A CLASS OF INVERSE SEMIGROUPS WITH BOOLEAN CONGRUENCE LATTICES

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Received June 25, 1988

Abstract

A construction of inverse semigroups whose idempotents form a (locally finite) tree and whose congruence lattices have the property P is given where P stands for one of the following properties of lattices: (dually) sectionally complemented, relatively complemented, modular and complemented, Boolean, respectively. These semigroups are completely characterized up to: congruence-free inverse semigroups (without zero), simple groups and locally finite trees. Furthermore, special sublattices of the congruence lattice easily can be studied: any two trace classes are isomorphic, and the lattices of all semilattice congruences and idempotent pure congruences, respectively are Boolean.

1. Introduction

The problem of characterizing the semigroups with Boolean congruence lattices has been solved for several classes of semigroups. Hamilton [9] and Auinger [1] studied the question for semilattices. Hamilton and Nordahl [10] considered commutative semigroups. Fountain and Lockley [6], [7] solved the problem for Clifford semigroups and idempotent semigroups, in [1], Auinger generalized their results to completely regular semigroups. Finally, Zhitomirskiy [16] studied the question for inverse semigroups.

In [3], Auinger proved that in general the problem can be treated in a similar way for the following lattice properties: sectionally complemented, relatively complemented, complemented and modular, Boolean, respectively and it can be shown that the same is true for the property dually sectionally complemented. From now on let P stand for any of these five properties. In [4] Auinger studied inverse semigroups whose congruence lattices have one of these properties P. In this paper, we use the results of [3] and [4] to give a description of inverse semigroups whose idempotents form a (locally finite) tree and whose lattice of congruences has the property P.

In chapter two we collect some results about semigroups S whose congruence lattice $C(S)$ is sectionally complemented, relatively complemented, modular and complemented, Boolean, respectively, and extend these results to the case when $C(S)$ is dually sectionally complemented.

In chapter three, we use the results of chapter 2 and a further result about inverse semigroups to give a construction of all inverse semigroups whose idempotents E form a tree and whose congruence lattices have the property P (Theorem 4). In this way these semigroups are completely characterized up to congruence free inverse semigroups (without zero), simple groups and locally finite trees. Furthermore, for the case when E forms a *locally finite* tree, a simplified version of the theorem can be given. Using this characterization, special congruences on such semigroups easily can be studied: it is shown that each trace class is isomorphic to the lattice of normal subgroups of the maximal group homomorphic image of S. Furthermore, the lattices of all semilattice congruences and idempotent pure congruences, respectively, are shown to be isomorphic to some power set lattices. We also obtain that for inverse semigroups whose idempotents form a tree the following properties of the congruence lattice are equivalent: (dually) sectionally complemented, relatively complemented, modular and complemented, respectively. Only the maximal group homomorphic image is responsible for the possible difference to the Boolean case.

2. Preliminaries

We first collect some definitions and results which we need for our investigations.

Throughout the paper, the congruence lattice of some semigroup S is denoted by $(C(S), \vee, \cap)$. $\varepsilon = \varepsilon_S$ is the identical relation on S, $\omega = \omega_S$ is the universal relation on S.

Definition. (i) Let X be an ordered set, $x, y \in X$. Then x *covers* y or y *is covered by* x, to be denoted by $x \succ y$, if $y < x$ and $y < z \leq x$ implies that $z = x$.

(ii) A semilattice is a *(locally finite) tree* if each interval $[x, y] = \{z \in X\colon\ x \leq z \leq y\}$ is a (finite) chain.

(iii) For any semigroup S let $S^* = S$ if S has no zero and $S^* = S/\{0\}$ if 0 is the zero of S.

Construction. Let X be a locally finite tree. To each $\alpha \in X$ associate a 0-simple semigroup $I_\alpha (I_\alpha \neq \{0\})$ satisfying $I_\alpha \cap I_\beta = 0$ if $\alpha \neq \beta$. For each $\alpha \in X^*$ let α' denote the unique element of X that is covered by α. For each $\alpha \in X^*$ let $f_\alpha : I_\alpha^* \to I_{\alpha'}^*$ be a partial homomorphism. Let $f_{\alpha,\alpha} := \mathrm{id}|I_\alpha^*$ and $f_{\alpha,\beta} := f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_n}$ if $\alpha = \alpha_1 \succ \ldots \succ \alpha_n \succ \beta$. Suppose that for any $a \in I_\alpha^*$, $b \in I_\beta^*$ there exists $\gamma \leq \alpha\beta$ such that the product $(af_{\alpha,\gamma})(bf_{\beta,\gamma})$ is defined in I_γ^*. Let $\delta(a, b)$ be the greatest element of X satisfying this condition. Let $S = \cup (I_\alpha^*\colon \alpha \in X)$ and define a multiplication on S by

$$ab := (af_{\alpha,\delta(a,b)})(bf_{\beta,\delta(a,b)}) \text{ for } a \in I_\alpha^*, b \in I_\beta^*.$$

Then S is a semigroup. Furthermore, for $a \in I_\alpha^*$, $I_\alpha^* = J_a$ and $S/\mathcal{J} \cong X$.

