PERIODICA POLYTECHNICA SER. TRANSP. ENG. VOL. 21, NO. 2, PP. 125–136 (1993)

INFLUENCE OF MOTION OF THE LOAD DISTRIBUTED OVER A GIVEN LENGTH ON BENDING MOMENT AND SHEAR FORCE IN A TRACK MODELLED AS TIMOSHENKO BEAM

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Received: Nov. 10, 1992

Abstract

The object of consideration is a track modelled by Timoshenko beam resting on inertia-less, viscoelastic foundation of non-linear characteristic. The beam is subjected to load distributed over a given length and moving at a constant velocity. Stationary vibrations of the beam are described by two ordinary differential equations of the fourth order determined from the system of two partial differential equations.

To analyze the influence of non-linearity, the method of approximation presented in [2] was applied. The study of the load motion velocity on extreme values of bending moments and shear forces in a track was considered taking into account the assumed types of non-linearities.

Keywords: track model, Timoshenko beam, vibrations.

1. Introduction

The railway track has a significant influence on the rail vehicles behaviour during motion. The more exact analysis of the track properties and in particular — the non-linear characteristic of foundation — is necessary in the case of the increasing velocity.

The paper is devoted to the problem of steady-state vibrations of the infinite beam of Timoshenko resting on a visco-elastic semi-space of non-linear characteristic and subjected to a load distributed over a given length, moving at a constant velocity. It is assumed that a track and train are modelled by beam and moving load, respectively.

Problems of dynamics of continuous systems under travelling load are related with noise generation. Particular attention is paid to high-speed transportation systems. The mathematical model of the system contains piece-wise characteristic of elastic force and dry friction of foundation. An approximate procedure of the solution is proposed for the case of beam interacting with non-linear foundation.

The analytical solution to linear case and load described by Heaviside function is provided in [1]. A vast survey of papers devoted to this type of problems can be found in a few papers, e. g. [3]. The aim of this paper is to obtain approximate solution for the case of beam interacting with non-linear visco-elastic foundation. To follow the way of the analysis of dynamic behaviour of the system, it is necessary to determine the response of the system to linear case. Then, for the form of travelling waves, one has to investigate into the influence of no-linear (bilinear) characteristic of the foundation on the range of frequencies and the qualitative changes of the solution. An approximate procedure, as in [2] and [4], is proposed for the solution of this problem. Some numerical results of the investigation will be presented.

2. The Differential Equations of Motion of the Beam

The motion of the Timoshenko beam in the Cartesian fixed coordinate system $\overline{O}\overline{x}_1\overline{y}$, in which $\overline{O}\overline{x}_1$ is axis of beam and $\overline{O}\overline{y}$–axis is downward, is described by the partial differential equations:

$$\kappa AG\left(\frac{\partial^2 y}{\partial \overline{x}_1^2}-\frac{\partial \psi}{\partial \overline{x}_1}\right)-\rho\frac{\partial^2 y}{\partial t^2}+p=0\,,$$

$$EI\frac{\partial^2 \psi}{\partial \overline{x}_1^2}+\kappa AG\left(\frac{\partial y}{\partial \overline{x}_1}-\psi\right)-\rho r^2\frac{\partial^2 \psi}{\partial t^2}-N\frac{\partial y}{\partial \overline{x}_1}+m=0\,, \tag{1}$$

with the following notation:

y – displacement in $\overline{y}$ direction,
ψ – angle of rotation of the beam due to pure shear,
EI – flexural rigidity,
κ – shear coefficient of Timoshenko,
A – cross-sectional area with moment of inertia I,
ρ – constant linear density,
N – constant tensile force,
t – time,
$m=m(\overline{x}_1,t)$ – external continuous load moment,
$p=p(\overline{x}_1,t)$ – external continuous load,
$r=\sqrt{\frac{I}{A}}$.

It is assumed:

$$p=p_v-p_0 \qquad \text{and} \qquad m=m_v-m_0\,,$$

where:
$p_v\,, m_v$ – given moving continuous load and continuous moment,
$p_0\,, m_0$ – load resulting from the reaction of beam foundation.

The assumed non-linear characteristic of foundation:

$$p_0 = c_p y + b_p \frac{\partial y}{\partial t} + p_0^*\,,$$

$$m_0 = c_m \psi + b_m \frac{\partial \psi}{\partial t} + m_0^*\,, \tag{2}$$

where:
c_p, c_m, b_p, b_m – coefficients of elasticity and damping of the linear characteristic of foundation,
p_0^*, m_0^* – non-linear terms.

