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CONTRIBUTIONS TO MATRIX METHODS FOR INVESTIGATING TECHNICAL SYSTEMS

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Abstract

In this paper a short survey of a set of 'Matrix Algorithms and Methods' (MAM) is given created by the author during the last three decades, published bit by bit in ZAMM, ISNM, IKM, Equa-diff, BAM, etc. by cca 100 papers (P), in several research bulletins (Bn) and postgraduate lecture notes (pLn), in some books (Bk) [look at the Reference 1,1 – 4, 9], applied in these works to investigate multilaterally various technical etc. systems, among others vehicle dynamic ones, too, and programmed into computer in certain languages (e.g. in PL_1, Pascal).

Keywords: MAM created for algebraic, analytic and stochastic tasks connected with the investigation of technical systems.

Introduction

The pieces of the above mentioned set MAM have — taken together and often one by one too — numerous mathematical domains to use (giving tasks for these MAM devices). Therefore, it was suitable to choose several important domains of use and to show — through these only — the essence, the structure, the performance, etc. of some MAM's algorithms and methods and to compare them with other (known) ones, for finding occasionally advantages for our MAM. Such four domains will be here:

1. algebras (lin. and non-lin. ones, with programmings),
2. ordinary analysis (mainly lin. and non-lin. diff. equs.),
3. functional analysis (transform of function systems into various bases), and
4. stochastics (random basic products, Markov chains).

For the technical (applied) domains, only titles are here available (at the end of the chapters). The references (at the end of this paper) allow the reader to supplement this survey — with theoretical, numerical, and applied details — into totality. Finally, remember here the late scientist Acad. Prof. E. Egerváry (died in 1958), the great master of applied mathematics and matrix theory; the author as his former co-worker achieved these MAM-results by 'following the master's footsteps' [19].

1. MAM for Algebras

Their main base is our Bn [12].

1.1 Let us transform an arbitrary matrix $\mathbb{A}_0 \widehat{=} {}^1[a_j]^m \subset E_n$ given on an orthonormal basis $\mathbb{B}_0$ (with $\mathbb{B}_0^*\mathbb{B}_0 = \mathbb{E}$), e.g. on $\mathbb{B}_0 = \mathbb{E} \widehat{=} {}^1[e_i]^n \subset E_n$, or its $a_{l(\not{L})} \longrightarrow e_k$ modified variant $\mathbb{A}_0^{[1]} = \mathbb{A}_0 - (a_l - e_k)e^l$ into the $e_{k(\not{L})} \longrightarrow a_l$ counter-changed basis $\mathbb{B}_1 \widehat{=} \mathbb{B}_{10} = \mathbb{E} + (a_l - e_k)e^k$, $|\mathbb{B}_1| = e^k a_l \widehat{=} a_{kl} \neq 0$ (pivot element), thus obtaining our first static/dynamic transform step STA$_1$ // DTA$_1$ [11] (at $\gamma \widehat{=} 1/a_{kl}$, $\mathbb{B}_1^{-1} = \mathbb{E} - \gamma(a_l - e_k)e^k$):

$$\underset{[S]}{\mathbb{A}_1} = \mathbb{B}_1^{-1}\mathbb{A}_0 = \mathbb{A}_0 - \gamma(a_l - e_k)a^k \,, \tag{1.11a}$$

$$\underset{[D]}{\mathbb{A}_1} = \mathbb{B}_1^{-1}\mathbb{A}_0^{[1]} = \mathbb{A}_0 - \gamma(a_l - e_k)(a^k + e^l) \,, \tag{1.11b}$$

or obtaining by p similar sequential // simultaneous steps, our transform spring $\widehat{\mathrm{DTA}}_p$ [11]

$$\Big(\text{at} \quad \mathbb{A}_{KL} = \mathbb{E}^K \mathbb{A}_L \,, \quad |\mathbb{A}_{KL}| = |\mathbb{B}_p| = \prod_{q=0}^{p-1} |\mathbb{B}_{q+1,q}| \widehat{=} \prod_{q=0}^{p-1} a_{k_q l_q}^{[q]} \neq 0 \,,$$

$$\Gamma_{KL} \widehat{=} \mathbb{A}_{KL}^{-1}\Big) \;:$$

$$\underset{[D]}{\mathbf{A}_p} \widehat{=} \mathbf{B}_{p,p-1}^{-1} \mathbf{A}_{p-1}^{[1]} = \mathbf{A}_0 - \sum_{q=0}^{p-1} \gamma_q (a_{l_q}^{[q]} - e_{k_q})(a_{[q]}^{k_q} - e^{l_q}) \,, \tag{1.12a}$$

$$\underset{[D]}{\mathbf{A}_p} \widehat{=} \mathbf{B}_p^{-1} \mathbf{A}_0^{[p]} = \mathbf{A}_0 - (\mathbf{A}_L - \mathbf{E}_K)\Gamma_{KL}(\mathbf{A}^K + \mathbf{E}^I) \,, \tag{1.12b}$$

which yields the blocked form of matrices $\mathbf{A}_0$ and $\underset{[D]}{\mathbf{A}_p}$:

$$\Big(\mathbf{A}_0 \supset \mathbf{A}_L \widehat{=} {}^0[a_{l_q}]^{p-1}, \qquad \mathbf{E}_K \widehat{=} {}^0[e_{k_q}]^{p-1} \subset \mathbf{E}\Big) \,,$$

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_{KL} & \mathbf{A}_{KJ} \\ \mathbf{A}_{IL} & \mathbf{A}_{IJ} \end{bmatrix} \qquad \text{into} \qquad \underset{[D]}{\mathbf{A}_p} = \begin{bmatrix} \Gamma_{KL} & \Delta_{KJ} \\ -\Sigma_{IL} & \Omega_{IJ} \end{bmatrix} \,, \tag{1.13a}$$

$$[\mathbf{A}_{IL}\Gamma_{KL}\hat{=}\Sigma_{IL}, \qquad \Delta_{KJ}\hat{=}\Gamma_{KL}\mathbf{A}_{KJ}, \qquad \mathbf{A}_{IJ} - \mathbf{A}_{IL}\Delta_{KJ}\hat{=}\Omega_{LI}] \,, \tag{1.13b}$$

namely with programs made in languages Fortran, PL_1, Pascal, etc.

1.2 Our DTA was compared multilaterally with other algorithms, e.g. with STA, (Gaussian) GMA, (Jordan's) JMA, (our symmetric) SMA (for $\mathbf{A}_0 = \mathbf{A}_0^*$ having the first step (1.11a), then

$$\underset{[G]}{\mathbf{A}_1} = \mathbf{A}_0 - \gamma a_l a^k \,, \tag{1.21a}$$

$$\underset{[J]}{\mathbf{A}_1} = a_{kl} \underset{[G]}{\mathbf{A}_1} \,, \tag{1.21b}$$

$$\underset{[Sy]}{\mathbf{A}_1} = \mathbf{A}_0 - \gamma(a_l - e_k)(a_l - e_k)^* \,, \tag{1.21c}$$

namely at so simple lin. alg. tasks as

a) ranking ($p = r$, if $\Omega_{IJ} = \mathbf{O}_{IJ}$);

b) norming

$$\left[\underset{[S,D]}{\mathbf{A}_r} \supset\right] \mathbf{A}_r^K \hat{=} [\Gamma_{KL}, \Delta_{KL}] \left\langle \underset{[S]}{\mathbf{A}_r^I} = \mathbf{O}^I \right\rangle ;$$

c) ordinary inversion of a regular matrix ($n = m = r$) by DTA in n steps (at different $k_q = l_q \in N \hat{=} \{1\ 2, \ldots, n\}$, $\forall a^{(q)}_{k_q l_q} \neq 0$ or in $\nu(< n)$ springs (with regular blocks $\underset{(\rho)}{\mathbf{A}}_{KK}$ of p_ρ^{th} order at $\sum_{\rho=1}^{\nu} p_\rho = n$), giving $\underset{[D]}{\mathbf{A}_n} = \mathbf{A}_0^{-1}$ [12], (remark: at $\forall a_{ii} = 0$, we start from a spring $\widehat{\text{DTA}}_p$ with $|\mathbf{A}_{KK}| \neq 0$, then continue by steps DTA_q with $a^{[q]}_{k_q k_q} \neq 0$;

$$\underset{[S]}{\mathbf{A}_0}{}_n = \mathbf{E}, \quad (!), \quad \text{so} \quad \left[\underset{[S]}{\mathbf{A}_0, \mathbf{E}}\right]_n = \left[\mathbf{E}, \mathbf{A}_0^{-1}\right] \quad \text{of type } n \times 2n \quad (!);$$

d – e) generalized inversion for a whole row // column ranked [12], moreover for an arbitrary matrix by repeated DTA [17], etc. ... (1.22a, b); at these a – e), the DTA gets more and more advanced.