Definition. A semigroup so constructed is a *tree of 0-simple semigroups* I_α, to be denoted by $S = (X; I_\alpha, f_{\alpha,\beta})$ If each I_α is congruence free (with zero) and not the null-semigroup of order two, then S is a *tree of congruence free semigroups*, if all I_α are Brandt semigroups, then S is a *tree of Brandt semigroups*.

Similar constructions appear in [2], [3], [11] and [13]. If X has a least element μ then by the conditions of the construction, I_μ^* is closed under multiplication and thus I_μ^* is a simple semigroup.

The construction seems to be quite "artificial" but it naturally appears in the investigations of semigroups with complemented congruence lattices as the following theorem proves:

Theorem 1. ([2]) A globally idempotent semigroup S has a complemented congruence lattice if and only if

(i) $S = (X; I_\alpha, f_{\alpha,\beta})$ a tree of 0-simple semigroups

(ii) S/ξ, the maximal simple homomorphic image of S has a complemented congruence lattice

(iii) if $x\zeta y$ and $x \neq y$ then $xf_\alpha \neq yf_\alpha$ $\quad (x, y \in I_\alpha^*)$

(iv) for $x \in I_\alpha^*$, $y \in I_\beta^*$ there exists $\gamma \in X$ such that $xf_{\alpha,\gamma}\ \zeta\ yf_{\beta,\gamma}$.

Here ξ and ζ denote the following congruences:

$$x\xi y \Leftrightarrow xf_{\alpha,\gamma} = yf_{\beta,\gamma} \text{ for some } \gamma\ (x \in I_\alpha^*, y \in I_\beta^*),$$

$$x\zeta y \Leftrightarrow J(uxv) = J(uyv) \text{ for all } u, v \in S.$$

Some further results:

Lemma 1. ([8]) A 0-simple semigroup S is congruence free if and only if for $x \neq y$ there exist $u, v \in S$ such that $uxv = 0$ and $uyv \neq 0$, or conversely.

Lemma 2. ([2]) Let $S = (X; I_\alpha, f_{\alpha,\beta})$ and $\alpha \geq \beta \geq \gamma \geq \delta$. If $x\ \varrho\ y$ for some $x \in I_\alpha^*, y \in I_\delta^*$ for some $\varrho \in C(S)$ then $z\ \varrho\ zf_{\beta,\gamma}$ for all $z \in I_\beta^*$.

Definition. A lattice L with a least element ε is *sectionally complemented* if each interval $[\varepsilon, \xi]$ in L is a complemented lattice. A lattice L with a greatest element ω is *dually sectionally complemented* if each interval $[\xi, \omega]$ in L is a complemented lattice. A lattice L is *relatively complemented* if each interval $[\xi, \eta]$ in L is a complemented lattice. It is well-known that Boolean $\Rightarrow$ modular and complemented $\Rightarrow$ relatively complemented $\Rightarrow$ (dually) sectionally complemented $\Rightarrow$ complemented.

Lemma 3. ([3]) Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of 0-simple semigroups. If $C(S)$ is sectionally complemented then I_α is congruence free for all $\alpha \in X^*$.

Definition. Let X be a locally finite tree with a least element μ. If $\alpha \succ \mu$ then α is an *atom*. Furthermore, let $X^{**} := X/\{\mu, \text{atoms}\}$ if μ is the least element of X and $X^{**} = X$ otherwise.

Lemma 4. Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of 0-simple semigroups. If $C(S)$ is dually sectionally complemented then I_α is congruence free for all $\alpha \in X^*$.

Proof. Let $\alpha \in X^*$ and $I = \cup(I_\gamma^*\colon \gamma < \alpha)$ then I is not empty and hence an ideal in S. S/I is a tree of 0-simple semigroups with tree $Y = X/\{\gamma \in X\colon \gamma < \alpha\}$. Since $xf_\alpha \in I$ for all $x \in I_\alpha^*$ we have that f_α/I is a constant mapping where f_α/I is the partial homomorphism on I_α^* in the representation of S/I as a tree of 0-simple semigroups. By hypothesis, $C(S/I)$ is complemented and so by theorem 1. (iii) no two elements of I_α^* can be ζ-equivalent in S/I, i.e. for any $x, y \in I_\alpha^*$ there exist $u, v \in S/I$ such that $J(uxv) \neq J(uyv)$. Since α is an atom in Y, $J_u, J_v \geq J_x = I_\alpha^*$ and so we may assume that $u, v \in I_\alpha^*$, otherwise we replace them by $uf_{\gamma,\alpha}$ and $vf_{\beta,\alpha}$, respectively. Again, since α is an atom in Y, $J(uxv) \neq J(uyv)$ is only possible if $J(uxv) = J(x) = I_\alpha^* \cup I$ and $J(uyv) \subseteq I$, or conversely which by lemma 1 implies that I_α is congruence free.

