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STOCHASTIC MODEL OF OPERATION PROCESS FOR MACHINERY OF INTERMITTENT DUTY

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Abstract

Stochastic character of the proper operation of several machinery prevents the variation with time of the developing mechanical characteristics from being described by a deterministic functional relationship. In this study a stochastic model of the *rpm* and torque process in a simple mechanical system will be described, assuming cycle periods of the operation process repetitions to be independent random variables of identical distribution function. With the knowledge of the characteristic curves of driving and braking torques, and of the cycle time distribution function, the stationary limit distribution functions of the operation process will be determined.

Keywords: Stochastic System Dynamics

Introduction

With a great deal of machinery, the variation with time of the developing mechanical characteristics cannot be given by means of a deterministic functional dependence owing to the statistical character of the designed operation. In this paper, the case is examined when the stochastic operational process of a mechanical system with one degree of freedom representing the machinery can be described by the succession of function graphs alternating according to a determinate probabilistic law, and this alternation has a recurrent character.

Such a recurrent operational process occurs, e. g. with the main power transmission system of the rapid railway vehicles equipped with a stage-selection device, as well as with the auxiliary machine units of traction vehicles, and with the majority of other aggregates (e. g. elevators) operating in intermittent duty.

Model Description

The examined mechanical model is formed from a disc fixed on an axle and having moment of inertia Θ (*Fig. 1*). Accelerating torque M_g varying according to a fixed time-function is acting upon the axle up to a time-length $g(\tau)$. It is assumed that solution function $n_g(t)$ belonging to the zero initial value of motion equation

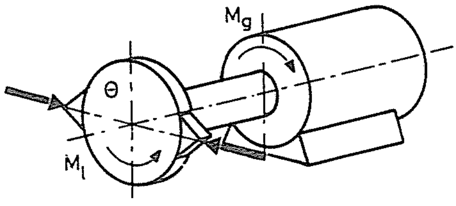

Fig. 1.

$$M_g(n) = c\,\Theta\dot{n} \tag{1}$$

is a strictly monotonously increasing one in interval $[0,\ g(t)]$. At point of time $g(t)$, the accelerating torque acting upon the system will be released, and braking torque M_l of fixed variation will be applied, under the influence of which, the *rpm* of the disc will decrease to zero during time $l(\tau)$ along strictly monotonously decreasing function $n_l(t-\tau)$. Function $n_l(t)$ is yielded from the solution belonging to zero initial value as given to motion equation

$$M_l(n) = c\,\Theta\dot{n}\,. \tag{2}$$

Consequently, in interval

$$\tau = g(\tau) + l(\tau) \tag{3}$$

the system is performing firstly accelerating then decelerating motion phases. Time interval τ is called the operation cycle of the system. One of the realizations of the operational process outlined above is obtained by giving the sequence of successive cycles τ_i according to *Fig. 2*. Points t_i will give the stopping points of the system. If $t \in [t_{i-1},\ t_i)$ then the *rpm* at time-point t is given by relationship:

$$n(t) = \min\Big\{n_g(t-t_{i-1}),\ \ n_l(t-t_i)\Big\}; \qquad t \in \big[t_{i-1},\ \ t_i\big)\,. \tag{4}$$

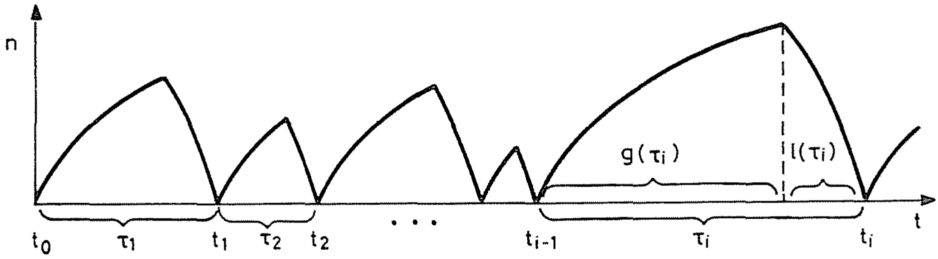

Fig. 2.