1.3 a) Again, our DTA gives for the gen. lin. unequality $\mathbf{A}_0 \boldsymbol{x} \leq \boldsymbol{a}_0$, or for its equivalent equality $(\mathbf{O} \leq) \boldsymbol{u} = \boldsymbol{a}_0 \cdot 1 + \mathbf{A}_0(-\boldsymbol{x}) \hat{=} \widehat{\mathbf{A}}_0(-\widehat{\boldsymbol{x}})$

— just at the rank $r = \rho(\mathbf{A}_0) = \rho(\underset{(r\times r)}{\mathbf{A}_{KL}})$ — the compatib. condition (CC) $(\mathbf{O} \leq) \underset{(\nu)}{u_I} = \ldots$ and the gen. solution (GS) for the main unknown $\underset{[r]}{x_L} = \ldots$ [12, 15]:

$$\widehat{v}_r \widehat{=} \begin{bmatrix} x_L \\ u_I \end{bmatrix} = \begin{bmatrix} \delta_{OK} & \vdots & \Gamma_{KL} & \Delta_{KL} \\ \varepsilon_{IO} & \vdots & -\Sigma_{IL} & \mathbf{O}_{IJ} \end{bmatrix} \begin{bmatrix} 1 \\ \ldots \\ -u_K \\ -x_J \end{bmatrix} \widehat{=} \underset{[D]}{\widehat{\mathbf{A}}_r} (-\widehat{y}_p)\,, \tag{1.31}$$

$$(\nu = n - r\,, \quad \mu = m - r)\,, \qquad (\mathbf{0} \leq u_K)\,.$$

b) For the gen. lin. equation (at $u = \mathbf{0}$),

$$\text{CC is} \quad \mathbf{0} = \varepsilon_{IO} \quad \text{and GS} \quad x_L = \delta_{OK} + \Delta_{KJ}(-x_J)\,. \tag{1.32}$$

– Its CC and GS by the STA, GMA and our SMA (for $\mathbf{A}_0 = \mathbf{A}_0^*$), as comparison is shown in [12, 14].

c) Then an iteration VSI is treated with a regular

$$\mathbf{A}_0 = \mathbf{C}_0^* \mathbf{C}_0 = \mathbf{A}_0^* \quad \text{at} \quad 0 < c^i c_i \widehat{=} a_{ii} \gg |a_{ij}| \widehat{=} |c^i c_j|$$

for

$$\left\langle a_{ii}^{-1} \right\rangle \cdot [\mathbf{A}_0,\, a_0] \begin{bmatrix} x \\ -1 \end{bmatrix} \widehat{=} \widehat{\mathbf{D}}_0 \widehat{x} = \mathbf{0}\,,$$

namely by sequential powering of our operator matrix

$$\widehat{\mathbf{S}} = \widehat{\mathbf{E}} - \sum_{i=1}^{n} \widehat{e}_i \widehat{\delta}^i \quad \text{at} \quad \widehat{\delta}^i = \widehat{d}^i - \sum_{j=1}^{i-1} d_{ij} \widehat{\delta}^j$$

[12], giving the convergent solving sequence

$$\widehat{x}^{[\nu]} = \widehat{\mathbf{S}}^{\nu} \widehat{x}[0] \quad (\text{e.g.} \quad x^{[0]} = d_0)\,. \tag{1.33$\alpha - \delta$}$$

1.4 The lin. programming (LP) tasks, namely e.g.

a) LP of production for max. profit,
b) LP of consumption for min. expense

$$\widehat{u} = \begin{bmatrix} u_0 \\ u \end{bmatrix} = \begin{bmatrix} a_{00} & a^0 \\ a_0 & \mathbf{A}_0 \end{bmatrix} \begin{bmatrix} 1 \\ -x \end{bmatrix} = \widehat{\mathbf{A}}_0(-\widehat{x})\,,$$

$$\widehat{v} \widehat{=} \begin{bmatrix} v_0 \\ v \end{bmatrix} = \begin{bmatrix} a_{00} & a^0 \\ a_0 & \mathbb{A}_0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} \widehat{=} \check{\mathbb{A}}_0 \widehat{y}$$

(at $a_0 > \mathbf{0}, \exists \quad a_{01} < 0$) || (at $a^0 > \mathbf{0}^*, \exists \quad a_{k0} < 0$)

$\mathbf{0} \leq x, \quad \mathbf{0} \leq u, \quad u_0 = \max!$ $(a_{00} = 0)$ $\mathbf{0} \leq y, \quad \mathbf{0} \leq v_p, \quad v_0 = \min!$

and

c) integer LP with $x = \text{int}!$ are related tasks to 1,3 and so well solvable
   a) by (primal) DTA steps // b) dual DTA_q one

$$\underset{[D]}{\widehat{\mathbb{A}}_{q+1}} = \widehat{\mathbb{A}}_q - \gamma_q \left(\widehat{a}_{l_q}^{[q]} - \widehat{e}_{k_q}\right) \left(\widehat{a}_{[q]}^{k_q} + \widehat{e}^{l_q}\right), \tag{1.42a}$$

$$\underset{[D]}{\check{\mathbb{A}}_{q+1}} = \check{\mathbb{A}}_q - \gamma_q \left(\check{a}_{l_q}^{[q]} + e_{k_q}\right) \left(\check{a}_{[q]}^{k_q} - e^{l_q}\right), \tag{1.42b}$$

for the special choice of pivots $a_{k_q l_q}^{[q]} \widehat{=} 1/\gamma_q \neq 0$ and with stop $a_{(p)}^0 \geq \mathbf{0}^* // a_0^{(p)} \geq \mathbf{0}$, then

c) by $\widehat{\text{DTA}}_p$ till max., then — after a (diophantic) supplement — by a DTA_{p+1} for int. Details are contained in our Bn. [13]. (1.43c)

1.5 a) Our generalized gDTA was proved suitable to solve a gen. nonlin. unequality

$$a_0(x) \approx a_0(x_0) + \mathbb{A}_0(x_0)\Delta x \leq 0,$$

or its equivalent equality

$$(\mathbf{0} \leq)\Delta u = a_0(x_0)(-1) + \mathbb{A}_0(x_0)(-\Delta x) \widehat{=} \widehat{\mathbb{A}}_0(x_0)(-\Delta \widehat{x}), \tag{1.51a, b}$$

by using diagonal pivot $a_{k_q k_p}^{[q]} \neq 0$ chosen to a 'bottle neck rule' and producing an approx. 'moving inversion'

$$\underset{[gD]}{\widehat{\mathbb{A}}_n} \left(x_0 + \sum_q \Delta x_q\right) \approx \widehat{\mathbb{A}}_0^{-1}(x_0),$$

namely at the Newton–Raphsonian conditions and at a convergency's acceleration compared with it [12 – 14]. (1.52a, b)

b) This gDTA seems to be useful also for nonlin. (convex) P, but also the original DTA is sometimes suitable for quadratic (Q) P

[12, 15], further — a special iteration (CVA) seems suitable at an irrational $(I)P$ studied in [11, 15]. (1.53a–c)

1.6 We created further (algebraic) algorithms, e.g. for

a) orthogonalizing a regular $\mathbf{A}_0$ (OMA),
b) triangularizing a determinant $|\mathbf{A}_0|$ (TAD), both are used in 4.3) to create ITA and OTA [14]. (1.61a–c)

1.7 As techn.-econ. applications of our upper MAM, let us mention, e.g.

a) for 1.45 various production and consumption (dieta, mixtura) LP tasks, transport and schedule LP tasks, then cutting and purchase-storage LP tasks in [11, 16],
b) for 1.52 some transport and electric network QP tasks [16],
c) 1.53 centre location LP tasks in [11] etc.

1.8 Our brochure [18] for matrix methods of graph theory used

a) real, or
b) binary ordinary, or
c $\left.{}^{\mathrm{r}}_{\mathrm{b}}\right)$ $\begin{matrix}\text{real}\\ \text{bin.}\end{matrix}$ direct matrix algebras.