By introducing the Cartesian coordinate system, Ox_1y is related to the moving loading by

$$\overline{y} = y \qquad \text{and} \qquad \overline{x}_1 = x_1 + vt\,,$$

where v – constant velocity of loading motion, under the assumption that the solutions are stationary in the rectangular coordinate system Ox_1y connected with the load front, the equations of motion of the beam (1) are as follows:

$$\kappa AG\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x_1^2} - \frac{\mathrm{d}\psi}{\mathrm{d}x_1}\right) - \rho v^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x_1^2} + p_v - c_p y + b_p v \frac{\mathrm{d}y}{\mathrm{d}x_1} - p_0^* = 0\,,$$

$$EI\frac{\mathrm{d}^2\psi}{\mathrm{d}x_1^2} + \kappa AG\left(\frac{\mathrm{d}y}{\mathrm{d}x_1} - \psi\right) - \rho r^2 v^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}x_1^2} - N\frac{\mathrm{d}y}{\mathrm{d}x_1} + m_v - c_m\psi +$$
$$+ b_m v \frac{\mathrm{d}\psi}{\mathrm{d}x_1} - m_0^* = 0\,. \tag{3}$$

Furthermore, as in [1], new dimensionless variables are introduced:

$$x = \frac{x_1}{r} \qquad \text{and} \qquad u = \frac{y}{y_s}\,,$$

where $y_s > 0$ is a given value, and the dimensionless coefficients are:

$$V = \left(\frac{v}{r}\right)\left(\frac{\rho}{c_p}\right)^{0.5}\,,$$

$$V_1 = \frac{\left(\frac{\kappa AG}{c_p}\right)^{0.5}}{r}, \qquad V_2 = \frac{\left(\frac{EA}{c_p}\right)^{0.5}}{r},$$

$$b = 0.5b_p(c_p\rho)^{-0.5}, \qquad B = \frac{0.5b_m(c_p\rho)^{-0.5}}{r^2},$$

$$S = \frac{N}{r^2 c_p}, \qquad C = \frac{c_m}{r^2 c_p}.$$

Similarly, dimensionless load can be expressed as follows:

$$\overline{p}_v = \frac{p_v}{p_s}, \qquad \overline{m}_v = \frac{m_v}{p_s r},$$

$$\overline{p}_0 = \frac{p_0}{p_s}, \qquad \overline{p}_0^* = \frac{p_0^*}{p_s},$$

in which

$$p_s = c_p y_s .$$

Assuming $m_0^* = 0$ and eliminating ψ from *Eqs.* (3), we obtain:

$$F\left[u(x)\right] + f\left[u(x)\right] = g_m\left[\overline{m}_v(x)\right] + g_p\left[\overline{p}_v(x)\right], \tag{4}$$

where F, f, g_m, and g_p denote differential operators:

F, g_m, g_p – linear,
f – non-linear.

Denoting $(') = \frac{\mathrm{d}}{\mathrm{d}x}$:

$$F\left[u(x)\right] = D\left(V^2\right)u^{(4)} - 2V\left[B\left(V^2 - V_1^2\right) + b\left(V^2 - V_2^2\right)\right]u^{(3)} +$$

$$+\left[V^2\left(V_1^2 + 1 + 4bB + C\right) - (S+C)V_1^2 - V_2^2\right]u'' -$$

$$-2V\left[b\left(V_1^2 + C\right) + B\right]u' + \left(V_1^2 + C\right)u,$$

$$f[u] = g_p\left[\overline{p}_o^*\right], \qquad \overline{p}_0^* = \overline{p}_0^*\left(u, u'\right),$$

$$g_m\left[\overline{m}_v(x)\right] = -V_1^2\overline{m}_v',$$

$$g_p\left[\overline{p}_v(x)\right] = \left(V^2 - V_2^2\right)\overline{p}_v'' - 2BV\overline{p}_v' + \left(V_1^2 + C\right)\overline{p}_v,$$

$$D\left(V^2\right) = \left(V^2 - V_1^2\right)\left(V^2 - V_2^2\right).$$

To perform thorough analysis, certain non-linearity has been taken into account. The non-linear characteristic of foundation in formula (2), related to the support of track, takes the form:

$$p_0 = p_e + p_d \,,$$

where p_e and p_d are elastic and damping components, respectively. Taking into consideration bilinear character of elasticity (*Fig. 1*) and linear and dry friction, we can assume:

$$p_e = \begin{cases} p_n + \xi c_p \left(y_n - y\right) & \text{for } y \le y_n \,, \\ c_p y & \text{for } y > y_n \,, \end{cases}$$

$$p_d = b_p \frac{\partial y}{\partial t} + p_T \operatorname{sgn} \frac{\partial y}{\partial t} \,,$$

where: $y_n < 0, \xi, p_T > 0$ – are given values,

$$p_n = c_p y_n < 0 \,.$$

Coefficient ξ describes inclination of piece-wise linear characteristic of elasticity. For $\xi = 1$, bilinear character of curve disappears.

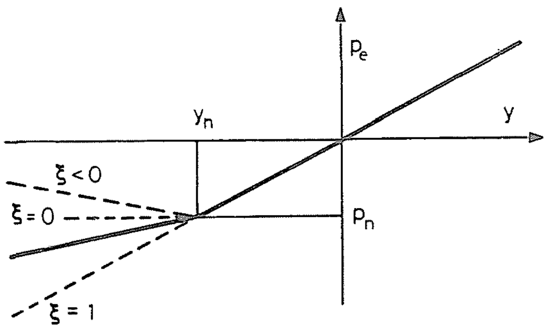

Fig. 1. Characteristic of elasticity

Thus non-linear component for *Eq.*(4) is expressed by the formula:

$$\overline{p}_0^* = (1 - \xi)\left(u_n - u\right) 1 \left(u_n - u\right) - \overline{p}_T \operatorname{sgn} V \operatorname{sgn} u' \,, \tag{5}$$

where: $1(z)$ – Heaviside function and

$$\overline{p}_T = \frac{p_T}{p_s} \,.$$

3. Formulas of Bending Moment and Shear Force

Bending moment and shear force in Timoshenko beam are described by the formulas:

$$M = -EI\frac{\mathrm{d}\psi}{\mathrm{d}x_1} \qquad \text{and} \qquad T = \kappa AG\left(\frac{\mathrm{d}y}{\mathrm{d}x_1} - \psi\right). \tag{6}$$

For the dimensionless variables x and u, dimensionless moment and force are introduced:

$$\overline{M} = \frac{M}{M_0} \qquad \overline{T} = \frac{T}{T_0},$$

where:

$$M_0 = EAy_s \qquad \text{and} \qquad T_0 = \kappa AG\left(\frac{y_s}{r}\right).$$

Then formulas (6) take the form:

$$\overline{M} = \left[\left(V^2 - V_1^2\right)u'' - 2bVu' + u - \overline{p}_v + \overline{p}_0^*\right]V_1^{-2},$$

$$\overline{T} = u' - \overline{\psi},$$

where $\overline{\psi} = (r/y_s)\,\psi$ is expressed by formula:

$$\overline{\psi} = \Big\{D\left(V^2\right)u^{(3)} - 2V\left[B\left(V^2 - V_1^2\right) + b\left(V^2 - V_2^2\right)\right]u'' +$$
$$+\left[V^2(1+4bB) - V_1^2\left(V_1^2 - S\right) - V_2^2\right]u' + 2BV\left(\overline{p}_v - \overline{p}_0^* - u\right) +$$
$$+\left(V^2 - V_2^2\right)\left(\overline{p}_0^* - \overline{p}_v\right)'\Big\}\left[V_1^2\left(V_1^2 + C\right)\right]^{-1}.$$

The presented expressions allow to calculate bending moment and shear force for known solution $u = u(x)$.

4. Solution to the Linear Case

Let

$$\overline{m}_v = \overline{p}_0^* = 0 \qquad \text{and} \qquad \overline{p}_v = q_0 1(-x),$$

where:

$$q_0 = \frac{q}{p_s}.$$

By taking into account that the linear approximation of *Eq.*(4) can be written in the form:

$$F\left[u(x)\right] = g_m\left[\overline{m}_v(x)\right] + g_p\left[\overline{p}_v(x)\right], \tag{7}$$

the solution $u_\infty(x)$ meeting the boundary conditions of infinite beam, has been achieved by the application of the method of the Fourier and Laplace transforms and the theorem of convolution to the equation of motion. This solution is expressed by formula:

$$u_\infty(x) = q_0 1(-x) + \sum_{j+1}^{4} 1\left(-x \operatorname{sgn} n_j\right) b_j h_j(x) ,$$

where:

$$n_j = \operatorname{Re} s_j ,$$

s_j – roots of the characteristic polynomial,
b_j – constants.