Since torque functions $M_g(n)$ and $M_l(n)$ are given, the realization function of the torques acting upon the system are obtained on the basis of formula:

$$M(t) = \begin{cases} M_g\Big[n(t)\Big] & \text{if} \quad \dot{n}(t) > 0 \\ M_l\Big[n(t)\Big] & \text{if} \quad \dot{n}(t) < 0 \end{cases}. \tag{5}$$

The stochastic model describing the variation with time of the mechanical characteristics can be created in case cycle-lengths τ_i are considered as equally distributed independent random variables defined on the set Ω of the elementary events. Let the common distribution function of random variables $\{\tau_i(\omega)\}_{i=1}^{\infty}$ yielded accordingly be designated by $F_\tau(x)$. Thus the system of stopping points $\{t_i(\omega)\}_{i=0}^{\infty}$ will be defined by partial sums of sequence $\{\tau_i(\omega)\}$ in the following way:

$$t_0(\omega) \equiv 0\,, \qquad t_i(\omega) = \sum_{j=1}^{i} \tau_j(\omega)\,. \tag{6}$$

Now those constitute a so called recurrent point process. The *rpm* of the examined system at time-point $t \in \{t_{i-1}(\omega),\ t_i(\omega)\}$ is characterized by the random variable determined by expression

$$\nu_t(\omega) = \min\Big\{n_g\Big(t - t_{i-1}(\omega)\Big);\ n_l\Big(t - t_i(\omega)\Big)\Big\} \tag{7}$$

while the torque applied to the system is described by random variable

$$\mu_t(\omega) = \begin{cases} M_g\Big(\nu_t(\omega)\Big) & \text{if} \quad \dot{\nu}(\omega) > 0 \\ M_l\Big(\nu_t(\omega)\Big) & \text{if} \quad \dot{\nu}(\omega) < 0 \end{cases}. \tag{8}$$

If time-point t traverses the positive semi-axis, then expressions (7) and (8) determine stochastic processes, consequently, the velocity and force action conditions of the model examined can be described by vector-valued stochastic process

$$\left\{\boldsymbol{\vartheta}_t(\omega)\right\} = \left\{\left[\nu_t(\omega),\ \ \mu_t(\omega)\right]^*\right\}. \tag{9}$$

First, consider the first order marginal distribution function of process $\{\nu_t(\omega)\}$. Let x designate a fixed level of rpm, and now examine the probability of $\nu_t(\omega) < x$. In interval $[\varphi(x),\ \ \psi(x))$ defined for value x by pair of relationships

$$\varphi(x) = t - n_g^{-1}(x); \qquad \psi(x) = t - n_l^{-1}(x) \tag{10}$$

contains at least one point of events t_i, then the occurrence of event $\{\nu_t(\omega) < x\}$ is obvious. On the contrary, if $\{\nu_t(\omega) < x\}$ has occurred, then $[\varphi(x)\psi(x))$ is 'not empty'. Consequently, required distribution function $F_{\nu_t}(x) = P\{\nu_t(\omega) < x\}$ can be determined on the basis of expression

$$F_{\nu_t}(x) = P\left\{\nu_t(\omega) < x\right\} = P\left\{\left[\varphi(x),\ \ \psi(x)\right) \text{ is 'not empty'}\right\}. \tag{11}$$

For us, the stationary probabilities yielded by means of limit transition $t \to \infty$ are of paramount importance. These probabilities are designated by P_∞. It can be pointed out that the stationary probability of the emptiness of $[\varphi(x),\ \ \psi(x))$ is given by formula

$$P_\infty\left\{\left[\varphi(x),\ \ \psi(x)\right) \text{ is 'empty'}\right\} = \frac{1}{T_1}\int\limits_{\psi(x)-\varphi(x)}^{\infty}\left[1 - F_\tau(y)\right]\mathrm{d}y\,, \tag{12}$$

where $T_1 = M\tau_i$ is the common expected value of random variables τ_i. On the basis of (12), the required marginal distribution is yielded in the form ([3], [5]):

$$F_\nu(x) = 1 - \frac{1}{T_1}\int\limits_{\psi(x)-\varphi(x)}^{\infty}\left[1 - F_\tau(y)\right]\mathrm{d}y\,. \tag{13}$$

Let us introduce the strictly monotonous function $\vartheta(x)$ by the following definition:

$$\vartheta(x) = \psi(x) - \varphi(x)\,.$$

On the basis of formula (13) the marginal density function $f_\nu(x)$ can also be determined:

$$f_\nu(x) = \frac{\mathrm{d}F_\nu(x)}{\mathrm{d}x} = \frac{1}{T_1} G_\tau \left[\vartheta(x)\right] \frac{\mathrm{d}\vartheta(x)}{\mathrm{d}x}\,, \tag{13.a}$$

where $G_\tau(\cdot) = 1 - F_\tau(\cdot)$.