Further on it treats practical problems associated with them, namely
ad a) seeking a (real) critical path = $\mathrm{CP}(\lambda, \mu)$, (1.81)
ad b) flow-crossing of hostile pairs as binary CP;
ad c ${}_{\mathrm{b}}$) cooperation of two discrete automats $G_a = (X_3, \mathbf{U})$ and $G_b = (Y_2, \mathbf{V})$ by simultaneous (parallel) transits: $G_{c_p} = (Z_6, \overset{\times}{\mathbf{W}})$ and by sequential (serial) ones: $G_{c_s} = (Z_6, \overset{+}{\mathbf{W}})$ at $Z_6 = X_3 \times Y_2$, where $\overset{\times}{\mathbf{W}} = \mathbf{U} \times \mathbf{V} = [u_{ij}\mathbf{V}]$, $\overset{+}{\mathbf{W}} = \mathbf{U} + \mathbf{V} = [u_{ij}\mathbf{E} + \delta_{ij}\mathbf{V}]$ as dir. prod. $(\times)$ // dir. sum $(+)$.. (1.82)

2. MAM for the Function's Analysis

Their main bases are our Bn [22] and Ps [23, 24, 28], which show persuasively the advantages of matrix methods in various (scalar) analytical problems (e.g. diff. equs) and in their applications.

2.1 Lin. diff. equations of n^{th} order (LDE_n)

$$L_n[y]\widehat{=}y^{[n]}+\sum_{k=0}^{n-1}p_k(t)y^{[k]}=y^{[n]}+p^*(t)y=\widehat{p}^*(t)\widehat{y}=\begin{cases}0 & (h)\\ x(t) & (ih)\end{cases}$$

(2.11a – c)

$$\left\langle \forall t\in T=(\alpha,\beta);\quad \forall p_k(t),\quad x(t)\in C_T;\quad y\widehat{=}{}_{n-1}^{0}\left[y^{[k]}\right]\right\rangle$$

– completed by identities $y^{[k+1]}-y^{[k+1]}=0$ and got under initial conditions (IC_n) $y^{[k]}(t_0)=y_0^{[k]}$ for $t_0\in T$ .... (2.12)
– occur in the matricial form [as a special case of

$$\dot{z}=\mathbb{A}(t)z+\mathbb{B}(t)x(t)] \tag{2.13}$$

$$\begin{matrix}(h)\\(ih)\end{matrix}L_{I_n}[y]\widehat{=}\dot{y}+\mathbb{P}(t)y=$$

$$\begin{cases}\mathbf{0}\\ x_n(t)\end{cases}\quad\begin{cases}\mathbb{P}(t)=e_np^*(t)-\sum_{j=1}^{n-1}e_je^{j+1}=e_np^*(t)-K_n\\ x_n(t)=e_nx(t)\end{cases}, \tag{2.14}$$

For its solution — at Lipschizian and helping limitation [22] — the existence and unicity (EU) are valid.

2.2 For hom. (h) $L_{I_n}[y]=$ having a (full) scalar // vector solving system

$$y^{(0)}(t)=[y_j(t)]^n//\mathbf{Y}(t)\widehat{=}{}^1[y_j(t)]^n={}_{n-1}^{0}\left[y^{(k)}(t)\right]$$

(Wronskian)

$$\left\langle \text{at}\quad L_n[y^{(0)}(t)]\equiv\mathbf{0}^*//L_{I_n}[\mathbf{Y}(t)]\equiv\mathbf{O}\right\rangle,$$

some theorems are proved [22]:

a) $$y(t)\widehat{=}y^{(0)}(t)c//y(t)\widehat{=}\mathbf{Y}(t)c$$

are particular solutions;

b) at lin. indep. (basic) one

$$y(t)\not\equiv 0\quad\text{at}\quad c\neq\mathbf{0}\quad\text{for}\quad\forall t\in T,$$

so (with EU) in case of regularity of $\mathbf{Y}(t)$:

$$|\mathbf{Y}(t)|\neq 0\quad\text{for}\quad\forall t\in T,\qquad y(t)//y(t)$$

is the general solution;

c) a lin. manifold of such (basic // regular) system

$$y_C^{(0)}(t) \hat{=} y^{(0)}(t)\mathbb{C} \subset \mathbb{Y}_C(t) \hat{=} \mathbb{Y}(t)\mathbb{C}$$

can be formed by an (arbitrary) regular matrix factor $\mathbb{C}(|\mathbb{C}| \neq 0)$ from an (ever existing and) known $\mathbb{Y}(t)$;

d) this $\mathbb{Y}_C(t)$ ever has an entity $\mathbb{Y}_\sim(t,t_0) \hat{=} \mathbb{Y}(t)\mathbb{C}_0$ being $\mathbb{E}$-normed for $t_0 \in T$ $\mathbb{Y}_\sim(t_0,t_0) \hat{=} \mathbb{Y}(t_0)\mathbb{C}_0 = \mathbb{E}$ consequ. $\Big\langle$ with $\mathbb{C}_0 = \mathbb{Y}_0^{-1}(t_0)$ counted by our DTA of 1.2c)$\Big\rangle$

$$\mathbb{Y}_\sim(t,t_0) \hat{=} \mathbb{Y}(t)\mathbb{Y}^{-1}(t_0) \hat{=} \mathbb{Y}(t)\mathbb{Z}(t_0)\,,$$

$$[Y_\sim(t,t_0) \hat{=} Y(t)Z(t_0) \neq 0 \qquad \text{for} \quad \forall t, t_0 \in T]\,,$$

where

$$\mathbb{Y}_\sim(t,t_0) \hat{=}_{n-1}^{0} \left[y^{(k)}(t) z_{(l)}(t_0)\right]^{n-1}$$

with

$$\mathbb{Y}_\sim(t_0,t_0) \hat{=}_{n-1}^{0} [\delta_{kl}]^{n-1} = \mathbb{E}$$

and

$$y_{\hat{n}}(t,t_0) \hat{=} \mathbb{Y}(t) z_{(n-1)}(t_0) \quad \text{with} \quad y_{\hat{n}}(t_0,t_0) = e_n$$

occur in Green type;

e) $\mathbb{Y}_\sim(t,t_0)$ is a resolvent matrix giving the conditional solution so:

$$y_0(t) = \mathbb{Y}_\sim(t,t_0) y_0 \quad [\text{at} \quad y_0(t_0) = \mathbb{E} y_0]\,. \qquad (2.21-25)$$

2.3 Some methodical contrib. [22] for hLDE$_n$;

a) a given, in T regular W-matrix (basic vector system)

$$\mathbf{Y}(t) \hat{=}\ _{n-1}^{0} \left[\boldsymbol{y}^{(k)}(t)\right]$$

determines, together with

$$\dot{\mathbf{Y}}(t) =\ _{n}^{1} \left[\boldsymbol{y}^{(i)}(t)\right]\,,$$

the corresponding hLDE:

$$\dot{\mathbf{Y}}(t) + \mathbf{P}(t)\mathbf{Y}(t) = \mathbf{O}$$

or its (contin.) coefficients:

$$\mathbb{P}(t) = -\dot{\mathbb{Y}}(t)\mathbb{Z}(t) \supset -y^{(n)}(t)\mathbb{Z}(t) = p^*(t) \ni -y^{(n)}(t)z_{(1)}(t) =$$

$$= p_{n-1}(t) = -\dot{Y}(t)Z(t) \qquad (2.31\ \alpha\text{–}\gamma)$$

[with derivation of W-determinant $Y(t)\widehat{=}|\mathbb{Y}(t)|$];

b) it results the Liouville-formulas:

$$Y(t) = Ce^{-\int p_{n-1}(t)\mathrm{d}t}\widehat{=}Ce^{-\widehat{p}_{n-1}(t)} \neq 0 \qquad (\text{for} \quad \forall t \in T), \qquad (2.31a)$$

$$Y_{\sim}(t, t_0) = e^{-[\widehat{p}_{n-1}(t)-\widehat{p}_{n-1}(t_0)]}\widehat{=}e^{-\Delta\widehat{p}_{n-1}(t,t_0)} \neq 0\,; \qquad (2.31b)$$

c) the DE

$$L_{I_n}[\mathbb{Y}]\widehat{=}\dot{\mathbb{Y}} + \mathbb{P}(t)\mathbb{Y} = \mathbb{O}$$

has not — generally — an exponential solution

$$\mathbb{Y}(t) = e^{-\int \mathbb{P}(t)\mathrm{d}t}\widehat{=}e^{-\widehat{\mathbb{P}}(t)}$$

being

$$-\mathbb{Y}(t)\mathbb{P}(t) + \mathbb{P}(t)\mathbb{Y}(t) \not\equiv \mathbb{O}, \qquad (2.32a)$$

but the associate DE

$$L_{I_n}[\mathbb{Z}]\widehat{=}\dot{\mathbb{Z}} - \mathbb{Z}\mathbb{P}(t) = \mathbb{O}$$

has such a solution:

$$\mathbb{Z}(t) = e^{+\widehat{\mathbb{P}}(t)}$$

being

$$\mathbb{Z}(t)\mathbb{P}(t) - \mathbb{Z}(t)\mathbb{P}(t) \equiv \mathbb{O}\,; \qquad (2.32b)$$

d) consequ. for

$$\dot{y} + \mathbb{P}(t)y = O\,, \quad \text{there is} \quad \mathbb{Y}(t)c \neq e^{-\widehat{\mathbb{P}}(t)}c\,,$$

but for

$$(e^1\dot{\mathbb{Z}}\widehat{=})\dot{z}^* = z^*\mathbb{P}(t)\,, \quad \text{or for} \quad \dot{z} - \mathbb{P}^*(t)z = \mathbf{0}\,,$$

there is

$$z(t) = \mathbb{Z}^*(t)c = e^{+\widehat{\mathbb{P}}^*(t)}\gamma\left[\widehat{=}\mathbb{Y}^{*^{-1}}(t)\gamma\right]. \qquad (2.33a,b)$$

