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A MATHEMATICAL INVESTIGATION OF THE DYNAMICS OF DRIVE-SYSTEMS OF RAILWAY TRACTION VEHICLES UNDER STOCHASTIC TRACK EXCITATION

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Abstract

In ZOBORY and NHUNG [10], some explicit conditions ensuring the existence of a stable stationary forced vertical vibration of the railway vehicle dynamic system model (see ZOBORY [7]) were derived. In this paper, we consider an elementary drive-system model. Not only the vertical displacement in the translatory subsystem of the model will be investigated, but also the angular displacements taking place in the torsional sub-system of the model.

The elementary dynamics of drive-systems of railway traction vehicles under stochastic track excitation may be described by an 8×8-system of random non-linear differential equations whose linearized system has constant coefficients. To ensure the existence and the stability of weakly stationary vertical and relative angular displacements in the model, we apply the Routh–Hurwitz criterion (see e. g. ARNOLD [1]) and some theorems elaborated by BUNKE [3] and NHUNG [4–6] to impose explicit conditions on the system parameters in terms of algebraic inequalities. These conditions guarantee that the four eigenvalues corresponding to the vertical displacements have negative real parts, and the eigenvalue zero corresponding to the relative angular displacements is simple and the other three eigenvalues have negative real parts.

The algebraic inequalities characterizing the existence and the stability of stationary motions can be easily checked on computers.

Motion Processes in the Translatory Sub-System

In [7] the following system of two second order random linear differential equations (equations of motion) is used to describe the vertical displacements z_t, z_{kt} (see ZOBORY [7]):

$$m\ddot{z}_t + k_v\dot{z}_t - k_v\dot{z}_{kt} + s_v z_t - s_v z_{kt} = 0\,,$$

$$(m_p + m_k)\ddot{z}_{kt} - k_v\dot{z}_t + (k_v + k_p)\dot{z}_{kt} - s_v z_t + (s_v + s_t)z_{kt} = \\ = s_p u_t + k_p\dot{u}_t + m_p\ddot{u}_t\,, \qquad (1.1)$$

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where m, m_p, m_k, k_v, k_p, s_v, s_p are parameters whose meaning can be seen in ZOBORY [7] and u_t is the random excitation caused by the vertical unevennesses in the track. It can be seen that *Eqs.* (1.1) don't depend on the track direction travelling speed $\dot{x}_0$ of the vehicle model. The stochastic process input u_t is assumed to be weakly stationary with spectral density function $g_{uu}(\omega)$ [8].

If there exists a weakly stationary output $(z_t,\ z_{kt})^{\mathrm{T}}$, then the normalized vertical wheel force process may be expressed by using $(z_t,\ z_{kt})^{\mathrm{T}}$ as

$$\tilde{T}_t = T_t - T_0 = m_k \ddot{z}_{kt} + k_v \dot{z}_{kt} + s_v z_{kt} - s_v z_t - k_v \dot{z}_t \,. \tag{1.2}$$

Thus, in this way, the forced vibrations of the system can already be analysed.

System (1.1) is equivalent to the following 4×4-system

$$\begin{bmatrix} \dot{z}_t \\ \ddot{z}_t \\ \dot{z}_{kt} \\ \ddot{z}_{kt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{s_v}{m} & -\frac{k_v}{m} & \frac{s_v}{m} & \frac{k_v}{m} \\ 0 & 0 & 0 & 1 \\ \frac{s_v}{m_p+m_k} & \frac{k_v}{m_p+m_k} & -\frac{s_v+s_p}{m_p+m_k} & -\frac{k_v+k_p}{m_p+m_k} \end{bmatrix} \begin{bmatrix} z_t \\ \dot{z}_t \\ z_{kt} \\ \dot{z}_{kt} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_p + m_k}(s_p u_t + k_p \dot{u}_t + m_p \ddot{u}_t) \end{bmatrix} . \tag{1.3}$$

The 4×4 coefficient matrix in (1.3) does not depend on speed $\dot{x}_0$. In concise form:

$$\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t + \mathbf{v}_t \,.$$

Proposition 1.1 (see Corollary 3.1 in [10])

Suppose that the following algebraic inequality is satisfied:

(i1)
$$(k_v s_p + k_p s_p)\,\{[k_v(m_p + m_k) + m(k_v + k_p)] \cdot$$
$$\cdot\,[k_v k_p + m(s_v + s_p) + s_v(m_p + m_k)] - m(m_p + m_k)(k_v s_p + k_p s_v)\} -$$
$$- s_v s_p\,[k_v(m_p + m_k) + m(k_v + k_p)]^2 > 0\,;$$