Lemma 5. ([3]) Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of congruence free semigroups I_α such that X has no least element. If $C(S)$ is complemented then for $x \in I_\alpha^*$, $y \in I_\beta^*$ there exists $\gamma \in X$ such that $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$, i.e. $\xi = \omega$.

Lemma 6. Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of 0-simple semigroups where X has a least element μ. If $C(S)$ has the property P then $C(I_\mu^*)$ has the property P.

Proof. For $P =$ dually sectionally complemented, this is an immediate consequence of $I_\mu^* \cong S/\xi$. The other cases are proved in [3].

Now we are able to give a new formulation of theorem 8 in [3]:

Theorem 2. ([3]) The congruence lattice of a globally idempotent semigroup S has the property P if and only if S is one of the following:

(i) S is simple and $C(S)$ has the property P

(ii) $S = (X; I_\alpha, f_{\alpha,\beta})$ a tree of congruence free semigroups such that for all $x \in I_\alpha^*$, $y \in I_\beta^*$ there exists $\gamma \in X$ satisfying $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$.

(iii) S is a retract ideal extension of a semigroup (i) by a semigroup (ii) with zero such that the retract homomorphism f is compatible with the partial homomorphisms f_α, i.e. $f_\alpha f = f$ for all $\alpha \in X^{**}$.

In this case $C(S) = C(X) \times C(I)$ where I is the kernel of S.

Condition (iii) says that S is a tree of 0-simple semigroups I_α such that X has a least element μ, I_α is congruence free for $\alpha \neq \mu$, and $C(I_\mu^*)$ has the property P.

3. Inverse semigroups

For inverse semigroups we use the notation of [14]. For any inverse semigroup S, $E = E_S$ denotes the semilattice of idempotents of S. For any congruence ϱ on S, the kernel of ϱ is defined by $\ker \varrho = \{a \in S\colon a \mathrel{\varrho} e$ for some

$e \in E\}$ and the trace of ϱ is tr $\varrho = \varrho|E$. Each congruence is uniquely determined by its kernel and trace. The relation $\varrho\ \Theta\ \xi \Leftrightarrow \text{tr}\ \varrho = \text{tr}\ \xi$ is a lattice congruence on $C(S)$, the congruence classes $\varrho\Theta$ are the trace classes. Some special congruences are the following:

$$a\ \sigma\ b \Leftrightarrow ae = be \text{ for some } e \in E,$$

is the *least group congruence* on S. Furthermore,

$$a\ \mu\ b \Leftrightarrow a^{-1}ea = b^{-1}eb \text{ for all } e \in E$$

is the *greatest idempotent separating congruence* on S. An inverse semigroup is an *antigroup* (fundamental inverse semigroup) if $\mu = \varepsilon$, i.e. the identity relation is the only idempotent separating congruence. A congruence ϱ is *idempotent pure* if ker $\varrho = E$; the greatest idempotent pure congruence on S is denoted by τ. Furthermore, ϱ is a semilattice congruence if S/ϱ is a semilattice, i.e. if ker $\varrho = S$. The least semilattice congruence on S is denoted by η. For an arbitrary subset $K \subseteq S$, $K\omega = \{a \in S: a \geq b \text{ for some } b \in K\}$ where the order relation on S is defined by $a \leq b \Leftrightarrow a = be$ for some $e \in E$.
In [4] the following theorem is proved:
Theorem 3. ([4]) Let S be an inverse semigroup. Then $C(S)$ has the property P if and only if
(i) S is isomorphic to a subdirect product of a group G and an antigroup A,
(ii) $C(G)$ and $C(A)$ both have the property P
(iii) for any $(x, e), (x, f) \in S$ (where $e, f \in E_A$, $x \neq 1$) there exist $(y_1, a_1), \ldots, (y_n, a_n) \in S$, $g \in E_A$ such that

$$x = y_1^{-1}x^{\varepsilon_1}y_1y_2^{-1}x^{\varepsilon_2} \ldots y_n^{-1}x^{\varepsilon_n}y_n \text{ where } \varepsilon_i \in \{-1, 1\} \text{ and}$$

$$e = a_1^{-1}fa_1a_2^{-1}f \ \ldots \ a_n^{-1}fa_ng.$$

Furthermore, in this case $C(S) \cong C(G) \times C(A)$.