2.4 a) The lin. transform

$$y = y_1(t) \int u dt \hat{=} y_1 \tilde{u} \quad (\neq 0 \quad \text{in} \quad T)$$

of a hom. LDE can be performed advantageously by our product form [23]

$$\hat{y} \hat{=} \,_n^0 \left[y^{(l)}\right] = \left[\sum_{\lambda=0}^{l} \binom{l}{\lambda} y_1^{(l-\lambda)} \tilde{u}^{(\lambda-1)}\right] = \overset{\rhd}{\mathbf{Y}}_1(t)\hat{u}_+ = \hat{y}_1(t)\tilde{u} + \overset{\geq}{\mathbf{Y}}_1(t)\hat{u}$$

with the (lower triangular) derivative matrix of Pascal type $\overset{\rhd}{\mathbf{Y}}_1(t)$, which results a (from n into $n-1$) subordered $\text{LDE}_{I_{n-1}}$ and W-det. (as SoTA_1):

$$U_{n-1}(t) = \frac{Y_n(t)}{y_1^n(t)} \neq 0 \qquad \text{in} \qquad T\,, \tag{2.42}$$

$$L_{I_n}[\hat{y}] \hat{=} \hat{p}^* \hat{y} = \hat{p}^*(t) \overset{\rhd}{\mathbf{Y}}_1(t)\hat{u}_+ = \hat{p}^*_,(t)\overset{\geq}{\mathbf{Y}}_1(t)\hat{u} = y_1(t)\cdot \hat{q}^*(t)\hat{u} \hat{=}$$
$$\hat{=} y_1(t) \cdot L_{I_{n-1}}[\hat{u}] = \mathbf{0}\,. \tag{2.43}$$

b) By repeating the steps SoTA_q, our total algorithm $\widehat{\text{SoTA}}_{n-1}$ [23] can be realized. E.g. at $n-4$, the $\widehat{\text{SoTA}}_3$ has the form:

$$L_4[y] \hat{=} p^* \overset{\geq}{\mathbf{Y}}_1 \overset{\geq}{\mathbf{U}}_2 \overset{\geq}{\mathbf{V}}_3 w = y_1 u_2 v_3 \cdot [\dot{w} + s_0 w] \hat{=} f_{1,2,3}(t) \cdot L_1[w] = 0\,, \tag{2.44}$$

$$Y \hat{=} c e^{-\hat{p}_3} = y_1^4 u_2^3 v_3^2 w_4 \neq 0 \qquad \text{in} \qquad T\,; \tag{2.45}$$

$$y_1, \quad u_2 = \frac{Y_2}{Y_1^2}\,, \qquad v_3 = \frac{Y_1 Y_3}{Y_2^2}\,, \qquad w_4 = \frac{Y_2 Y_4}{Y_3^2}\,. \tag{2.46}$$

2.5 For the inhom. (ih) $\text{LDE}_{i,n}$ (2.13) and IC_{I_n} (2.14), some theorems are proved [22]:

a)

$$y(t)_x \hat{=} y(t) + y_1(t)_x = \mathbf{Y}(t)c + y_1(t)_x \quad [Y(t) \neq 0 \quad \text{for} \quad \forall t \in T\,,$$

$$L_{I_n}[y_1(t)_x] \equiv x_n(t)]$$

is the general solution,

$$y_0(t)_x = \mathbb{Y}_{\sim}(t, t_0)[y_0 - y_1(t_0)_x] + y_1(t)_x$$

is the y_0-conditional one;

b) the $y_1(t_0)_x$ $\mathbb{0}$-conditional one can be found by Lagrange's variations of const.:

$$y(t)_x = \mathbb{Y}(t)c(t) \quad [\text{at} \quad c(t) = ?] \,,$$

$$L_{I_n}[\mathbb{Y}c] \hat{=} \mathbb{0} + \mathbb{Y}\dot{c} = x_n \hat{=} x e_n \,,$$

$$\dot{c} = \mathbb{Z}x_n \,, \qquad c = \int_{t_0}^{t} \mathbb{Z}x_n \mathrm{d}\tau \,, \tag{2.51}$$

finally

$$y_1(t)_x = \mathbb{Y}(t) \int_{t_0}^{t} \mathbb{Z}(\tau)x_n(\tau)\mathrm{d}\tau = \int_{t_0}^{t} \mathbb{Y}_{\sim}(t,\tau)x_n(\tau)\mathrm{d}\tau =$$

$$\int_{t_0}^{t} y_{\tilde{n}}(t,\tau)x(\tau)\mathrm{d}\tau$$

$$[\text{at} \quad y_1(t_0)_x = \int_{t_0}^{t_0} \ldots \mathrm{d}\tau = \mathbb{0}] \,. \tag{2.52}$$

2.6 a) At the LDE_{I_n} of const. coeffs

$$L_{I_n}[y] \hat{=} \dot{y} + \mathbf{P}y = \begin{cases} \mathbb{0} \\ P_\nu(t)e^{\alpha t}e_n \end{cases} \tag{2.61a}$$

$$\left[\mathbf{P} = e_n p^* - \mathbf{K}_n \,, \qquad P \hat{=} |\mathbf{P}| = p_0 \,, \qquad \mathbf{K}_n^k = \sum_{j=1}^{n-k} e_j e^{j+k} \right] \tag{2.61b}$$

the trial for hom. LDE $y(t) = e^{-\lambda t}v$ leads us to the eigen problem [22, 25]

$$\mathbf{P}_n(\lambda)v \hat{=} (\lambda\mathbf{E} - \mathbf{P})v = \mathbf{0}$$

at the condition

$$|\mathbf{P}_n| \hat{=} |\lambda\mathbf{E} - e_n p^* + \mathbf{K}_n| = \lambda_0^n + p^*\boldsymbol{\lambda} = 0$$

with the roots

$$\lambda_{\bar{j}} \widehat{=} -\lambda_j \quad \text{and} \quad v_{\bar{j}} \widehat{=} -\boldsymbol{\lambda}_j \widehat{=} \, {}^{0}_{n-1}\left[(-\lambda_j)^k\right] ; \qquad (2.62\text{a–d})$$

then — e.g. at single

$$\lambda_j \in -\Lambda_\backslash \quad \text{and} \quad \mathbf{V}\Lambda_\backslash - \mathbf{P}\mathbf{V} = \mathbf{O}$$

with

$$\mathbb{Y}(t) \widehat{=} \mathbf{V}e^{-\Lambda_\backslash t} = \left[v_j e^{-\lambda_j t}\right]$$

and

$$\mathbb{Y}_\sim(t, t_0) \widehat{=} \mathbf{V}e^{-\Lambda_\backslash (t-t_0)}\mathbf{W} = e^{-\mathbf{P}(t-t_0)} \widehat{=} \mathbb{Y}_\sim(t-t_0)$$

$$\text{at} \quad \mathbf{W} = \mathbf{V}^{-1} . \qquad (2.63\text{a–d})$$

Determinant $V_n \widehat{=} {}^1|\boldsymbol{\lambda}_{\bar{j}}|^n$ can be counted quickly by our triangularizing algorithm TAD [22]; e.g. for $n = 4$ and with

$$\Delta_{ji} = -\lambda_j + \lambda_i , \quad \Delta_i = \prod_{i<j\leq n} \Delta_{ji}$$

at single λ_j

$$V_4 \widehat{=} \left|\mathbf{V}_4 + \boldsymbol{\lambda}_1(e^* - e^1)\right| = \Delta_1 V_3 \widehat{=} \left|\mathbf{V}_3 + \boldsymbol{\lambda}_2(e^* - e^2)\right| =$$

$$\Delta_1\Delta_2 V_2 = \Delta_1\Delta_2\Delta_3 \cdot 1 \neq 0 .$$

For the more general case with $\lambda = -\lambda_\rho$ with $\nu_\rho \geq 1$, $\sum_{\rho=1}^{r} \nu_\rho = n$, look at our [22].