(i2) The random excitation process u_t in (1.1) and (1.3) is a.s. differentiable up to the third order and weakly stationary so that

$$\mathrm{E}|u_t| < \infty\,, \qquad \mathrm{E}|\dot{u}_t| < \infty\,, \qquad \mathrm{E}|\ddot{u}_t| < \infty\,. \tag{1.4}$$

Then the random process defined by

$$\mathbf{x}_t^0 = \int_{-\infty}^{t} \exp\left(\mathbf{A}(t-s)\right) \mathbf{v}_s \mathrm{d}s\,, \tag{1.5}$$

where $\mathbf{A}$ is the system matrix in (1.3) and

$$\mathbf{v}_t = \left[0,\, 0,\, 0,\, \frac{1}{m_p + m_k}(s_p u_t + k_p \dot{u}_p + m_p \ddot{u}_t)\right]^{\mathrm{T}} \tag{1.6}$$

is a weakly stationary solution (output) of (1.3), which is globally stable in the sense that for any other solution $\mathbf{x}_t = [z_t,\, \dot{z}_t,\, z_{kt},\, \dot{z}_{kt}]^{\mathrm{T}}$ of (1.3) we have

$$\lim_{t\to\infty} \|\mathbf{x}_t - \mathbf{x}_t^0\| = 0 \qquad \text{(a.s.)}\,.$$

This proposition may be proved by using a theorem of H. BUNKE [3. p. 51] and the Routh-Hurwitz criterion (see ARNOLD [1]). An extension of Proposition 1.1 to the case when the random perturbation u_t in (1.1) and (1.3) is only asymptotically weakly stationary, i. e. u_t is close to some weakly stationary process as $t \to \infty$, has been done (see Corollary 3.2 in [10]) by applying a theorem of NHUNG [4, 6].

Motion Processes in the Torsional Sub-System

Let φ_t denote the angular displacement of the drive side rotating disc, and let φ_{kt} denote the angular displacement of the rotating disc modelling the driven wheelset (see *Fig. 1* in ZOBORY [7]). Both φ_t and φ_{kt} are related to the initial static state. The relative angular displacements are determined by

$$\Phi_{kt} = \varphi_{kt} - \dot{\varphi}_{k0} \cdot t\,, \qquad \Phi_t = \varphi_t - \dot{\varphi}_0 \cdot t - \frac{M_0}{s_c}\,.$$

Using a linearization technique, we obtain the pair of second order random linear differential equations for Φ_t and Φ_{kt} (see (21) in ZOBORY [7]):

$$\begin{gathered} \Theta\ddot{\Phi}_t + (k_c - \gamma)\dot{\Phi}_t - k_c\dot{\Phi}_{kt} + s_c\Phi_t - s_c\Phi_{kt} = 0\,, \\ \Theta_k\ddot{\Phi}_{kt} - k_c\dot{\Phi}_t + (k_c + T_0\beta r^2)\dot{\Phi}_{kt} - s_c\Phi_t + \Phi_{kt} = -\mu_0 r\tilde{T}_t\,, \end{gathered} \tag{2.1}$$

where $\tilde{T}_t$ is given in (1.2). It is very important to note that in *Eqs.* (2.1), values γ, μ_0 and β depend on the track-directional travelling speed $\dot{x}_0$. This fact means that stability of the motion of the torsional sub-system may depend on the value of $\dot{x}_0$ as a parameter.

Union of Sub-Systems and Motion Equations

From the second equation in (1.1) we get

$$\ddot{z}_{kt} = \frac{1}{m_p + m_k}\left[s_v z_t + k_v \dot{z}_t - (s_v + s_p) z_{kt} - (k_v + k_p)\dot{z}_{kt} + \right.$$

$$\left. + m_p \ddot{u}_t + k_p \dot{u}_t + s_p u_t\right] . \tag{2.2}$$

Substitute $\ddot{z}_{kt}$ in (2.2) into (1.2) and, after that, $\tilde{T}_t$ in (1.2) into (2.1). We finally obtain the following linearized 8×8-system of random linear differential equations for $\mathbf{x}_t := [z_t, \dot{z}_t, z_{kt}, \dot{z}_{kt}, \Phi_t, \dot{\Phi}_t, \Phi_{kt}, \dot{\Phi}_{kt}]^{\mathrm{T}}$:

$$\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t + \mathbf{v}_t , \tag{2.3}$$

where coefficient matrix $\mathbf{A}$ can be seen on the following page and

$$\mathbf{v}_t = \left[0,\ 0,\ 0,\ \frac{1}{m_p + m_k}(m_p \ddot{u}_t + k_p \dot{u}_t + s_p u_t),\ 0,\ 0,\ 0,\ - \right.$$

$$\left. - \frac{\mu_0 r m_k}{\Theta_k (m_p + m_k)}(m_p \ddot{u}_t + k_p u_t + s_p u_t)\right]^{\mathrm{T}} . \tag{2.5}$$

Thus, $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \vdots & \mathbf{0} \\ \dots & \dots & \dots \\ \mathbf{A}_3 & \vdots & \mathbf{A}_2 \end{bmatrix} , \tag{2.6}$$

$$
\mathbf{A} = \left[\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{s_v}{m} & -\frac{k_v}{m} & \frac{s_v}{m} & \frac{k_v}{m} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{s_v}{m_p+m_k} & \frac{k_v}{m_p+m_k} & -\frac{s_v+s_p}{m_p+m_k} & -\frac{k_v+k_p}{m_p+m_k} & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{s_c}{\Theta} & \frac{\gamma-k_c}{\Theta} & \frac{s_c}{\Theta} & \frac{k_c}{\Theta} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{\mu_0 r s_v m_p}{\Theta_k m_p+m_k} & \frac{\mu_0 r k_v m_p}{\Theta_k m_p+m_k} & -\frac{\mu_0 r(m_p s_v-m_k s_p)}{\Theta_k(m_p+m_k)} & -\frac{\mu_0 r(-m_k k_p+m_p k_v)}{\Theta_k(m_p+m_k)} & \frac{s_c}{\Theta_k} & \frac{k_c}{\Theta_k} & -\frac{s_c}{\Theta_k} & -\frac{k_c+T_0\beta r^2}{\Theta_k}
\end{array}\right] \qquad (2.4)
$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{0}$ are 4×4-matrices and $\mathbf{A}_1$, $\mathbf{A}_2$ have the same structure. Because of

$$\det[\mathbf{A} - \lambda\mathbf{I}] = \det[\mathbf{A}_1 - \lambda\mathbf{I}_1]\det[\mathbf{A}_2 - \lambda\mathbf{I}_2],$$

$\mathbf{A}$ is a Hurwitz matrix iff so are $\mathbf{A}_1$ and $\mathbf{A}_2$. The condition (i1) in Proposition 1.1 guarantees that $\mathbf{A}_1$ is Hurwitz. Here we mean by a Hurwitz matrix any matrix whose eigenvalues have negative real parts. Note that matrix $\mathbf{A}_2$ corresponding to the angular part in the model admits zero as an eigenvalue. The following conditions ensure that the zero eigenvalue is simple and the other three eigenvalues of $\mathbf{A}_2$ have negative real parts:

(i3) $$\Theta(k_c + T_0\beta r^2) - \Theta_k(\gamma - k_c) > 0,$$

(i4) $$T_0\beta r^2 - \gamma > 0,$$

(i5) $$\left[\Theta(k_c + T_0\beta r^2) - \Theta_k(\gamma - k_c)\right]\left[-\gamma(k_c + T_0\beta r^2) + \right.$$
$$\left. + k_c T_0\beta r^2 + \Theta s_c + \Theta_k s_c\right] + \Theta\Theta_k s_c(\gamma - T_0\beta r^2) > 0.$$

Thus, the inequalities (i1 – i5) and (1.4) are sufficient conditions for the existence and stability of a weakly stationary vertical vibration and relative angular displacements in the model [7].

Remark. The model investigated in ZOBORY [7] and in this paper is rather simple. However, it is necessary to emphasize that our method here may be used for more complicated models, e. g. in ZOBORY and PÉTER [9]. The main idea is as follows: Suppose that a railway vehicle dynamic system is described by a system of (stochastic) non-linear differential equations with random inputs. A linearization technique may be used, again. The situation now is that the system matrix $\mathbf{A}$ in the linearized part may be random and dependent on time. The Lyapunov exponents (cf. ARNOLD and WIHSTUTZ [2]) of stochastic non-homogeneous linear differential equations now play an important role similar to that of eigenvalues of the constant matrix $\mathbf{A}$ in (1.3) and (2.4).

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