Now we use theorems 2 and 3 to construct the inverse semigroups S where E_S is a tree and $C(S)$ has the property P.

We first treat the case (ii) of theorem 2. Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of congruence free inverse semigroups such that E_S forms a tree. Then $E_{I_\alpha^*}$ is a tree and by [14], IV. 3. 11, E_{I_α} is 0-disjunctive which by [14], IV. 3. 13, is equivalent to: for idempotents $e < f$ there exists $g \in E_{I_\alpha^*}$, $0 \neq g \leq f$ and $ge = 0$. Since $E_{I_\alpha^*}$ is a tree, $g \leq e$ or $e \leq g$, then $g > ge = 0$ implies $e \leq g$ and $e = eg = 0$. We have thus obtained that each idempotent is primitive, i.e. I_α is a Brandt semigroup. Since I_α has no proper congruence, we obtain that $I_\alpha \cong K_\alpha \times K_\alpha \cup \{0\}$ for some set K_α, a Brandt semigroup over the trivial group. Partial homomorphisms among the nonzero parts of such semigroups are given by the following

Lemma 7. ([5]) Let $\varphi: K_\alpha \to K_\beta$ be an arbitrary mapping. Then $f: K_\alpha \times K_\alpha \to K_\beta \times K_\beta$, defined by $(i, j)f = (i\varphi, j\varphi)$ is a partial homomorphism and conversely, each partial homomorphism $f\colon I_\alpha^* \to I_\beta^*$ can be so constructed.

We are able to give the following

Proposition 1. Let $S = (X;\ I_\alpha, f_{\alpha,\beta})$ be a tree of Brandt semigroups I_α over the trivial group. Then $C(S)$ has the property P; in particular, $C(S)$ is Boolean.

Proof. By theorem 2, it remains to show that for $x \in I_\alpha^*, y \in I_\beta^*$ there exists $\gamma \in X$ such that $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$. Since $xf_{\alpha,\alpha\beta}$ and $yf_{\beta,\alpha\beta} \in I_{\alpha\beta}^*$, we may assume that $\alpha = \beta$. Let $(i, j), (l, k) \in I_\alpha^*$. There exists γ such that the product $(i, j)f_{\alpha,\gamma}(k, l)f_{\alpha,\gamma}$ is defined in $I_\gamma^* = K_\gamma \times K_\gamma$, thus $j\varphi_{\alpha,\gamma} = k\varphi_{\alpha,\gamma}$ where $\varphi_{\alpha,\gamma}\colon K_\alpha \to K_\gamma$ is the defining mapping for $f_{\alpha,\gamma}$. By analogy, $i\varphi_{\alpha,\delta} = l\varphi_{\alpha,\delta}$ for some δ and thus $(i, j)f_{\alpha,\beta} = (l, k)f_{\alpha,\beta}$ for $\beta = \min(\gamma, \delta)$.

We now consider the case (*i*) of theorem 2. By theorem 3, S is isomorphic to a subdirect product of a group G and an antigroup A. If S is a group it is well known that necessary and sufficient in order that $C(S)$ has the property P is that G is a direct sum of simple groups (in which no Abelian factor appears twice for the Boolean case). For the case when S is an antigroup we show

Proposition 2. Let S be a simple antigroup such that E_S is a tree and $C(S)$ is complemented. Then S is congruence free.

Proof. Let $\varrho \in C(S)$, $\varrho \neq \varepsilon$ and let ξ be a complement of ϱ.

(i) For any $e, f \in E$, $e > f$ we have $e\ \varrho \vee \xi f$. Since E is a tree there exist unique $n \in N$, $e_0, \ldots, e_n \in E$ such that $e = e_0 > \ldots > e_n = f$ and $e = e_0 \Theta_1 e_1 \Theta_2 \ldots \Theta_n e_n = f$ where $\Theta_i \in \{\varrho, \xi\}$ and $\Theta_i \neq \Theta_{i+1}$. Let $n(e, f) := n$; then for $e > f > g$ $n(e, g) \geq n(e, f) + n(f, g) - 1$.

