0$ are principally characterized by greater value of amplitudes.

The propagation of wave packets, as shown in *Figs. 3, 4, 5,* shows also the symmetry for $\dot{\varphi}_0 = 0$ and asymmetry for $\dot{\varphi}_0 \neq 0$. The calculations show that the wave packets undergo strong dispersion.

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