Process $\{\nu_t(\omega)\}$ is differentiable at almost each point of time t, and thus derivative process $\{\dot{\nu}_t(\omega)\}$ can be interpreted. The sign conditions of process $\{\dot{\nu}_t(\omega)\}$ play an important role in the examination of the dynamic characteristics. Since events $\{\dot{\nu}_t(\omega) > 0\}$ and $\{\dot{\nu}_t(\omega) < 0\}$ constitute a total system of events, consequently, a number of results can be achieved by the decomposition with respect to this system of events. When using a lengthy mathematical deduction, the probability of the two above events can be expressed by the following formula:

$$P_\infty\{\dot{\nu} > 0\} = \frac{1}{T_1} \int_0^\infty \left[1 - F_\tau \left[g^{-1}(u)\right]\right] \mathrm{d}u = p_+ \tag{14}$$

and

$$P_\infty\{\dot{\nu} < 0\} = 1 - \frac{1}{T_1} \int_0^\infty \left[1 - F_\tau \left[g^{-1}(u)\right]\right] \mathrm{d}u = p_-\,. \tag{15}$$

For the determination of the conditional distribution functions of *rpm* process, the knowledge of the probability of event $\{\nu_t(\omega) < x,\ \dot{\nu}_t(\omega) > 0\}$ is required. As a result of lengthy deduction [3], required probability $q_+(x)$ is yielded in the form:

$$P_\infty\{\nu < x,\ \dot{\nu} > 0\} = \frac{1}{T_1} \int_0^{n_g^{-1}(x)} \left[1 - F_\tau \left[g^{-1}(u)\right]\right] \mathrm{d}u = q_+(x)\,, \tag{16}$$

and hence, on the basis of the total probability theorem

$$P_\infty\{\nu < x,\ \dot{\nu} < 0\} = 1 - q_+(x) = q_-(x)\,. \tag{17}$$

Consequently, the conditional distribution functions of the *rpm* process can be calculated with the help of formulas:

$$F_\nu\left(x|\dot{\nu} > 0\right) = \frac{q_+(x)}{p_+}\,; \qquad F_\nu\left(x|\dot{\nu} < 0\right) = \frac{q_-(x)}{p_-}\,. \tag{18}$$

The stationary distribution function of torque process $\{\mu_t(\omega)\}$ will be yielded on the basis of the total probability theorem, as a result of combining conditional distribution functions

$$F_\mu(y|\dot\nu > 0) = P_\infty\{M_g(\nu) < y|\dot\nu > 0\} = P_\infty\left\{\nu > M_g^{-1}(y)|\dot\nu > 0\right\} =$$

$$1 - F_\nu^+\left(M_g^{-1}(y)|\dot\nu > 0\right) \tag{19}$$

and

$$F_\mu(y|\dot\nu < 0) = P_\infty\{M_l(\nu) < y|\dot\nu < 0\} = P_\infty\left\{\nu < M^{-1}(y)|\dot\nu <\right\} =$$

$$F_\nu\left(M_l^{-1}(y)|\dot\nu < 0\right) \tag{20}$$

with respect to conditions $\{\dot\nu > 0\}$ and $\{\dot\nu < 0\}$, resp., and applying weighing factors p_+ and p_-, the distribution function in question will have the following form:

$$F_\mu(y) = F_\mu(y|\dot\nu > 0)\,p_+ + F_\mu(y|\dot\nu < 0)\,p_- . \tag{21}$$

In formula (19), F_ν^+ indicates the right-hand side limit value of F_ν. With the deduction of the conditional distribution functions, the invertibility of torque functions $M = f(n)$ was assumed, which is ensured, e. g. in the case of the strictly monotonous and continuous properties of them. More general torque functions can be examined similarly by means of the distinction of cases.