(ii) Let $e, f \in E$, $e > f$. Since S is simple, there exists $a \in S$ such that $e = aa^{-1}$ and $a^{-1}a \leq f$. By [14], IV.2.3, the mapping $\delta^a\colon g \mapsto a^{-1}ga$ is an isomorphism between the principal ideals $\{g \leq aa^{-1}\}$ and $\{g \leq a^{-1}a\}$ of E. Since S is an antigroup and $C(S)$ is complemented, by [4], $\sigma = \omega$, i.e. $\ker \sigma = E\omega = S$. So there exists $h \in E$ such that $h \leq a$. This implies that $g\delta^a = g$ for all $g \leq h$. Since $e\delta^a \leq f < e$ we have $h \leq f$. Now choose $e_0, \ldots, e_n \in E$ such that $e = e_0 > e_1 > \ldots > e_n = h$ and $e_0 \Theta_1 e_1 \Theta_2 \ldots \Theta_n e_n$ with $\Theta_i \in \{\varrho, \xi\}$ and $\Theta_i \neq \Theta_{i+1}$. $f \geq e_0\delta^a \Theta_1 e_1 \delta^a \Theta_2 \ldots \Theta_n e_n \delta^a = e_n$. Since $e_0\delta^a > e_1\delta^a > \ldots > e_n\delta^a$ and $\Theta_i \neq \Theta_{i+1}$ we obtain $n(f, h) \geq n$. Then $n = n(e, h) \geq n(e, f) + n(f, h) - 1 \geq n + n(e, f) - 1$ which implies that $n(e, f) = 1$. Thus for arbitrary $e > f$ either $e\ \varrho\ f$ or $e\ \xi\ f$ holds. Since $\varrho \neq \varepsilon$ there exist $e > f$ with $e\ \varrho\ f$. Let $g, h \in E, g > h$. Suppose that $g\ \xi\ h$. S is a simple antigroup, so E is subuniform: there exists $i \in E$ such that $i < hf$. By the foregoing argument, $e\ \varrho\ i$ and $g\ \xi\ i$ and so $hf\ \varrho \cap \xi\ i$, a contradiction. Therefore, $\operatorname{tr} \xi = \varepsilon$ and thus $\varrho = \omega$ which proves that S is congruence free.

An immediate consequence of this proof is.

Corollary 1. Let S be a simple antigroup such that E_S is a locally finite tree and $C(S)$ is complemented. Then S is the trivial semigroup.
Summarizing the results we can formulate

Theorem 4. Let S be an inverse semigroup. Then E_S forms a tree and $C(S)$ has the property P if and only if S is (isomorphic to) one of the following:

(i) a congruence free semigroup A (without zero) where E_A is a tree

(ii) a direct sum of simple groups (in which no Abelian factor appears twice for the Boolean case)

(iii) a subdirect product I of a semigroup G as in (ii) and a semigroup A as in (i) such that for any (x, e), $(x, f) \in I$ with $e, f \in E_A$, $x \neq 1$, there exist $(y_1, a_1), \ldots, (y_n, a_n) \in I$, $g \in E_A$ such that

$x = y_1^{-1}x^{\varepsilon_1}y_1y_2^{-1}x^{\varepsilon_2} \ldots y_n^{-1}x^{\varepsilon_n}y_n$ where $\varepsilon_i = \pm 1$ and

$e = a_1^{-1}fa_1a_2^{-1}fa_2 \ldots a_n^{-1}fa_ng$

(iv) $S = (X; I_\alpha, f_{\alpha,\beta})$, a tree of Brandt semigroups over the trivial group

(v) $S = (X; I_\alpha, f_{\alpha,\beta})$, a tree of 0-simple semigroups where X has a least element μ, $I_\mu^* = I$ is a semigroup as in (i) — (iii) and each I_α is a Brandt semigroup over the trivial group for $\alpha \neq \mu$.

Proof. It remains to show that for the case (v), E_S is a tree. Let e, f be uncomparable elements of E_S. We may assume that $e \in I_\alpha^*$ and $f \in I_\mu^*$. If $g \geq e$ then $g \in I_\beta^*$ for some $\beta \geq \alpha$ and $gf_{\alpha,\beta} = e$. By the uncomparability of e and f, $f \leq ef_{\alpha,\mu}$. Then $gf_{\beta,\mu} = ef_{\alpha,\mu} \geq f$ and thus g is no upper bound for f.

For the case (v) where I is a semigroup as in (iii),

$$C(S) \cong C(X) \times C(I) \cong P(X^*) \times N(G) \times C_2 \cong P(X) \times N(G)$$

where $P(Z)$ is the power set of Z and $N(G)$ is the lattice of the normal subgroups of G and $C_2 = \{\varepsilon, \omega\}$ the chain of two elements. For the simpler cases the analogous results hold (some factors may be missing). A consequence is that for inverse semigroups whose idempotents form a tree the properties (dually) sectionally complemented, relatively complemented, modular and complemented, respectively, are equivalent, and the difference to the Boolean case only depends on S/σ, the maximal group homomorphic image of S.
For semigroups whose idempotents form a locally finite tree we obtain a simplified version of the theorem (using corollary 1):

Corollary 2. Let S be an inverse semigroup. Then E_S forms a locally finite tree and $C(S)$ has the property P if and only if S is (isomorphic to) one of the following:

(i) a direct sum of simple groups (in which no Abelian factor appears twice for the Boolean case)

(ii) a tree of Brandt semigroups over the trivial group
(iii) an ideal extension of a semigroup (i) by a semigroup (ii) with zero.

