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LATERAL DYNAMICS OF RAILWAY WHEELSETS RUNNING ALONG A CONTINUOUS ELASTIC SUPPORT TRACK

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Abstract

There is an important role in the lateral dynamical processes of the railway wheelset running along an elastic support track. This investigation shows a method for examining the wheel/track lateral dynamical model when the track is described as a homogeneous discrete elastic support beam, and the lateral creep force as a function of the lateral velocity of the wheelset and the track, exciting the wheelset, and the track, too. So the wheelset and the track form a lateral dynamical system. It can be described by a second-order general differential equation, and two fourth-order partial differential equations for the wheelset and the tracks, and those can be solved by using a linear creep force excitation as the connection conditions between the joint differential equations.

Mr. Chairman, Dear Colleagues!

I will show you a method to investigate into a dynamical system from the point of view of stability. The system consists of a railway wheelset running at velocity v, and a left- and right-side rail as an elastic support beam with stiffness coefficient k. There is a lateral force acting on the wheelset and the rails, too, which can be written by using the linear Kalker method, so it is a creep force. The wheelset can move only in lateral direction, but it cannot rotate around the vertical axle z. The conicity of the wheel profile is not considered in this investigation. The rails are beams with anchored ends at the infinity, and can be derived continuously at least twice everywhere. The model of the wheel-rail system can be seen in *Fig. 1.* The wheelset has an elastic support with damping k_k, stiffness s_k, and mass m_k, the lateral displacement is function $y_k(t)$ considering a moving coordinate system with velocity v. The differential equation system in *Fig. 1*:

$$m_k\ddot{y}_k(t) = -k_k\dot{y}_k(t) - s_k y_k(t) + f_{yl}(t) + f_{yr}(t)\,, \qquad (1)$$

$$IE\frac{\partial^4 y_l(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y_l(x,t)}{\partial t^2} + k y_l(x,t) = f_{yl}(t)\delta(x - vt) \qquad (2)$$

and boundary conditions:

$$\lim_{x\to\pm\infty} y_l(x,t) = 0\,,$$

$$IE\frac{\partial^4 y_r(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y_r(x,t)}{\partial t^2} + ky_r(x,t) = f_{yr}(t)\delta(x-vt) \tag{3}$$

and

$$\lim_{x\to\pm\infty} y_r(x,t) = 0\,,$$

where m_k – the mass of the wheelset,
k_k – damping of the lateral support of the wheelset,
s_k – stiffness of the lateral support of the wheelset,
$f_y(t)$ – coupling force between the wheel and the rail-creep force,
$y_k(t)$ – lateral displacement of the wheelset, as a function of time,
$y(x,t)$ – lateral displacement of the left- and right-side rails at points x and t,
I – inertia of the cross-section of the rail for axle z,
E – Young modulus of the material of the rail,
ρ – density of the material of the rail,
A – area of the cross-section of the rail,
$f_y(t)$ – coupling force between the wheel and the rail-creep force,
k – stiffness of the elastic support [N/m^2].

The coordinate system of the rails is a stationary coordinate system, and creep force $f(t)$ connects the wheelset and the rails. The essence of the solution is to solve the second-order differential equation for the wheelset and the fourth-order partial differential equation of rails, when $f(t)$ is an unknown force, and finally, force $f(t)$ is a function of the first derivative of the relative displacement of the wheelset and the rail at point $x = vt$. The connection condition is

$$f_y(t) = -\frac{f_{22}}{v}\left(\frac{\mathrm{d}y_k(t)}{\mathrm{d}t} - \frac{\partial y(x,t)}{\partial t}\right)\,, \tag{4}$$

where f_{22} – Kalker coefficient,
v – velocity of the wheelset.