So far, the distribution functions of the torques and *rpms* were investigated distinctly from each other. However, as far as the dimensioning of the machinery is concerned, the bivariate distribution function of the two mechanical characteristics is needed. Similarly to the train of thoughts used throughout above, conditional distribution functions

$$H_{\nu,\mu}(x, y|\dot\nu > 0) = P_\infty\{\nu < x,\ \ \mu < y|\dot\nu > 0\} \tag{22}$$

and

$$H_{\nu,\mu}(x, y|\dot\nu < 0) = P_\infty\{\nu < x,\ \ \mu < y|\dot\nu < 0\} \tag{23}$$

can be calculated from formulae:

$$H_{\nu,\mu}(x,y|\dot{\nu}>0) = \begin{cases} F_\nu(x|\dot{\nu}>0) & \text{if } y > M_g(0), \\ F_\nu(x|\dot{\nu}>0) - F_\nu\left(M_g^{-1}(y)|\dot{\nu}>0\right) & \text{if } M_g(x) < y < M_g(0), \\ 0 & \text{if } y < M_g(x) \end{cases} \quad (24)$$

and

$$H_{\nu,\mu}(x,y|\dot{\nu}<0) = \begin{cases} F_\nu(x|\dot{\nu}<0) & \text{if } y > M_l(x), \\ F_\nu\left(M_l^{-1}(y)|\dot{\nu}<0\right) & \text{if } y \le M_l(x), \\ 0 & \text{if } y < M_l(0), \end{cases} \quad (25)$$

respectively. The stationary distribution function of vector process $\{\boldsymbol{\vartheta}_t(\omega)\} = \{[\nu_t(\omega),\ \mu_t(\omega)]^*\}$ based upon *rpms* and torques is yielded by formula:

$$H_{\nu,\mu}(x,y) = H_{\nu,\mu}(x,y|\dot{\nu}>0)\,p_+ + H_{\nu,\mu}(x,y|\dot{\nu}<0)\,p_-, \quad (26)$$

which follows from the total probability theorem.

Possible Generalizations

The generalization of the model examined so far is advisable to take place in two directions. On the one hand, in the course of the individual operational cycles, the slope of *rpm* functions can be considered as varying, while on the other hand, rest cycles can also occur between the individual operational cycles, and it is also desired that they should be reckoned with in the course of investigations.

First, the case of the cycle-wise varying slope of *rpm* functions should be examined. For the sake of simplicity of discussion, the moment of inertia of the disc should be considered as varying cycle-wise. Thus, during the operational cycles of length $\tau_i(\omega)$, let the moment of inertia be characterized by random variables $\Theta_i(\omega)$. Concerning sequence $\{\Theta_i(\omega)\}$, let it be assumed that its elements are independent from on another, and those of sequence $\{\tau_i(\omega)\}$ and their bivariate distribution function is $F_\Theta(z)$. Under the mentioned conditions, one of the realizations of the *rpm* process is shown in *Fig. 3.*

Let $\{\nu_t'(\omega)\}$ designate the *rpm* process of cycle-wise varying slope, then the marginal distribution function belonging to it is obtained by the following Stieltjes integral

$$F_{\nu'}(x) = \int_0^\infty \left\{1 - \frac{1}{T_1}\int_{\psi(x,z)-\varphi(x,z)}^\infty \left[1 - F_\tau(y)\right]\mathrm{d}y\right\}\mathrm{d}F_\Theta(z). \quad (27)$$

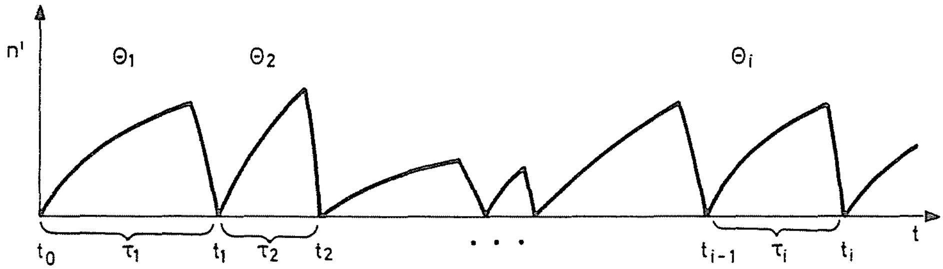

Fig. 3.