Functions $h_j(x)$ have the forms:

$$\exp\left(n_j x\right) \quad \text{for} \quad \operatorname{Im} s_j = 0$$

or

$$\exp\left(n_j x\right)\cos\left(k_j x\right) \quad \text{and} \quad \exp\left(n_j x\right)\sin\left(k_j x\right)$$

for

$$k_j = \left|\operatorname{Im} s_j\right| > 0 .$$

For the dimensionless moving load of constant value on a given section $x \in \langle x_a; x_b \rangle$:

$$\overline{p}_v(x) = q_0 \left\{1\left[-\left(x - x_b\right)\right] - 1\left[-\left(x - x_a\right)\right]\right\} ,$$

the solution $u_0(x)$ takes the form:

$$u_0(x) = u_\infty\left(x - x_b\right) - u_\infty\left(x - x_a\right) . \tag{8}$$

5. Analysis of the Non-Linear Equation of Beam

The approximate solution of non-linear equation (4) was obtained by the method given in [4] for the Bernoulli–Euler beam. This method was generalized for the Timoshenko beam in [2].

The stationary solution of non-linear equation (4) is approximated by the successive terms of the functional series $u_k(x)$, $k = 0, 1, 2, \ldots$. Function $u_0(x)$ is described by (8). To determine successive functions $u_k(x)$, the series of linear differential equations is used:

$$F\left[\Delta u_{k+1}(x)\right] = g_p\left[\overline{p}_{vk}(x)\right] , \qquad k = 0, 1, 2, \ldots , \tag{9}$$

where:

$$\Delta u_{k+1}(x) = u_{k+1}(x) - u_0(x),$$
$$g_p\left[\overline{p}_{vk}(x)\right] = -f\left[u_k(x)\right],$$
$$\overline{p}_{vk}(x) \approx -\overline{p}_0^*\left[u_k(x), u_k'(x)\right].$$

The *Eqs.*(9) are similar to (7).

Approximation results from the fact that the form $\overline{p}_{vk}(x)$ should enable the usage of known analytical solutions of linear differential equation. Accordingly, $\overline{p}_{vk}(x)$ is taken in the form of limited sum of dimensionless moving continuous loads of constant values on given sections. It allows us to use the solutions of form (8) and the principle of superposition for linear differential equations (9).

The solution was estimated as based on the following criterion:

$$\max_{x\in\langle-\infty;+\infty\rangle} |u_{k+1}(x) - u_k(x)| \le \varepsilon, \qquad k = 0, 1, 2, \ldots,$$

where $\varepsilon > 0$ – is considered as the permissible error.

6. Results of the Numerical Analysis

The influence of the velocity of moving load for chosen parameters characterizing non-linearity on bending moment and shear force in beams has been considered.

In *Fig. 2,* two exemplary functions of dimensionless displacements $u(x)$ determined numerically for $V = 18$ and other parameters:

$$q_0 = 2, \qquad V_1 = 45, \qquad V_2 = 90, \qquad b = 0.2,$$

$$x_a = -100, \qquad x_b = B = C = S = 0, \tag{10}$$

for linear and non-linear cases are presented.

For curve k_1 (linear case) it was assumed:

$$\overline{p}_0^* = 0, \tag{11}$$

for curve k_2 (non-linear case) the non-linearity (5) has been taken into account for the following parameters:

$$\xi = 0, \qquad \overline{p}_T = 0, \qquad u_n = -0.14 \qquad \text{and} \qquad \varepsilon = 0.02. \tag{12}$$