b) The inhom. LDE_{I_n} (2.61b) — e.g. at $\nu_\alpha \geq 1$ of $\lambda = -\alpha$ — will be solved so (for $k = 0$):

$$y_{11}(t)_x = \int_{t_0}^{t} y_\sim(t-\tau)x(\tau)\mathrm{d}\tau = \ldots = t^{\nu_\alpha} e^{-\alpha t} Q_{\nu_\alpha}(t) . \qquad (2.65)$$

c) It should be noted that such LDE_{I_n} is sometimes replaced in practice [21, 25] by difference equation got e.g. by

$$y_\Delta \widehat{=} {}^1_n[y_i] , \quad \ddot{y}_\Delta \widehat{=} -\frac{1}{\tau^2}\mathbf{C}y_\Delta$$

$$\left(\text{at} \quad \Delta t \hat{=} \tau \quad \text{and with s.c. continuant} \right.$$

$$\left. \mathbf{C} = \begin{bmatrix} -1 & 2 & 0 & \dots \\ 2 & -1 & 2 & \dots \\ 0 & 2 & -1 & 2\dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \right) . \tag{2.66}$$

2.7 a) The special state LDE_{I_n} $\dot{z} = \mathbb{A}z + \mathbf{B}x(t)$ $\left(\text{with} \quad z \hat{=} {}^1_n[z_i] \neq {}^0_{n-1}[z^{(i-1)}]\right.$, generally) used often by us [28] has again expon. solutions $z(t) = ue^{-\lambda t}$ for hom. case, namely with the (more general) eigen problem [24, 28]

$$\left[(\dot{z} - \mathbb{A}z)e^{-\lambda t} \hat{=} \right] (\lambda\mathbb{E} - \mathbb{A})u \hat{=} \mathbb{A}_n(\lambda)u = \mathbb{0}$$

at char. equ.

$$\left|\mathbb{A}_n(\lambda)\tilde{\mathbb{A}}_n(\lambda)\right| = D_n(\lambda) \hat{=} \prod_{\rho=1}^{r} (\lambda - \lambda_\rho)^{\alpha_\rho} = 0 , \tag{2.71}$$

$$\left\langle \sum_{\rho=1}^{r} \alpha_\rho = n , \qquad D_n(\mathbb{A}) \equiv \mathbb{O} \right\rangle \tag{2.72}$$

and min. equ. [with *g.c.d.* $\Theta(\lambda)$]

$$\Delta_m(\lambda) \hat{=} \frac{D_n(\lambda)}{\Theta(\lambda)} \hat{=} \prod_{\rho=1}^{r} (\lambda - \lambda_\rho)^{\beta_\rho} = 0 ,$$

$$\left\langle 1 \le \beta_\rho \le \alpha_\rho , \qquad r \le \sum_{\rho=1}^{r} \beta_\rho = m \le n , \qquad \Delta_m(\mathbb{A}) \equiv \mathbb{O} \right\rangle .$$

E.g. in the extra-min. case $\forall \beta_\rho = 1$, there is

$$\mathbf{A} = \sum_{\rho=1}^{r} \mathbf{U}_\rho \cdot \lambda_\rho \mathbf{E}_{\alpha_\rho} \cdot \mathbf{V}^\rho \hat{=} \mathbf{U}\Lambda_\backslash \mathbf{V} \qquad (\text{at} \quad \mathbf{V} \hat{=} \mathbf{U}^{-1}) , \quad \text{const.} \tag{2.73a}$$

$$\mathbb{Z}(t) = \mathbf{U}e^{\Lambda_\backslash t} = e^{\mathbf{A}t}\mathbf{U} , \tag{2.73b}$$

so

$$z_0(t) = e^{\mathbf{A}(t-t_0)}z_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)} \cdot \mathbf{B}x(\tau)\mathrm{d}\tau . \tag{2.73c}$$

b) The general state LDE$_n$ $\dot{z} = \mathbf{A}(t)z + \mathbf{B}(t)x(t)$ has generally no expon. and exact solution; the resolvent matrix can be sought from the hom. matrix integral equ.

$$\mathbb{Z}_\sim(t, t_0) = \mathbb{E} + \int_{t_0}^{t} \mathbb{A}(\tau)\mathbb{Z}_\sim(\tau, t_0)\mathrm{d}\tau\,;$$

by successive approximation

$$\mathbb{Z}_{\tilde{k}+1}(t, t_0) = \mathbb{E} + \int_{t_0}^{t} \mathbb{A}(\tau)\mathbb{Z}_{\tilde{k}}(\tau, t_0)\mathrm{d}\tau\,. \qquad (2.74)$$

But our proposed trial

$$\mathbb{Z}(t) = \mathbb{U}(t)e^{\widehat{\Lambda_\backslash}(t)} \text{ at } \dot{\mathbb{U}} = \mathbb{A}_1(t)\mathbb{U} \quad [\text{with } \mathbb{A}_1(t) \widehat{=} \mathbb{A}(t) - \mathbb{A}_0(t)] \qquad (2.75)$$

often results in such solution through the time-variant eigenproblem

$$\mathbb{U}(t)\Lambda_\backslash(t) - \mathbb{A}_0(t)\mathbb{U}(t) = \mathbb{O}\,;$$

by choosing $\mathbb{A}_0(t)$ suitably, so it is very useful in numerous systems.

2.8 We gave MM to solve some non-lin. DE$_{I_n}$ [23]:

a) The Bernoulli type:

$$\dot{\mathbf{Z}}_\backslash = \mathbb{A}(t)\mathbf{Z}_\backslash + \mathbf{B}(t)\mathbf{Z}_\backslash^\alpha \quad [\alpha \neq 0,\ 1], \quad \dot{z} = \mathbb{A}(t)z + \mathbf{B}(t)z^\alpha$$

$$\left[\text{with} \quad \mathbf{Z}_\backslash^\alpha e \widehat{=}\ {}^1_n[z_i^\alpha \cdot 1] = [z_i^\alpha] \widehat{=} z^\alpha\right]\,;$$

a right-multiplication by $(1-\alpha)\mathbf{Z}_\backslash^{-\alpha}$ and a transform $\mathbf{\Sigma}_\backslash = \mathbf{Z}_\backslash^{1-\alpha}$ linearizes it:

$$\dot{\mathbf{\Sigma}} = \mathbf{A}_\alpha(t)\mathbf{\Sigma} + \mathbf{B}_\alpha(t)| \cdot e\,, \quad \dot{\sigma} = \mathbf{A}_\alpha(t)\sigma + b_\alpha(t)\,, \qquad (2.81\text{a,b})$$

which — as the former type — can be often solved by our trial (2.75).

The Riccati type [23] and its transform by

$$\mathbf{Z}_\backslash = \tilde{\mathbf{Z}}_\backslash + \mathbf{Z}_1^\backslash(t)\,, \quad \text{then by} \quad \mathbf{\Sigma}_\backslash = \tilde{\mathbf{Z}}^{-1}\left(\tilde{\mathbf{Z}}_\backslash e = \tilde{z}\right)\,: \qquad (2.82\text{a})$$

$$\dot{\mathbf{Z}}_\backslash = \mathbb{A}(t)\mathbf{Z}_\backslash + \mathbf{B}(t)\mathbf{Z}_\backslash^2 + \mathbf{C}(t)\,, \tag{2.82b}$$

$$\dot{\tilde{\mathbf{Z}}}_\backslash = \mathbb{A}_{b1}(t)\tilde{\mathbf{Z}}_\backslash + \mathbf{B}(t)\tilde{\mathbf{Z}}_\backslash^2\,, \tag{2.82c}$$

$$\dot{\Sigma}_\backslash = -\mathbb{A}_{b1}(t)\Sigma_\backslash - \mathbf{B}(t).$$

The Duffing type [27]:

$$\dot{\mathbb{Z}}_\backslash = \mathbb{A}(t)\mathbb{Z}_\backslash + \mathbb{B}(t)\mathbb{Z}_\backslash^3 \qquad \langle \alpha = 3 \rangle\,, \tag{2.83a}$$

$$\dot{\mathbf{\Sigma}}_\backslash = -2\mathbb{A}(t)\mathbf{\Sigma}_\backslash - 2\mathbb{B}(t)\,. \tag{2.83b}$$

b) The local linearization for a nonlin. state vDE_{I_n} $\dot{z} = (fz)$, then
c) to perform a step of the Runge–Kutta method in a nonlin. state vDE_{I_n} $\dot{z} = f(z, x, t)$ by our algorithm of 4 substeps is treated in [24].
d) We sometimes made dynamic optimization for such state vDE_{I_n} and with an object $f_0(z, x, t)$ by the variational task [26]:

$$I(z, x, t) \widehat{=} \int_0^T [f_0 + \lambda^*(f - z)]\mathrm{d}t = \int_0^T [H_\lambda(z, x, t, \lambda) - \lambda^*\dot{z}]\mathrm{d}t = \text{Min!} \tag{2.89}$$

2.9 Let some technical applications of our upper MAM be mentioned; e.g. ad 2.6: analysis of a turbine axle for critical ω, bending analysis of chain bridge, motor vehicle as vibrating system; bending, buckling, vibrating bars [21, 25]; ad 2.7 – 8: various systems with theoretical and electrical problems [28], dyn. optimization of lin. control system at quadr.criterion [26], nonlin. vibrations with bifurcations [27] and still more.