Since the displacement of the wheelset is a function of time only, it was examined in a moving coordinate system, and the displacements of the

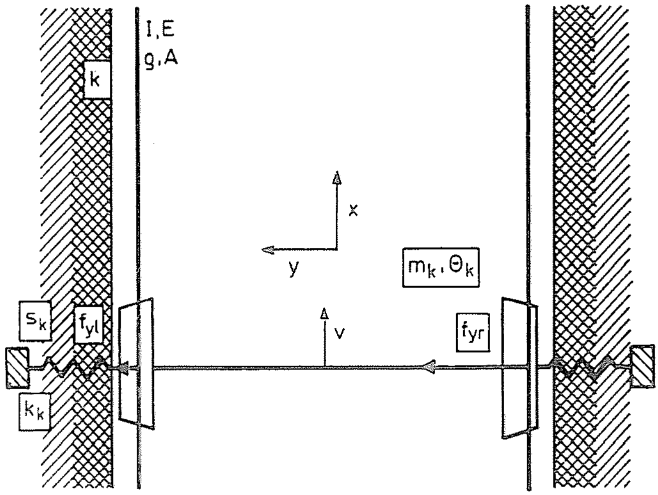

Fig. 1. The wheel-rail system

rails were examined in a stationary coordinate system. To transform it into the moving system, it should be considered at point $x = vt$.

Since our system is a linear one, for obtaining a simpler solution, it can be supposed, that the creep force has a special form, as follows:

$$f_y(t) = C_f e^{\lambda t}, \tag{5}$$

where C_f – complex constant,
λ – eigenvalue of the wheelset-track system.

Before substituting it intodifferential equation system, (1) – (3) the *Laplace transform* method is used for (2) and (3) by applying the following general form:

$$Y(p,t) = \int_0^\infty y(x,t)e^{-px}\mathrm{d}x\,. \tag{6}$$

If it applies, then the following second order differential equation is obtained:

$$\rho A\ddot{Y}(p,t) + \left(IEp^4 + k\right)Y(p,t) = C_f e^{\lambda t} e^{-pvt}, \tag{7}$$

and it can be solved formally in the form:

$$Y(p,t) = De^{(\lambda - pv)t}, \tag{8}$$

where

$$D = \frac{C_f}{IEp^4 + \rho Av^2p^2 - 2\rho A\lambda vp + k + \rho A\lambda^2}, \tag{9}$$

by and substituting it into (8), it will have the form:

$$Y(p,t) = \frac{C_f}{IEp^4 + \rho Av^2p^2 - 2\rho A\lambda vp + k + \rho A\lambda^2} \cdot e^{(\lambda - pv)t} . \qquad (10)$$

To use it for the left- and right-side rail [it means, to use it for (2) and (3)], we obtain the solution at the Laplace plane in a form:

$$Y_{sl}(p,t) = \frac{C_f}{IEp^4 + \rho Av^2p^2 - 2\rho A\lambda vp + k + \rho A\lambda^2} \cdot e^{(\lambda - pv)t} , \qquad (11)$$

$$Y_{sr}(p,t) = \frac{C_f}{IEp^4 + \rho Av^2p^2 - 2\rho A\lambda vp + k + \rho A\lambda^2} \cdot e^{(\lambda - pv)t} . \qquad (12)$$

By applying (5) for (1), the solution will have the form of the following function at the coordinate system of the wheelset:

$$y_k(t) = \frac{2C_f}{m_k\lambda^2 + k_k\lambda + s_k} \cdot e^{\lambda t} . \qquad (13)$$