In formula (27), $\psi(x,z)$ and $\varphi(x,z)$ resp. designate the quantities according to (10) with fixed $\Theta = z$. The characteristics of process $\{\mu_t'(\omega)\}$ belonging to $\{\nu_t'(\omega)\}$, as well as those of vector process $\{\boldsymbol{\vartheta}_t'(\omega)\}$ can be determined analogously on the basis of those written above.

As for the examination of the second possibility of generalization, let it be assumed that there occur rest cycles $\{\tau_i'(\omega)\}$ between operating cycles $\{\tau_i(\omega)\}$. As for the sequence of random variables $\{\tau_i'(\omega)\}_{i=1}^{\infty}$ it is assumed that its elements are independent from each other, as well as from those of sequence $\{\tau_i(\omega)\}$, and that they have identical distribution function $F_{\tau'}(x)$. One of the realizations of *rpm* process $\{\nu_t''(\omega)\}$ yielded is shown in *Fig. 4*. If the common expected values of variables τ_i' is designated by T_2, then the stationary probability of the operation or the state of rest of the system can be given by pair of relationships

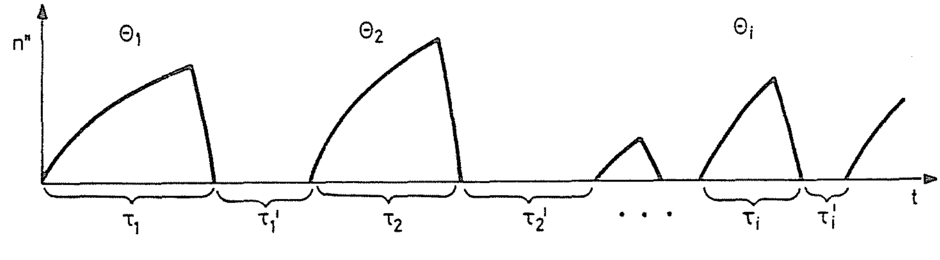

Fig. 4.

$$P_\infty\left\{\nu'' > 0\right\} = \frac{T_1}{T_1+T_2}; \qquad P_\infty\left\{\nu'' = 0\right\} = \frac{T_2}{T_1+T_2}. \tag{28}$$

The stationary marginal distribution of *rpm* process $\{\nu_t''(\omega)\}$ can be given by relationship

$$F_{\nu''}(x) =$$

$$\frac{T_2}{T_1+T_2} + \frac{T_1}{T_1+T_2}\int_0^\infty \left\{ 1 - \frac{1}{T_1} \int_{\psi(x,z)-\varphi(x,z)}^{\infty} \Big[1 - F_\tau(y)\Big] \mathrm{d}y \right\} \mathrm{d}F_\Theta(z) , \quad (29)$$

as it can be proved from the total probability theorem.

The analysis of torque process $\{\mu_t''(\omega)\}$ can be carried out on the basis of the defining relationship:

$$\mu_t''(\omega) = \begin{cases} M_g\Big(\nu_t''(\omega)\Big) & \text{if} \quad \dot{\nu}_t'' > 0 \, , \\ 0 & \text{if} \quad \dot{\nu}_t'' = 0 \, , \\ M_l\Big(\nu_t''(\omega)\Big) & \text{if} \quad \dot{\nu}_t'' < 0 \, , \end{cases} \quad (30)$$

with the method introduced in the foregoing. The determination of the stationary probabilities of vector process $\{\boldsymbol{\vartheta}_t''(\omega)\}$ can be performed in a quite similar way.

The stationary distribution function obtained by the analysis of the stochastic model of the operational process ensures the reliable implementation of stress dimensioning and the dimensioning with respect to the operation requirement of the machinery examined. As far as the sphere of problems associated with the dimensioning with respect to operation is concerned, the optimization of the energetic conditions of the drive system can be underlined, while with respect to that of stress dimensioning, the determination of the equivalent loads required for life-dimensioning can be put into the foreground.

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