3. MAM for the Functional Analysis

Their main basis is given by our papers and bulletins [31 – 33], which illustrate well — e.g. also under complex circumstances of the Hilbert-space for Lebesgue sense quadratically integrable functions L_T^2 — the elasticity, efficiency and other advantages of matrix methods.

3.1 Functions system (FS), functions bases (FSs) and their norm dyads (NDs).

a) Having at the interval

$$T \widehat{=} [a, b] \quad \text{a FS} \quad a^*(t) \widehat{=} {}^0[a_j(t)]^m \subset L_T^2$$

and a clinogonal (cg) FB

$$b^*(t) \widehat{=} [b_n^*(t),\, b_\nu(t)] =^0 [b_i(t)]^{n+\nu} \subset L_T^2\,,$$

$$[\text{so lin. indep.:} \quad b^*(t)c \neq 0 \quad \text{at} \quad c \neq \mathbf{0} \quad \text{for} \quad t \in T]\,,$$

where $m < M$, or $m \to \infty$ and $\nu \to \infty$. Either of the systems can be characterized by its (systems // basis) norm dyad (SND//BND) and by their (common or) cross norm dyad (having a Gram-type):

$$\overset{\cap}{\mathbf{A}} \widehat{=} [a, a^*] = \int_{(T)} a(t)a^*(t)\mathrm{d}t = \left[\int_{(T)} a_j(t)a_l(t)\mathrm{d}t\right] = [(a_j, a_l)] \widehat{=}^0 \left[\overset{\cap}{a}_{jl}\right]^m$$

$$\left\langle \text{at} \quad \forall \overset{\cap}{a}_{jj} \widehat{=} \alpha_j^2 < \alpha^2\,, \quad \overset{\cap}{\mathbf{A}} \lessgtr \pm [\alpha_j \alpha_l] = \pm \alpha\, \alpha^* \lessgtr \pm \alpha^2 [1] \widehat{=} \pm \alpha^2 \mathbf{U} \right\rangle\,, \tag{3.11a}$$

$$\overset{\cap}{\mathbf{B}} \widehat{=} [b, b^*] = \ldots \widehat{=}\ ^0\left[\overset{\cap}{b}_{ik}\right]^\infty \lessgtr \pm \beta^2 \mathbf{U}\,, \tag{3.11b}$$

$$\overset{\cap}{\mathbf{C}} \widehat{=} [b, a^*] = \ldots \widehat{=}\ ^0\left[\overset{\cap}{c}_{ij}\right]^m \lessgtr \pm \alpha\beta \mathbf{U}$$

$$\langle B_n \widehat{=} \det(\mathbf{B}_n) > 0 \quad \text{for} \quad \forall n \in N_0: \quad \text{def. pos}\rangle\,. \tag{3.11c}$$

E.g a cgFB consists of polynomials

$$b_i(t) \widehat{=} \sum_{h=0}^{1} c_h t^h \in b^*(t)\,,$$

then a FS by powers

$$a_j(t) \widehat{=} t^j \in a^*(t)\,. \tag{3.12}$$

b) It is advantageous to choose an orthogonal (og) FB $b_\perp^* \widehat{=} d^*(t)$, or a s.c. orthonormal (on) one $b_1^* \widehat{=} e^*(t)$ because

$$\overset{\cap}{\mathbf{B}} \widehat{=} \left[\overset{\cap}{b}_{ik}\right]_\perp = \langle \overset{\cap}{b}_{ii} \rangle \widehat{=} \overset{\cap}{\mathbf{D}}\,, \qquad \overset{\cap}{\mathbf{B}}_1 \widehat{=} [b_{ik}]_1 = \langle 1 \rangle \widehat{=} \overset{\cap}{\mathbf{E}} (= \mathbf{E})\,. \tag{3.13a,b}$$

E.g. an ogFB consists of the (spherical) Legendre's polynomials so:

$$d^*(t) \ni d_i(t) \widehat{=} \frac{\left[(t^2-1)^i\right]^{(i)}}{2^i i!} = \sqrt{\frac{2}{2i+1}}\, e_i(t) \in e^*(t) \overset{\cap}{\mathbf{D}}^{\frac{1}{2}}\,. \tag{3.14a,b}$$

3.2 a) The (exact, linear) basic product form (BPF) of $a^*(t)$ on $b^*(t)$ is sought so:

$$a^*(t)\hat{=}{}^0[a_j(t)]^m = \left[\sum_{j=0}^{\infty} b_i(t)a_{ij}\right] = [b^*(t)a_j] = [b_n^*(t), b_\nu^*(t)]\begin{bmatrix}\mathbf{A}_n\\ \mathbf{A}_\nu\end{bmatrix} =$$

$$= b_n^*(t)\mathbb{A}_n + b_\nu^*(t)\mathbb{A}_\nu \hat{=} b^*(t)\mathbb{A} \tag{3.21}$$

and its (exact) unknown coefficient $\mathbb{A}$ is obtained from the amplified equ.

$$\overset{\cap}{\mathbb{C}}\hat{=}(b, a^*) = (b, b^*)\mathbb{A}\hat{=}\overset{\cap}{\mathbb{B}}\,\mathbb{A} \quad \text{so} \quad \mathbb{A} = \overset{\cap}{\mathbb{B}}^{-1}\overset{\cap}{\mathbb{C}}\hat{=}\int_{(T)} \beta(t)a^*(t)\mathrm{d}t \tag{3.22}$$

with the inverse (i)FB

$$\beta(t) = \mathbf{B}^{-1}b(t) \in L_T^2 \quad \text{at} \quad (\beta, \beta^*)\hat{=}\overset{\cap}{\mathbf{B}}^{-1}. \tag{3.23}$$

b) The approximate BPF of $a^*(t)$ on $b_n^*(t) \subset b^*(t)$ is searched as

$$a^*(t) \approx b_n^*(t)\tilde{\mathbf{A}}_n \hat{=} \tilde{a}_n^*(t)$$

and the unknown $\tilde{\mathbf{A}}_n$ by minimizing the error norm dyad (END) $\overset{\cap}{\mathbf{H}}_n(\tilde{\mathbf{A}}_n)$ with the error

$$h^*(t)\hat{=}[a^*(t) - a_n^*(t)] + [a_n^*(t) - \tilde{a}_n^*(t)]\hat{=}a_\nu^*(t) + \Delta\tilde{a}_n^*(t):$$

$$\overset{\cap}{\mathbf{H}}_n(\tilde{\mathbf{A}}_n)\hat{=}(h_n, h_n^*) =$$

$$= \mathbf{A}_\nu^*\overset{\cap}{\mathbf{B}}_\nu\mathbf{A}_\nu - \mathbf{A}_\nu^*\overset{\cap}{\mathbf{B}}_{\nu n}\Delta\tilde{\mathbf{A}}_n - \Delta\tilde{\mathbf{A}}_n^*\left[\overset{\cap}{\mathbf{B}}_{\nu n}\mathbf{A}_\nu - \overset{\cap}{\mathbf{B}}_n\Delta\mathbf{A}_n\right] = \min! \tag{3.26a}$$