To write it into the standing coordinate system, notice that $y_k(t)$ is zero when $x \neq vt$, and is non-zero when $x = vt$. This function can be characterized by the Dirac delta function,

$$y_k(x,t) = y_k(t)\delta(x - vt) . \qquad (14)$$

As it can be seen, the new $y_k(x,t)$ is a function of time and distance x, so it can be transformed onto the Laplace plane, like the displacement of the rails by using the form:

$$Y_k(p,t) = L_x\{y_k(x,t)\} = y_k(t)e^{-pvt} . \qquad (15)$$

Then we apply the condition equation (4) in the Laplace plane, substituting the partial derivative with respect to time

$$-\frac{f_{22}}{v}\left[\frac{\partial Y_k(p,t)}{\partial t} - \frac{\partial Y_{sl}(p,t)}{\partial t}\right] = f_l(t)e^{-pvt} , \qquad (16)$$

$$-\frac{f_{22}}{v}\left[\frac{\partial Y_k(p,t)}{\partial t} - \frac{\partial Y_{sr}(p,t)}{\partial t}\right] = f_r(t)e^{-pvt} . \qquad (17)$$

Note that there is no connection between parameter p and time t. Substituting (5) into (16) and (17), we obtain the following formulas:

$$-\frac{f_{22}}{v}\left[\frac{2C_f}{m_k\lambda^2 + 2k_y\lambda + 2s_y} \cdot e^{\lambda t}e^{-pvt}(\lambda - pv) - \right.$$

$$-\frac{C_f}{IEp^4+\rho Av^2p^2-2\rho A\lambda vp+k+\rho A\lambda^2}\cdot e^{\lambda t}e^{-pvt}(\lambda-pv)\Big]=C_fe^{\lambda t}e^{-pvt}\,, \tag{18}$$

$$-\frac{f_{22}}{v}\left[\frac{2C_f}{m_k\lambda^2+2k_y\lambda+2s_y}\cdot e^{\lambda t}e^{-pvt}(\lambda-pv)-\right.$$

$$\left.-\frac{C_f}{IEp^4+\rho Av^2p^2-2\rho A\lambda vp+k+\rho A\lambda^2}\cdot e^{\lambda t}e^{-pvt}(\lambda-pv)\right]=C_fe^{\lambda t}e^{-pvt}\,. \tag{19}$$

By adding (18) and (19) and simplifying them, the following equation is obtained:

$$-\frac{f_{22}}{v}2C_fe^{\lambda t}e^{-pvt}(\lambda-pv)\left[\frac{1}{m_k\lambda^2+2k_y\lambda+2s_y}-\right.$$

$$\left.-\frac{1}{IEp^4+\rho Av^2p^2-2\rho A\lambda vp+k+\rho A\lambda^2}\right]=2C_fe^{\lambda t}e^{-pvt}\,, \tag{20}$$

and rearranging and reducing it to the common denominator, we obtain a fourth-degree equation with variable λ:

$$\rho A\lambda^4+\left[\rho Ak_k+\rho A-2\frac{f_{22}}{v}\rho Avpm_k-\frac{f_{22}}{v}m_k\right]\lambda^3+$$

$$\left[\left(IEp^4+\rho Av^2p^2+k\right)m_k+\rho As_k-2\rho Avpk_k-\frac{f_{22}}{v}k_k-\right.$$

$$\left.-2\frac{f_{22}}{v}\rho Avp\right]\lambda^2+\left[\left(IEp^4+\rho Av^2p^2+k\right)k_k+\frac{f_{22}}{v}\left(IEp^4+\rho Av^2p^2+k\right)-\right.$$

$$\left.-2\rho Avps_k-\frac{f_{22}}{v}s_k\right]\lambda+\left[\left(IEp^4+\rho Av^2p^2+k\right)s_k\right]=0\,. \tag{21}$$

As parameter p is unknown, we need another equation. Since Laplace transformed functions (11) and (12) use the inverse Laplace transform, we apply the residuum theorem, which means to determine the roots of these denominators, so we obtain the solution:

$$IEp^4+\rho Av^2p^2-2\rho A\lambda vp+k+\rho A\lambda^2=0\,. \tag{22}$$

(21) and (22) together give a non-linear equation system. To solve it, first of all λ should be calculated, since λ is the eigenvalue of the system, and depending on the real part of it, the rail-wheelset system can be stable or unstable. Equation system (21) and (22) can be solved by using numerical method. As *Fig. 2* shows, by using the following data there are some