Proof. By Corollary 1, the cases (i) and (iii) of theorem 4 violate the condition that E_S is locally finite. Clearly the idempotents of a tree of Brandt semigroups form a locally finite tree. Finally, each ideal extension of a group G is a retract ideal extension. It remains to prove that $f_\alpha f = f$ for all $\alpha \in X^{**}$ where f denotes the retraction. This is clear for idempotents since G has only one idempotent. Let $(i, j) \in K_\alpha \times K_\alpha$ and $(j, i)f_\alpha = (j\varphi, i\varphi)$; let $(i, j)f = x$ and $(j, i)f_\alpha f = y$. Then $(j, i)f_\alpha(i, j) = (j, i) f_\alpha(i, j)f_\alpha = (j\varphi, i\varphi)(i\varphi, j\varphi) = (j\varphi, j\varphi) = (j, j)f_\alpha$. Now $yx = (j, i)f_\alpha f(i, j)f = [(j, i)f_\alpha(i, j)f_\alpha] f = (j, j)f_\alpha f = 1$. Thus $(i, j)f = x = y^{-1} = [(j, i)f_\alpha f]^{-1} = (j, i)^{-1}f_\alpha f = (i, j)f_\alpha f$.

Remark. As it can be seen in [3], the semigroups characterized by corollary 2 are exactly the completely semisimple inverse semigroups whose congruence lattice has the property P. For the Boolean case, this was obtained by ZHITOMIRSKIY [16].

For the case (v) of theorem 4 we now study some special congruences of $C(S)$. For the less complicated cases (i) — (iv) the analogous results can be obtained immediately.

Using the results of [3] and [4], each congruence ϱ on S can be identified with a triple $(\varrho_X, \varrho_G, \varrho_A) \in C(X) \times C(G) \times C(A)$ given by:

$\alpha \varrho_X \beta \Leftrightarrow x \varrho x f_{\alpha,\alpha\beta}$ and $y \varrho y f_{\beta,\alpha\beta}$ for all $x \in I_\alpha^*$, $y \in I_\beta^*$

$x \varrho_G y \Leftrightarrow (x, e) \varrho (y, e)$ for some $e \in E_A$

$a \varrho_A b \Leftrightarrow (1, a^{-1}fa) \varrho (1, b^{-1}fb)$ for all $f \in E_A$

where the pairs mean elements of I_μ^* in its representation as a subdirect product of G and A.

It is easy to see that the trace class of some congruence ϱ is given by $\varrho\Theta = \{(\varrho_X, \xi, \varrho_A): \xi \in C(G)\}$. Any two trace classes are isomorphic lattices. In particular, $[\sigma, \omega] = \{(\omega, \xi, \omega): \xi \in C(G)\}$ and $[\varepsilon, \mu] = \{(\varepsilon, \xi, \varepsilon): \xi \in C(G)\}$.

Now we study semilattice congruences: it is clear that $\varrho_G = \omega$ and $\varrho_A = \omega$ for any semilattice congruence ϱ. Thus the lattice of all semilattice congruences $[\eta, \omega]$ is isomorphic to an interval of $C(X)$. In [1] it is shown that $C(X) \cong P(X^*)$ for any locally finite tree X. From this we can obtain that $[\eta, \omega]$ is isomorphic to the power set lattice of some subset of X.

Lemma 8. Let η be the least semilattice congruence on S. If $|I_\alpha^*| > 1$ for $\alpha \in X^*$ then $x \eta x f_\alpha$.

Proof. By [14], III, $\eta = \mathcal{J}^*$, the congruence generated by Green's relation $\mathcal{J}$. Let $x \neq y \in I_\alpha^*$ then $x \eta y$. Since I_α is congruence free there exist $u, v \in I_\alpha^*$ such that $uxv \in I_\alpha^*$ and $uyv \in I_\beta^*$ for some $\beta < \alpha$, or conversely. By lemma 2 we obtain that $x \eta x f_\alpha$ for all $x \in I_\alpha^*$.

Corollary 3. Let $\alpha > \beta$; if $|I_\delta^*| > 1$ for all $\alpha \geq \delta > \beta$ then $x \eta x f_{\alpha,\beta}$ for all $x \in I_\alpha^*$.

Notation. $X^{++} = \{\alpha \in X^*: \quad |I^*_\alpha| = 1\}$

Theorem 5. $S/\eta \cong X^{++} \cup \{\mu\}$ and $[\eta, \omega] \cong P(X^{++})$.