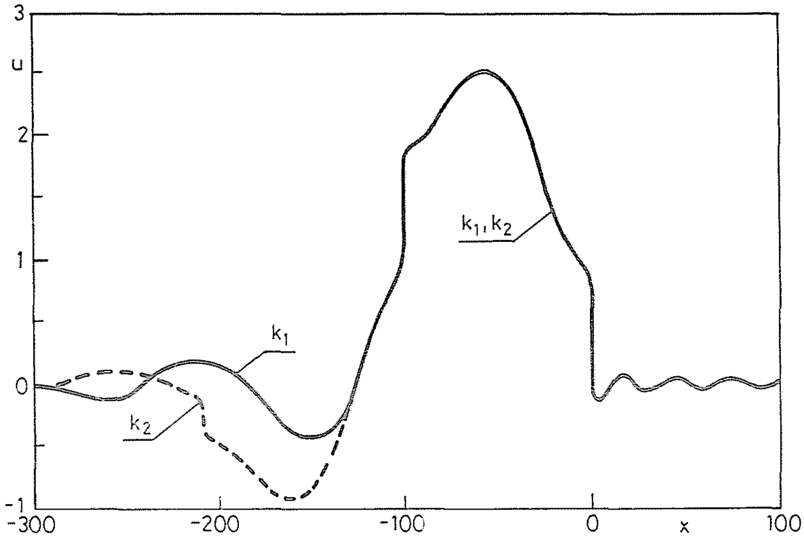

Fig. 2. Displacement curves

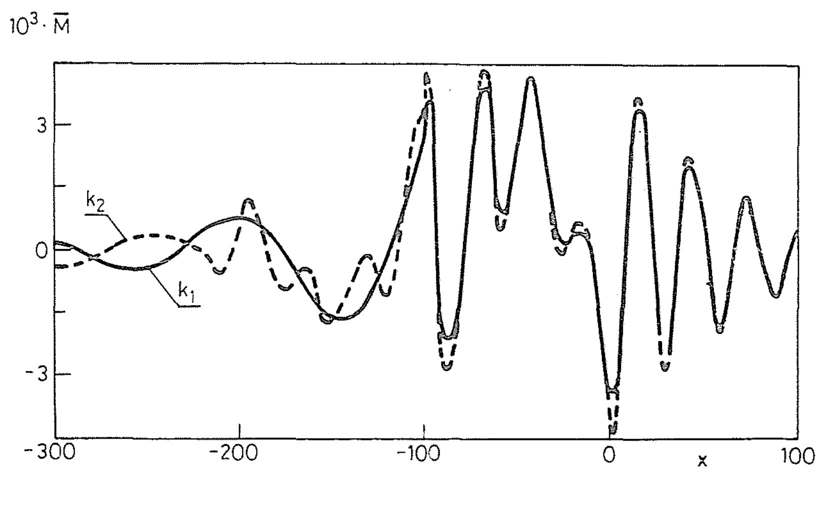

Fig. 3. Moment curves

In *Figs. 3* and *4*, respectively, dimensionless functions: bending moment and shear force versus x for $V = 18$ are presented. For curves k_1, condition (11) was assumed and parameters (10) were chosen. For curves k_2, parameters (10) and (12) were assumed.