diagrams obtained as a function of stiffness k characterizing the elastic support of the rail, while the constant parameter of the computation was velocity v. It can be seen that there exists a critical value of k, which is less than $10^5\,\mathrm{N/m^2}$, the system is unstable, otherwise it is stable. If stiffness increases, the solutions approach the limit case belonging to the rigid rail. It can be also seen that there are two λ values belonging to given v and k values. One of them is the higher, and the other is the lower one. If velocity v increases, the real part of λ increases, too, so the stability reserve decreases. The λ solution had only real values, the imaginary part was zero. This examination can be developed by considering the conicity of the wheel profile, and other non-linearities in the system. For example, the non-conical or worn wheel-rail profiles and other components of the vehicle, too, can be considered in addition to the wheelset itself.

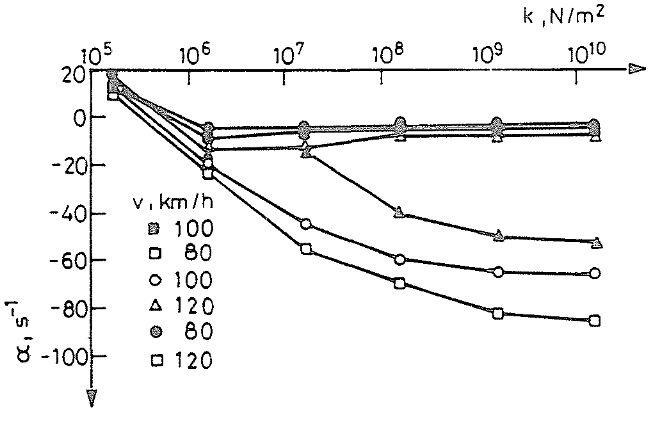

Fig. 2. The real parts of the eigenvalues

Data for calculation: rail profile UIC 54

$$I = 4.175 \cdot 10^6\,\mathrm{m}^4\,; \qquad E = 2.1 \cdot 10^{11}\,\mathrm{N/m^2}\,; \qquad \rho A = 54\,\mathrm{kg/m^3}\,;$$

$$m_k = 2500\,\mathrm{kg}\,; \quad k_k = 2000\,\mathrm{Ns/m}\,; \quad s_k = 1 \cdot 10^6\,\mathrm{N/m}\,; \quad f_{22} = 5 \cdot 10^6\,\mathrm{N}\,.$$

References

1. FORTIN, J-P.: Dynamic Track Deformation. *French Railway Review*, North Oxford Academic, Vol. 1, No. 1, 1983.
2. NILSEN, J.: Non-linear Vehicles Traversing a Rail Structure. Thesis for the Degree of Licentiate of Engineering, Göteborg, 1991.

3. NILSEN, J. – ABRAHAMSSON, J. S.: Coupling of Physical and Modal Components for Analysis of Moving Non-linear Dynamic Systems on General Beam Structures. Division of Solid Mechanics, Chalmers University of Technology, Göteborg, 1991.
4. TIMOSHENKO, S.: Method of Analysis of Statical and Dynamical Stresses in Rail. East-Pittsburgh, Pennsylvania, USA, 1983.
5. HORVÁTH, A.: Sínleerősítések kialakítása és méretezése. Műszaki Könyvkiadó, Budapest, 1984.
6. FÜLÖP, L. – ZÁBORI, Z.: Befogott vasúti kerékpár határciklus amplitúdó vizsgálata. *Járművek, Mezőgazdasági Gépek*, Vol. 37, No. 6, Budapest, 1990.
7. KALKER, J. J.: On the Rolling Contact of Two Elastic Bodies in the Presence of Dry Friction. Research Report, Technics Hogeschool Delft, 1973.
8. WICKENS, A. H.: A Refined Theory of the Lateral Stability of a Four Wheeled Railway Vehicle Having a Flexible Undamped Suspension. *Research Report*, 1966.