Proof. For $\alpha \in X$ we define

$$\alpha + : = \begin{cases} \alpha \text{ if } |I^*_\alpha| = 1 \\ \beta \text{ if } \alpha = \alpha_0 \succ \alpha_1 \succ \ldots \succ \alpha_n = \beta, |I^*_\beta]| = 1 \text{ and } |I^*_{\alpha_i}| > 1 \text{ for } i < n \\ \mu \text{ if } |I^*_\beta| > 1 \text{ for all } \beta \leq \alpha. \end{cases}$$

Then $\alpha{+}{+} = \alpha+$ and $\alpha \geq \beta \Rightarrow \alpha+ \geq \beta+$. For $x \in I^*_\alpha$, $y \in I^*_\beta$ define

$$x \varrho y \Leftrightarrow \alpha+ = \beta+.$$

We prove that $\varrho = \eta$. Since $x \varrho x f_{\alpha,\alpha+} = (xf_{\alpha,\alpha+})^2$ if $\alpha+ \neq \mu$ or $x \varrho x f_{\alpha,\mu} \varrho (xf_{\alpha,\mu})^2$, ϱ is a semilattice congruence and hence $\eta \leq \varrho$.

Let $x \varrho y$, i.e. $\alpha \succ \alpha_1 \succ \ldots \succ \alpha+, \beta \succ \beta_1 \ldots \succ \beta+$ and $\alpha+ = \beta+$, $|I^*_\delta| > 1$ for all $\delta \in (\alpha+, \alpha] \cup (\alpha+, \beta]$ and $|I^*_{\alpha+}| = 1$ or $\alpha+ = \mu$. Since $u \,\eta\, v$ for all $u, v \in I^*_\mu$, by corollary 3 we obtain that $x \,\eta\, x f_{\alpha,\alpha+} \,\eta\, y f_{\beta,\alpha+} \,\eta\, y$, thus $x \,\eta\, y$.

Now we show that the mapping F: $x\eta \mapsto \alpha+$ for $x \in I^*_\alpha$ is an (order) isomorphism between S/η and $X^{++} \cup \{\mu\}$. Clearly, F is surjective. If $(x\eta)F = (y\eta)F$, i.e. $\alpha+ = \beta+$ where $x \in I^*_\alpha, y \in I^*_\beta$ then $x \varrho y$ and thus $x\eta = y\eta$. Let $x\eta \geq y\eta$; then $y\eta = (xy)\eta$, $\beta+ = \delta(x, y)+ \leq \delta(x, y) \leq \alpha$ and thus $\beta+ \leq \alpha+$. Conversely, let $(x\eta)F = \alpha+ \geq \beta+ = (y\eta)F$. Let $u \in I^*_{\alpha+}$, $v \in I^*_{\beta+}$; $I^*_{\beta+}$; is closed under multiplication, thus $uv = (uf_{\alpha+,\beta+})v \in I^*_{\beta+}$.

So we get $(x\eta)(y\eta) = (u\eta)(v\eta) = [(uf_{\alpha+,\beta+})v]\eta = v\eta = y\eta$ thus $x\eta \geq y\eta$. $S/\eta \cong X^{++} \cup \{\mu\}$ implies that $[\eta, \omega] \cong C(S/\eta) \cong C(X^{++} \cup \{\mu\})$. In [1] it is proved that $C(X) \cong P(X^*)$ for locally finite trees X, so we obtain that $[\eta, \omega] = P(X^{++})$.

Using the triple representation of a congruence ϱ on S, we get

Corollary 4. A congruence ϱ on S is a semilattice congruence if and only if

(i) $\varrho_G = \omega$

(ii) $\varrho_A = \omega$

(iii) $\alpha \,\varrho_X\, \alpha+$ for all $\alpha \in X^*$.

Remark. If X is a locally finite tree without a least element it may happen that $\alpha+$ is not defined for some α. In this case we define $\alpha+$ to be $\alpha+ : = \mu \notin X$, then the theorem can be proved in the same way as above. If X has no least element μ but $\alpha+$ *is* defined for all α then we can use the same proof as above which shows that in this case $S/\eta \cong X^{++}$ and again $[\eta, \omega] \cong P(X^{++})$. Theorem 5 is a special case of the following fact: let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of 0-simple semigroups and $X^{++} = \{\alpha \in X^*: I^*_\alpha$ is closed under multiplication$\}$ then $S/\eta \cong X^{++}$ $(\cup \{\mu\})$ and $[\eta, \omega] \cong P(X^{++})$

Now we want to study idempotent pure congruences.

Lemma 9. Let ϱ be an idempotent pure congruence on S then $\varrho_G = \varepsilon$. If I^*_μ, the kernel of S is not E-unitary then $\varrho_A = \varepsilon$.

Proof. Let I^*_μ be given in its representation as a subdirect product of a group and an antigroup. Let $x \, \varrho_G \, y$, i.e. $(x, e) \, \varrho \, (y, e)$ for some $e \in E_A$. Then $(1, e) \, \varrho \, (yx^{-1}, e)$, i.e. $x = y$ since ϱ is idempotent pure.

