ON OUTPUT BEHAVIOUR OF MEALY-AUTOMATA

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Let $\mathbf{A} = (A, X, Y, \delta, \lambda)$ be a *Mealy-automaton*, with state set A, input set X, output set Y, transition function δ: $A \times X \to A$ and output function λ: $A \times X \to Y$. In this paper we assume that the output function λ is surjective. The Mealy-automaton $\mathbf{A}$ is *finite*, if the sets A, X and Y are finite.

For a non-empty set Z, Z^* and Z^+ denote the free monoid and the free semigroup over Z, respectively, that is, $Z^+ = Z^* - \{e\}$ where e is the empty word of Z^*.

We extend the functions δ and λ in form δ: $A \times X^* \to A^*$ and λ: $A \times \times X^* \to Y^*$ as follows:

$$\delta(a, e) = a, \ \delta(a, px) = \delta(a, p)\delta(ap, x),$$

$$\lambda(a, e) = e, \ \lambda(a, px) = \lambda(a, p)\lambda(ap, x),$$

where $a \in A$, $p \in X^+$ and $x \in X$, furthermore ap denotes the last letter of $\delta(a, p)$.

The automaton without outputs $\mathbf{A}_{pr} = (A, X, \delta)$ is called the *projection* of $\mathbf{A}$.

The Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ is said to be *cyclic*, if the projection $\mathbf{A}_{pr}$ of $\mathbf{A}$ is cyclic with a generating element a_0, that is, for every $a \in A$ there exists $p \in X^*$ such that $a_0 p = a$. $\mathbf{A}$ is called *strongly connected*, if every state $a \in A$ is a generating element of $\mathbf{A}_{pr}$.

If $r \in Y^+$ then $\overline{r}$ denotes the last letter of r.

The Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ is said to be *output-cyclic*, if there exists $a_0 \in A$ such that

$$\forall y \in Y, \ \exists p \in X^+: \ y = \overline{\lambda(a_0, p)}.$$

a_0 is called an *output-generating element* of $\mathbf{A}$. $\mathbf{A}$ is called *output-strongly connected*, if for every elements $a \in A$ and $y \in Y$ there exists $p \in X^+$ such that $y = \overline{\lambda(a, p)}$. The Mealy-automaton $\mathbf{A}' = (A', X, Y, \delta', \lambda')$ is called an A-subautomaton of $\mathbf{A} = (A, X, Y, \delta, \lambda)$, if $A' \subseteq A$ and $\delta' = \delta|_{A'}$, $\lambda' = \lambda|_{A'}$ are the restriction of δ, λ to $A' \times X$. $\mathbf{A}'$ is called *output-full* if λ' is surjective.

Let $\mathbf{A}=(A, X, Y, \delta, \lambda)$ and $\mathbf{A}' = (A', X', Y', \delta', \lambda')$ be arbitrary Mealy-automata. Then we say that the system (α, β, γ) consisting of the mappings $\alpha\colon A \to A'$, $\beta\colon X \to X'$ and $\gamma\colon Y \to Y'$ is a *homomorphism* of $\mathbf{A}$ into $\mathbf{A}'$ if for arbitrary $a \in A$ and $x \in X$:

$$\alpha(\delta(a, x)) = \delta'(\alpha(a),\ \beta(x))$$

and

$$\gamma(\lambda(a, x)) = \lambda'(\alpha(a), \beta(x))$$

hold. If α, β and γ are onto mappings then $\mathbf{A}'$ is called a *homomorphic image* of $\mathbf{A}$. If α, β and γ are one-to-one mappings the system (α, β, γ) is called an *isomorphism*, and the automata $\mathbf{A}$ and $\mathbf{A}'$ are said to be *isomorphic.* If β and γ are identical mappings on the sets X and Y, respectively, then the homomorphisms (isomorphisms) of such type are called *A-homomorphisms* (*A-isomorphisms*).