Its necessary condition and optimal solution

$$\mathring{\tilde{\mathbf{A}}}_n \quad \left(\text{at} \quad \Delta\mathring{\tilde{\mathbf{A}}}_n = \mathbf{A}_n - \mathring{\tilde{\mathbf{A}}}_n \quad \text{and} \quad \nu\hat{=}\Delta n \to \infty\right) \tag{3.26b}$$

are as follows:

$$\dot{\overset{\cap}{\mathbf{H}}}_n(\mathring{\tilde{\mathbf{A}}}_n)\hat{=}\overset{\cap}{\mathbf{B}}_{n\nu}\mathbf{A}_\nu - \overset{\cap}{\mathbf{B}}_n\Delta\mathring{\tilde{\mathbf{A}}}_n = \mathbf{O},$$

$$\underset{\circ}{\tilde{\mathbb{A}}}_n = \mathbb{A}_n - \overset{\cap}{\mathbf{B}}{}_n^{-1} \overset{\cap}{\mathbf{B}}{}_{n\nu} \mathbb{A}_\nu \widehat{=} \mathbb{F}(\mathbb{A}_\nu) . \tag{3.26c}$$

c) This (optimal) approximate coefficient shows the very pleasant permanency

$$\underset{\circ}{\tilde{\mathbb{A}}}_n = \mathbb{A}_n , \quad \text{so} \quad \boldsymbol{a}_n^*(t) - \tilde{\boldsymbol{a}}_n^*(t) = \mathbf{0} , \quad \boldsymbol{h}_n^*(t) = \boldsymbol{a}_\nu^*(t)$$

for the orthogonality of FB

$$\boldsymbol{b}^*(t) = \boldsymbol{d}^*(t) , \quad \text{then} \quad \overset{\cap}{\mathbf{B}} = \overset{\cap}{\mathbb{D}} \quad \text{(diagonal), so}$$

$$\overset{\cap}{\mathbb{D}}{}_{n\nu} = \overset{\cap}{\mathbf{B}}{}_{n\nu} = \mathbf{O} , \tag{3.27a}$$

resulting a more simple form:

$$\left(\underset{\circ}{\tilde{\mathbb{A}}}_n \widehat{=} \right) \mathbb{A}_n = \overset{\cap}{\mathbb{D}}{}_n^{-1} (\boldsymbol{d}_n, \boldsymbol{a}^*) \widehat{=} \overset{\cap}{\mathbb{D}}{}_n^{-1} \int_{(T)} \boldsymbol{d}_n(t) \boldsymbol{a}^*(t) \mathrm{d}t = (\boldsymbol{\delta}_n, \boldsymbol{a}^*) \widehat{=}$$

$$\widehat{=} \int_{(T)} \boldsymbol{\delta}_n(t) \boldsymbol{a}^*(t) \mathrm{d}t , \tag{3.27b}$$

$$\overset{\cap}{\mathbf{H}}{}_n^{\perp} (\mathbb{A}_n) = \mathbb{A}_\nu^* \overset{\cap}{\mathbb{D}}{}_\nu \mathbb{A}_\nu \widehat{=} \sum_{l=n+1}^{\infty} a_\nu^{l\,*} \underset{(\nu)}{\mathrm{d}_{ll}} a_\nu^l = (\mathbb{A}_\nu^* \boldsymbol{d}_\nu, \boldsymbol{d}_\nu^* \mathbb{A}_\nu) \widehat{=}$$

$$\widehat{=} (\boldsymbol{a}_\nu, \boldsymbol{a}^\nu) \widehat{=} \overset{\cap}{\mathbb{A}}{}_\nu , \tag{3.27c}$$

as new $\dfrac{\text{coeff.}}{\text{END}}$ - formula.

In the case of totality, the limit of END at $n \to \infty$ is as follows (with $\overset{\cap}{\mathbb{A}}{}_\nu = \overset{\cap}{\mathbb{A}} - \overset{\cap}{\mathbb{A}}{}_n$):

$$\lim_{n\to\infty} \overset{\cap}{\mathbf{H}}{}_n^{\perp} (\mathbb{A}_n) = \lim_{n\to\infty} (\overset{\cap}{\mathbb{A}} - \overset{\cap}{\mathbb{A}}{}_n) \widehat{=} \overset{\cap}{\mathbb{A}} - \mathbb{A}^* \overset{\cap}{\mathbb{D}} \mathbb{A} = \mathbf{O} , \quad \text{so}$$

$$\overset{\cap}{\mathbb{A}} = \mathbb{A}^* \overset{\cap}{\mathbb{D}} \mathbb{A} . \tag{3.28a}$$

d) On the basis of (3.28a) and $\boldsymbol{a}^*(t) = \boldsymbol{d}^*(t)\mathbb{A} = \boldsymbol{e}^*(t)\mathbb{A}'$, our orthogonal/orthonormal Basis Factorisation Algorithm (og-BFA/on-BFA)

$$\lim_{n\to\infty} \overset{\cap}{\mathbf{H}}{}_n^{\perp} \widehat{=} \lim_{n\to\infty} \left(\overset{\cap}{\mathbb{A}}{}_n - \sum_{i=0}^{n} \overset{\cap}{d}{}_{ii} \boldsymbol{a}_*^i \boldsymbol{a}^i \right) = \overset{\cap}{\mathbb{A}} - \sum_{i=0}^{n} \overset{\cap}{d}{}_{ii} \frac{\boldsymbol{a}_{(i)}^{i*} \boldsymbol{a}_{(i)}^{i}}{\alpha_i \cdot \alpha_i} = \mathbf{O} ,$$

$$\left(\alpha_i \hat{=} \sqrt{\overset{\cap}{d}_{ii}\, \overset{\cap}{a}{}_{ii}^{(i)}} \neq 0\right) \tag{3.29a}$$

gives the coefficient matrix

$$\mathbf{A} \hat{=}_{\infty}^{0} [a^i]^m = \left[\frac{\overset{\cap}{a}{}_{(i)}^{\,i}}{\alpha_i}\right] \hat{=} \mathbf{A}_{\triangledown} = \overset{\cap}{\mathbf{D}}{}^{-1/2} \mathbf{A}'_{\triangledown}$$

(lower triangular [32]).

3.3 Transform an FS into various FBS.

a) Having a FS

$$a^*(t) \hat{=}^0 [a_j(t)]^m \quad \text{given on-FB} \quad b_0^*(t) = e^*(t) \hat{=}^0 [e_i(t)]^\infty$$

in BPF

$$a^*(t) = e^*(t)\mathbf{A}'_0\,, \quad \langle \text{with} \quad \overset{\cap}{\mathbf{C}} \hat{=} (e, a^*) = (e, e^*)\mathbf{A}'_0 \hat{=} \mathbf{E}\mathbf{A}_0 = \mathbf{A}_0 \rangle$$

one can try

α) an unilateral change on-FB:

$$b_1^*(t) = e^*(t) + [a_l(t) - e_k(t)]e^k \quad \text{at} \quad 0 \neq a_{kl} = (e_k, a_l) = \frac{1}{\gamma_{kl}} \tag{3.31}$$

and

β) a bilateral one, namely also a counter-change in FS, too:

$$\mathbf{a}_{(1)}^*(t) = \mathbf{a}^*(t) - [a_l(t) - e_k(t)]e^l\,, \tag{3.32}$$

obtaining so new BPFs

a)

$$a^*(t) = b_1^*(t) \underset{(S)}{\mathbf{A}_1} \tag{3.33}$$

and

b)

$$a_{(1)}^*(t) = b_1^*(t) \underset{(D)}{\mathbf{A}_1} \tag{3.34}$$

with the new coefficient matrices given by our former STA_1 of (1.11a) and DTA_1 of (1.11b), then at the p^{th} din. transform with $\underset{(D)}{\mathbb{A}_p}$ given by our $\widehat{\mathrm{DTA}}_p$ of (1.12a,b). Of course, all their vectors of ∞ dimens. (e.g. $\forall a'_j \in l^2_\infty$) must have finite norms (e.g. $a'^*_j a'_j < \infty$).

(3.35)

b) If one can make a (regular) full bilateral change in on-FB: $b^*_m(t) = a^*(t)$ and in cg-FS (being lin. indep.) $a^*_m(t) = e^*_m(t)$, then it can be written:

$$e^*_m(t)_{(m+1)} = a^*(t)\underset{(D)}{\mathbb{A}'_m} = a^*(t)\underset{(m+1)^2}{\mathbb{A}'^{-1}_0}$$

at

$$\mathbb{E}_m \hat{=} \mathbb{A}'^{*-1}_0 \overset{\cap}{\mathbb{A}}_0 \mathbb{A}'_0\,, \quad \alpha(t) = \overset{\cap}{\mathbb{A}}{}^{-1}_0 a(t) = \mathbb{A}'^{-1}_0 e_m(t) \tag{3.36}$$

at

$$(\alpha, a^*) = \mathbb{A}'^{-1}_0 \mathbb{E}_m \mathbb{A}'_0 \hat{=} \mathbb{E}_m\,, \tag{3.37}$$

consequ. the inverse FS $\alpha(t)$ of the original cg-FS $a(t) = \mathbb{A}'^*_0 e_m(t)$ produced by $\mathbb{A}'^{-1}_0$ ⟨made by our main diagonal $\widehat{\mathrm{DTA}}_{m\dot{+}1}$ of (1.22c)⟩ and on-Fb $e_m(t)$ [32].

c) Let our matricial investigations [33] into the lin. integral equ. (LIE) be mentioned, e.g. to treat the type

$$y(t) - \lambda \int_{(T)} K(t,\tau)y(\tau)\mathrm{d}\tau = x(t) \tag{3.38}$$

with the kernel $K(t,\tau) \hat{=} a^*(t)b(\tau)$, then to approach arbitrary LIE by corresponding lin. alg. equ. (LAE), further on to apply two on-FSs with a certain type etc.