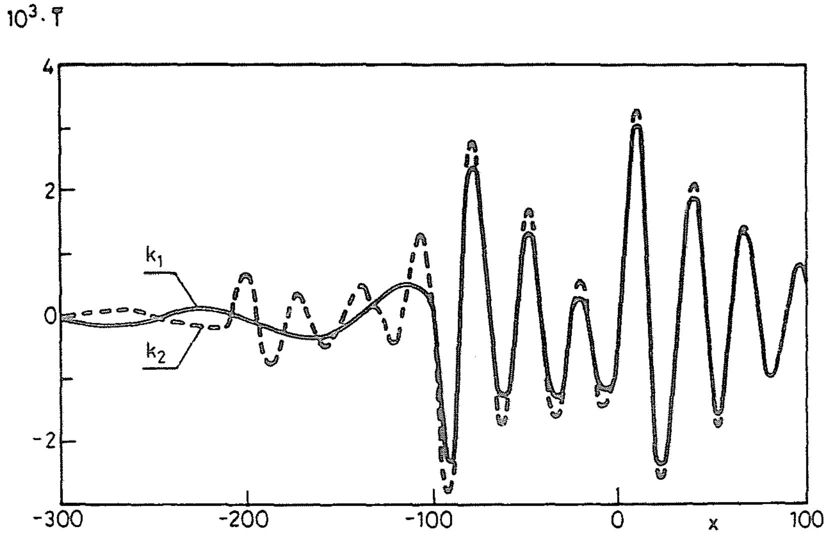

Fig. 4. Shear force curves

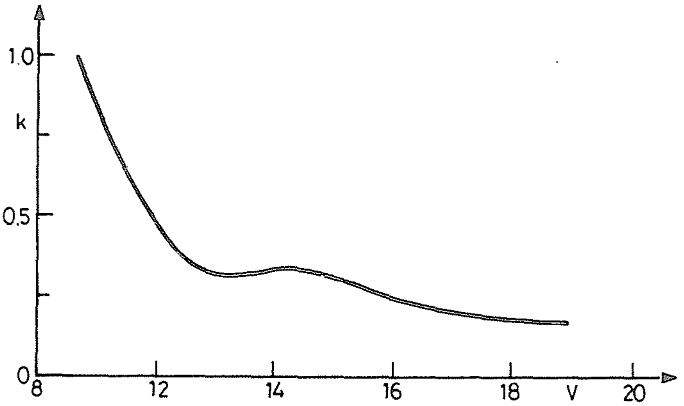

Fig. 5. The variation of k_1 on V

In *Fig. 5*, function

$$k(V) = \frac{u_n}{u_m}$$

is presented, where:

$$u_m = \max_{x \in (-\infty;+\infty)} u_0(x),$$

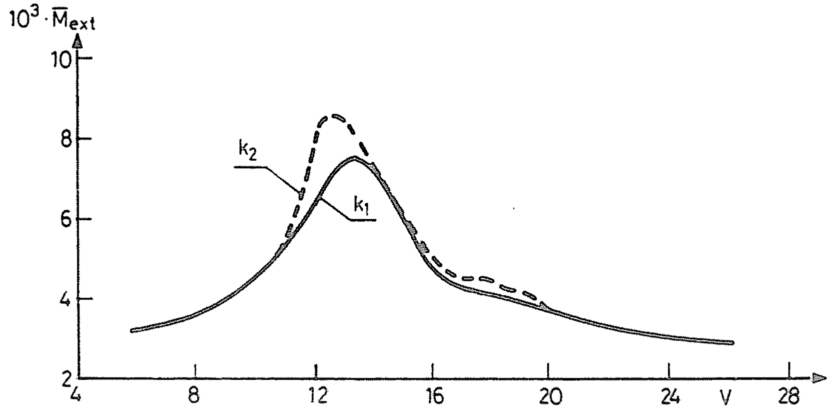

Fig. 6. Extreme values of moments

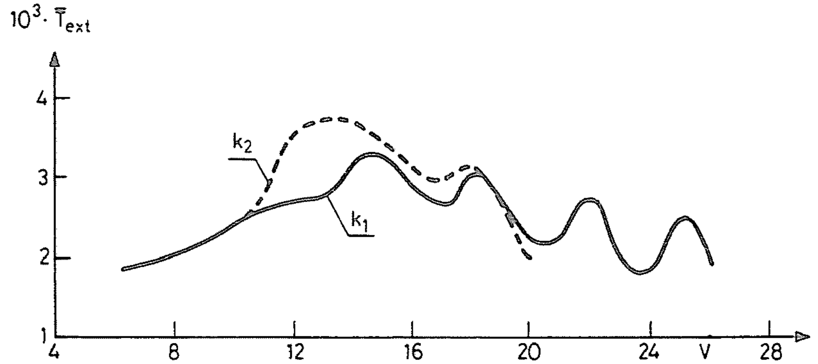

Fig. 7. Extreme values of shear forces

($u_0(x)$ for $V = 18$ is presented by curve k_1 in *Fig. 2*).
In *Figs. 6* and *7*, extreme dimensionless values of bending moments and shear forces in the beam versus dimensionless velocity V are presented, respectively:

curve k_1 for the condition (11) and parameters given by (10) – linear case

and curve k_2 for parameters according to (10) and (12) – non-linear case.

7. Concluding Remarks

In this paper, the proposal of investigation into the train motion on railway track has been presented based on the infinite Timoshenko beam model subjected to a load distributed over a given length, moving at a constant velocity.

The presented method takes into account non-linearities of foundation and enables to investigate into the influence of parameters of a mechanical system on wave propagation in a beam under moving load . The knowledge on the properties of solution allows to control the system and minimize noise generation.

The considered problems concerning the mechanical models may find applications in modern systems of rail transportation.

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