Theorem 1. *A Mealy-automaton $\mathbf{A}$ is output-cyclic if and only if $\mathbf{A}$ has an output-full cyclic A-subautomaton.*

Proof. Let the Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ be output-cyclic. Let a_0 be an output-generating element of $\mathbf{A}$. Furthermore, let $A_0 = \{a_0p \mid p \in X^*\}$. If $y \in Y$ then there are $p \in X^*$ and $x \in X$ such that

$$y = \overline{\lambda(a_0, px)} = \lambda(a_0p, x).$$

This means that $\mathbf{A}_0 = (A_0, X, Y, \delta_0, \lambda_0)$ is an output-full cyclic A-subautomaton of $\mathbf{A}$, where $\delta_0 = \delta|_{A_0}$ and $\lambda_0 = \lambda|_{A_0}$. Conversely, let the Mealy-automaton $\mathbf{A}' = (A', X, Y, \delta', \lambda')$ be an output-full cyclic A-subautomaton of $\mathbf{A}$. If a_0 is a generating element of $\mathbf{A}'$, then for every $a \in A'$ there is $p \in X^*$ such that $a = a_0p$. If $y \in Y$ then there are $a \in A'$, $x \in X$ such that $y = \lambda(a, x)$. Thus

$$y = \lambda(a, x) = \lambda(a_0p, x) = \overline{\lambda(a_0, px)}.$$

This means that a_0 is an output-generating element of $\mathbf{A}$.

Corollary 1. *Every cyclic Mealy-automaton is output-cyclic.*

A Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ is *covered* by the Mealy-automata $\mathbf{A}_i = (A_i, X, Y, \delta_i, \lambda_i)(i \in I)$ if $A = \bigcup_{i\in I} A_i$, $\delta|A_i = \delta_i$ and $\lambda|A_i = \lambda_i$.

Corollary 2. *A Mealy-automaton $\mathbf{A}$ is output-strongly connected if and only if it is covered by its certain output-full cyclic A-subautomata.*

Corollary 3. *Every strongly connected Mealy-automaton is output-strongly connected.*

We note that a homomorphic image of an output-cyclic (output-strongly connected) Mealy-automaton is output-cyclic (output-strongly connected), too.

The equivalence relation τ on the state set A of the Mealy-automaton $A = (A, X, Y, \delta, \lambda)$ is called a *congruence* on $\mathbf{A}$, if for every $p \in X^+$:

$$(a, b) \in \tau \Rightarrow [(ap, bp) \in \tau \quad \text{and} \quad \overline{\lambda(a, p)} = \overline{\lambda(b, p)}] \quad (a, b \in A).$$

We define the following relation σ on A:

$$(a, b) \in \sigma \Leftrightarrow [\forall p \in X^+ : \overline{\lambda(a, p)} = \overline{\lambda(b, p)}].$$

It is evident that σ is a congruence on $\mathbf{A}$ and if τ is a congruence on $\mathbf{A}$, then $\tau \leqq \sigma$.

The Mealy-automaton $\mathbf{A}$ is called *simple*, if

$$[\forall p \in X^+ : \overline{\lambda(a, p)} = \overline{\lambda(b, p)}] \Rightarrow a = b,$$

that is, σ is the equality relation on A. With other words, every A-homomorphisms of $\mathbf{A}$ are A-isomorphism of $\mathbf{A}$.

$\mathbf{A}$ is called *state-independent*, if for every $p, q \in X^+$ and $b \in A$:

$$bp = bq \Rightarrow [\forall a \in A : ap = aq].$$

Similarly, $\mathbf{A}$ is *output-independent* if for every $p, q \in X^+$ and $b \in A$:

$$\overline{\lambda(b, p)} = \overline{\lambda(b, q)} \Rightarrow [\forall a \in A : \overline{\lambda(a, p)} = \overline{\lambda(a, q)}].$$

Theorem 2. *If the simple Mealy-automaton is output-independent, then it is state-independent.*

Proof. Let $ap = aq (a \in A;\ p, q \in X^+)$. Then for every $r \in X^+$:

$$\overline{\lambda(a, pr)} = \overline{\lambda(ap, r)} = \overline{\lambda(aq, r)} = \overline{\lambda(a, qr)}.$$

But $\mathbf{A}$ is output-independent, thus for every $b \in A$:

$$\overline{\lambda(bp, r)} = \overline{\lambda(b, pr)} = \overline{\lambda(b, qr)} = \overline{\lambda(bq, r)}.$$

Since $\mathbf{A}$ is simple, therefore $bp = bq$.