Since A is congruence free, $\varrho_A = \varepsilon$ or $\varrho_A = \omega$. If $\varrho_A = \omega$ then $(x, a) \, \varrho (y, b) \Leftrightarrow x \, \varrho_G \, y$ and $a \, \varrho_A \, b \Leftrightarrow x = y$. In this case $\varrho | I^*_\mu$ is the least group congruence on I^*_μ. Since ϱ is idempotent pure this implies that I^*_μ is E-unitary.

Proposition 3. A congruence ϱ on S is idempotent pure if and only if
(i) $\varrho_G = \varepsilon$
(ii) $\varrho_A = \varepsilon$ if I^*_μ is not E-unitary
(iii) if $\alpha \, \varrho_X \, \beta$ then $f_{\alpha,\alpha\beta}$ and $f_{\beta,\alpha\beta}$ map non idempotent elements onto non idempotent elements.

Proof. Let ϱ be idempotent pure and suppose that $\alpha \, \varrho_X \, \beta$ and $xf_{\alpha,\alpha\beta} \in E$ for some $x \notin E$. $\alpha \, \varrho_X \, \beta$ implies that $x \, \varrho \, xf_{\alpha,\alpha\beta}$ which is a contradiction.

Conversely, suppose that (i) — (iii) hold. ϱ is idempotent pure on I^*_μ since $\varrho | I^*_\mu = \varepsilon$ or $\varrho | I^*_\mu = \sigma | I^*_\mu$ but in the latter case I^*_μ must be E-unitary. Suppose that $x \, \varrho \, e$ for $x \in I^*_\alpha$, $x \notin E$, $e \in E \cap I^*_\beta$. Then $\alpha \, \varrho_X \, \alpha\beta \, \varrho_X \, \beta$ and $xf_{\alpha,\alpha\beta} \, \varrho \, ef_{\beta,\alpha\beta}$. By condition (iii) $xf_{\alpha,\alpha\beta}$ is not idempotent, so we may assume $\alpha = \beta$. Since $\varrho | I^*_\mu$ is idempotent pure, $\alpha > \mu$. $x \neq e$ and the congruence freeness of I_α by lemma 2 imply that $z \, \varrho \, zf_\alpha$ for all $z \in I^*_\alpha$. Then $\alpha \, \varrho_X \, \alpha'$ where $\alpha \succ \alpha'$ and by condition (iii) f_α maps non idempotents onto non idempotents. Now we apply the same procedure to $xf_\alpha \, \varrho \, ef_\alpha$ and we repeat this argument for $\alpha \succ \alpha_1, \alpha_1 \succ \alpha_2, \ldots$ until $\alpha_n = \alpha+$. (For the case (iv) where $\alpha+$ maybe is not defined, we repeat this procedure until $xf_{\alpha,\gamma} = ef_{\alpha,\gamma}$.) Then $\alpha \, \varrho_X \, \alpha+$ and $xf_{\alpha,\alpha+} = ef_{\alpha,\alpha+} \in E$ or $\alpha+ = \mu$ and $xf_{\alpha,\mu} \, \varrho \, ef_{\alpha,\mu}$. The first case is a contradiction to the assumption on the mappings f_α, the second to the assumption that $\varrho | I$ is idempotent pure.

The lattice $\{\varrho_X \colon \varrho \text{ is idempotent pure}\} = [\varepsilon, \tau_X]$ is an interval of the lattice $C(X) \cong P(X^*)$ and thus is isomorphic to some power set lattice: $[\varepsilon, \tau_X] \cong P(\{\xi \colon \xi \leq \tau_X, \xi \text{ is an atom in } C(X)\})$.
The atoms in $C(X)$ are given by the relations
$\{(\alpha, \alpha'), (\alpha', \alpha)\} \cup \varepsilon_S$ where $\alpha \succ \alpha'$ so we have obtained the following theorem

Notation. $X^+ = \{\alpha \in X^* \colon xf_\alpha \notin E \text{ if } x \notin E\}$

Remark. For $\alpha \in X^{**}$ a necessary and sufficient condition in order that $\alpha \in X^+$ is that f_α is injective.

Theorem 6. The lattice of all idempotent pure congruences is Boolean. In particular,

$$[\varepsilon, \tau] \cong \begin{cases} P(X^+) \times C_2 & \text{if the kernel of } S \text{ is no group and } E\text{-unitary} \\ P(X^+) & \text{otherwise} \end{cases}$$

Remark. Immediately from the triple representation $(\varrho_X, \varrho_G, \varrho_A)$ it can be seen that each complement of a semilattice congruence is idempotent pure but the converse does not hold, even if $C(S)$ is Boolean.

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