(3.39a,b)

3.4 For their technical applications, there are interesting e.g.

a) Fourier analysis of a multivariate vibrating system n the og-FB $d(t) \hat{=} [\sin k\omega t, 1, \cos k\omega t]$ [32];
b) expansion of solution by eigen-og-FS with a certain (parametrical) hLDE of bar [32];
c) expansion of symmetric kernel by eigen-on-FS at LIE of 2nd kind [32], etc.

4. MAM for Stochastics

4.1 We mention from [41] the random variable (rv) $\xi = g(\omega)$ on $\Omega = \{\omega\}$ with state space/values $X \ni x_i = g(\omega_i)$ ($i = 1, 2, \ldots, n$), the random function (rf) $\xi(t) = g(\omega, t)$ on Ω, for $T = \{t\}$ with realizations $X \supset x_i(t) = g(\omega_i, t)$ and sections $X \supset \xi_j = g(\omega, t_j)$ $(j = 1, 2, \ldots, m)$ at stochastic processes/chains (of contin./discr. X)

(4.11)

then with distribution $F(x, t) \widehat{=} P\left[\underset{(m)}{\xi(t)} < x\right]$ (at $m = 1, 2, \ldots$), stationarity $F(x, t + \vartheta e) = F(x, t)$, expectation $m_\xi(t)$, variance $\sigma_\xi^2(t)$, covariance $c_{\xi\xi'}(t, t')$ etc.;

(4.12)

the vector rf $\boldsymbol{\xi}(t)$ and the Gaussian one.

(4.13)

We refer from [41] to random derivatives, integrals, diff. equs, too.

(4.14)

4.2 a) We proposed the random basic product (rbp) of a scalar rf $\xi(t)$ [42]:

$$\overset{\circ}{\xi}(t) \widehat{=} \xi(t) - m_\xi(t) = \sum_{k=1}^{\infty} x_k(t) \xi_k \widehat{=} \boldsymbol{\xi}^* \boldsymbol{x}(t) \qquad (4.21)$$

$$\left\langle \text{at} \quad l_T^2 \ni \boldsymbol{x}(t), \quad \boldsymbol{\xi} = ? \right\rangle$$

with parameters

$$m_\xi \widehat{=} [m_{\xi_k}] = \mathbf{0}, \qquad (4.22a)$$

$$\mathbf{C}_{\xi\xi} \widehat{=} [c_{\xi_k \xi_l}] = \langle c_{\xi_k \xi_k} = v_{\xi_k} \rangle \widehat{=} \mathbf{V}_\xi^\backslash \qquad (4.22b)$$

and functions

$$c_{\xi\xi'}^*(t, t') = \boldsymbol{x}^*(t) \mathbf{V}_\xi^\backslash(t'), \qquad (4.22c)$$

$$v_\xi(t) \widehat{=} \sigma_\xi^2(t) = c_{\xi\xi}(t, t), \qquad (4.23a)$$

$$\left\langle \text{at} \quad V_\xi^\backslash = \prod_k v_{\xi_k} > 0 \right\rangle, \qquad (4.23b)$$

$$c_{\xi'\xi}^*(t) = \boldsymbol{x}^*(t) \mathbf{V}_\xi^\backslash. \qquad (4.23c)$$

b) One can often approach it [42]:

$$\overset{\circ}{\xi}(t) \approx \overset{\circ}{\xi}_n(t) = \sum_{k=1}^{n} \xi_k z_k(t) \widehat{=} \boldsymbol{\xi}_n^* \boldsymbol{z}_n(t)$$

with demand

$$0 \leq \sigma^2_{\rho_n}(t) \widehat{=} M\left[\rho^2_n(t)\right] = \min!\,, \qquad \left\langle \text{at} \quad \rho_n(t) \widehat{=} \overset{\circ}{\xi}(t) - \overset{\circ}{\xi}_n(t)\right\rangle ,$$

having min. $\sigma^2_{\rho_n}(t)$ in case of permanency $z_n(t) = x_n(t) \subset x(t)$ and $\breve{\sigma}^2_{\rho_n}(t) \xrightarrow[\text{at } n\to\infty]{} 0$ for totality.

(4.24)

c) The Rbp for a vector rf $\overset{\circ}{\boldsymbol{\xi}}(t)$ is also constructed in our [42].

4.3 a) From the (probable/statistical) sample

$$\overset{\circ}{\xi}{}^{*}(t) \widehat{=} {}^{1}\left[\overset{\circ}{\xi}(t_j)\right]^m // \overset{\circ}{\Sigma}(t) \widehat{=} {}^{1}_{n}\left[\overset{\circ}{x}_i(t_j)\right]^m,$$

the (indep./orthogon.) Rbv/m

$$\boldsymbol{\xi}^{*} \widehat{=} {}^{1}[\xi_j]^m // \boldsymbol{\Sigma}_{\perp} \widehat{=} {}^{1}_{n}[x_{ij}]^m$$

of the requested properties

$$(m_\xi = \mathbf{0}\,, \quad \mathbf{C}_{\xi\xi} = \mathbf{V}_\xi // \mathbf{M}_\Sigma = \mathbf{O}\,, \quad \mathbf{C}_{\Sigma\Sigma} = \mathbf{V}_\Sigma)\,,$$

can be produced as

$$\boldsymbol{\xi}^{*} = \overset{\circ}{\boldsymbol{\xi}}{}^{*}(t)\mathbf{Y}_{\triangledown}^{-1}(t)$$

$$\left\langle \text{with} \quad \mathbf{Y}_{\triangleleft}(t) = 1 \quad \text{and} \quad y_{ij}(t) \underset{(i \geq j)}{=} \delta_{ij} \right\rangle$$

and

$$\boldsymbol{\Sigma}_{\perp} = \overset{\circ}{\boldsymbol{\Sigma}}(t)\mathbf{Y}_{\triangledown}^{-1}(t) \tag{4.31a,b}$$

suitably by our algorithms ITA/OTA of recurrent formulas [42]

$$\overset{\circ}{\boldsymbol{\xi}}{}^{*}{}_{j}(t) = \overset{\circ}{\boldsymbol{\xi}}{}^{*}{}_{j-1}(t) - \xi_j y^j(t) // \overset{\circ}{\boldsymbol{\Sigma}}_j(t) = \overset{\circ}{\boldsymbol{\Sigma}}_{j-1}(t) - x_j y^j(t)$$

$$\langle \text{at} \quad j = m \quad \text{with} = \mathbf{0}//O\rangle\,. \tag{4.32a,b}$$

OTA_m can be used for forecasting, too.

4.4 We gave useful MM also for the Markovian chains e.g. its vector (v) vAE, vDaE and vDE with basic properties

$$p(t+\tau) = \mathbf{P}(\tau|t)p(t)\,, \qquad \frac{\Delta p}{\Delta t} = \frac{\Delta \mathbf{P}}{\Delta t}p(t)\,; \qquad \dot{p}(t) = \dot{\mathbf{P}}(|t)p(t)$$

$$\Big\langle \text{for} \quad T_{\text{cont}} \quad \text{only;} \quad p(\vartheta) \geq \mathbf{0}\,, \quad \mathbf{P}(\tau|t) \geq \mathbf{O}\,; \quad e^{*}p(\vartheta) = 1\,,$$

$$\Delta t \widehat{=} \tau\,, \qquad e^{*}\mathbf{P}\;(\tau|t) = e^{*}\,; \qquad \lim_{\tau\to+0}\mathbf{P}(\tau|t) = \mathbf{E}\,; \qquad e^{*}\dot{\mathbf{P}}(|t) = \mathbf{0}^{*}\Big\rangle\,,$$

then occasional properties (superposed sequentially) for practical case homogeneous: $\dot{\mathbf{P}}_{|}$, $\cap$ rare: $\dot{\mathbb{P}}_{\llcorner}$, $\cap$ posteffectless: $\dot{\mathbb{P}}_{\searrow}$ (biprimitive), $\cap$ increasing: $\dot{\mathbb{P}}_{\searrow}$ (primitive); ergodic:

$$p(t) \to p\,, \qquad \text{so} \qquad \dot{p}(t) \to \mathbf{0} \qquad \text{at} \qquad t \to \infty\,. \tag{4.46a–e}$$

4.5 As their technical applications, let e.g.

a) various Markov chains be mentioned for mass service, demography, storage, inventory etc. problems [43, 44];
b) optimization of multivariate informational systems [45, 48];
c) analysis of various stochastic systems and their processes [42, 46, 47] etc.

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