In the following example it is shown that the converse of Theorem 2 does not hold.

Example 1. We define the state-independent simple Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ such that

$$A = \{1, 2\},\ X = \{x_1, x_2\},\ Y = \{y_1, y_2\},$$

$$\delta(1, x_1) = \delta(1, x_2) = 2,\ \delta(2, x_1) = \delta(2, x_2) = 1,$$

$$\lambda(1, x_1) = \lambda(1, x_2) = y_1,\ \lambda(2, x_1) = y_1,\ \lambda(2, x_2) = y_2.$$

$\mathbf{A}$ is not output-independent.

Let $\varrho_{\mathbf{A},a}$ be a right congruence on X^+ defined by

$$\forall p, q \in X^+ : [(p, q) \in \varrho_{\mathbf{A},a} \Leftrightarrow ap = aq]$$

for every $a \in A$. The *Myhill—Nerode congruence* $\varrho_{\mathbf{A}}$ of $\mathbf{A}_{pr}$ is defined by $\varrho_{\mathbf{A}} = \bigcap_{a \in A} \varrho_{\mathbf{A},a}$.

Let $\breve{\varrho}_{\mathbf{A},a}$ be an equivalence on X^+ defined by

$$\forall p, q \in X^+ \colon \; [(p, q) \in \breve{\varrho}_{\mathbf{A},a} \Leftrightarrow \overline{\lambda(a, p)} = \overline{\lambda(a, q)}]$$

for every $a \in A$. The left congruence $\breve{\varrho}_{\mathbf{A}}$ on X^+ is defined by $\breve{\varrho}_{\mathbf{A}} = \bigcap_{a \in A} \breve{\varrho}_{\mathbf{A},a}$. The *Peák-congruence* $\varrho'_{\mathbf{A}}$ of $\mathbf{A}$ is defined by $\varrho'_{\mathbf{A}} = \varrho_{\mathbf{A}} \cap \breve{\varrho}_{\mathbf{A}}$. The factor semigroup $S(A) = X^+ | \varrho'_{\mathbf{A}}$ is called the *characteristic semigroup of the Mealy-automaton* $\mathbf{A}$.

Theorem 3. *The characteristic semigroup of a simple output-independent Mealy-automaton is left cancellative.*

Proof. Indeed, for all $p, q, r \in X^+$ we get that

$$(rp, rq) \in \varrho'_{\mathbf{A}} \Leftrightarrow [(rp, rq) \in \varrho_{\mathbf{A}} \quad \text{and} \quad (rp, rq) \in \breve{\varrho}_{\mathbf{A}}] \Leftrightarrow \forall a \in A \colon$$

$$[arp = arq \quad \text{and} \quad \overline{\lambda(a, rp)} = \overline{\lambda(a, rq)}].$$

Thus

$$\forall a \in A \colon \; \overline{\lambda(ar, p)} = \overline{\lambda(a, rp)} = \overline{\lambda(a, rq)} = \overline{\lambda(ar, q)}.$$

But $\mathbf{A}$ is output-independent, therefore

$$\forall b \in A \colon \; \overline{\lambda(b, p)} = \overline{\lambda(b, q)}.$$

By Theorem 2, $\mathbf{A}$ is state-independent, thus

$$[\forall a \in A \colon \; arp = arq] \Rightarrow [\forall b \in A \colon \; bp = bq].$$

This means that $(p, q) \in \varrho'_{\mathbf{A}}$.

In Example 2 it is shown that the converse of Theorem 3 does not hold.

Example 2. Let

$$A = \{1, 2, 3\}, \; X = \{x\}, \; Y = \{y_1, y_2\},$$

$$\delta(1, x) = 2, \; \delta(2, x) = 3, \; \delta(3, x) = 1,$$

$$\lambda(1, x) = \lambda(2, x) = y_1, \; \lambda(3, x) = y_2.$$

The Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ is simple, but it is not output-independent. Let τ be an equivalence on X^*. Then $\tau[p]$ denotes the τ-class containing $p \in X^*$. We get that

$$\varrho_{\mathbf{A}}[e] = \{e, x^3, x^6, \ldots\}, \; \varrho_{\mathbf{A}}[x] = \{x, x^4, x^7, \ldots\}, \; \varrho_{\mathbf{A}}[x^2] = \{x^2, x^5, x^8, \ldots\},$$

$$\breve{\varrho}_{\mathbf{A}}[x] = \varrho_{\mathbf{A}}[x], \; \breve{\varrho}_{\mathbf{A}}[x^2] = \varrho_{\mathbf{A}}[x^2], \; \breve{\varrho}_{\mathbf{A}}[x^3] = \varrho_{\mathbf{A}}[e] - \{e\}.$$

Thus $\varrho_A = \varrho_A$. This means that $S(A)$ is left cancellative.

Let $\mathrm{A} = (A, X, Y, \delta, \lambda)$ be a Mealy-automaton. $G(\subseteq A)$ is an ***output-generating system*** of A, if

$$Y = \{\overline{\lambda(a, p)} \mid a \in G,\ p \in X^+\}.$$

A is called *characteristically output-free*, if there exists an output-generating system G of A such that

$$\overline{\lambda(a, p)} = \overline{\lambda(b, q)} \Rightarrow [a = b \quad \text{and} \quad (p, q) \in \breve{\varrho}_A]$$

$(a, b \in G, p, q \in X^+)$. In this case, G is called a *characteristically output-free system*.

Example 3. Let

$$A = \{a_0, a_1, a_2\},\ X = \{x_1, x_2\},\ Y = \{y_1, y_2\},$$

$$\delta(a_0, x_i) = a_i,\ \delta(a_1, x_i) = a_2,\ \delta(a_2, x_i) = a_1,$$

$$\lambda(a_0, x_i) = y_1,\ \lambda(a_1, x_i) = \lambda(a_2, x_i) = y_2\ (i = 1, 2).$$

Since $\lambda(a_0, x_1) = \lambda(a_0, x_2) = y_1$ and

$$\forall p \in X^+ - X\colon\ \overline{\lambda(a_0, p)} = y_2,$$

therefore $\mathrm{A} = (A, X, Y, \delta, \lambda)$ is a characteristically output-free cyclic Mealy-automaton.

We denote the cardinality of a set B by $|B|$.

Theorem 4. *Every characteristically output-free system G of a Mealy-automaton $\mathrm{A} = (A, X, Y, \delta, \lambda)$ is minimal among the output-generating systems of A in the usual sense. Furthermore, if A is finite, then*

$$|G| = \frac{|Y|}{|X^+/\breve{\varrho}_A|}. \tag{1}$$

Proof. Let G be a characteristically output-free system of A. Assume that there is an output-generating system G' of A such that $G' \subset G$. If $a \in G - G'$ then for every $p \in X^+$ there are $a_0 \in G'$ and $q \in X^+$ such that

$$\overline{\lambda(a, p)} = \overline{\lambda(a_0, q)}.$$

But G is a characteristically output-free system, thus $a = a_0$. It is impossible. This means that G is a minimal output-generating system of A.

The mapping $\varphi\colon G \times X^+/\breve{\varrho}_A \to Y$ such that

$$\varphi(a, \breve{\varrho}_A[p]) = \overline{\lambda(a, p)}\ (a \in G,\ p \in X^+)$$

is a one-to-one mapping of $G \times X^+/\breve{\varrho}_A$ onto Y. Therefore, if A is finite, then (1) is true.

The Mealy-automaton $\mathbf{A} = (A, X\ \ Y, \delta, \lambda)$ is the *direct sum* of the Mealy-automata $\mathbf{A}_i = (A_i, X, Y_i, \delta_i, \lambda_i)(i \in I)$ if $A = \bigcup_{i \in I} \mathbf{A}_i$, $Y = \bigcup_{i \in I} Y_i$, $\delta|A_i = \delta_i$ and $\lambda|A_i = \lambda_i$. Furthermore, for every $i \neq j (\in I) A_i \cap A_j = \Phi$ and $Y_i \cap Y_j = \Phi$.

Theorem 5. *The simple Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ is characteristically output-free if and only if there is an A-subautomaton of $\mathbf{A}$ such that it is a direct sum of isomorphic characteristically output-free cyclic Mealy-automata.*

Proof. Let the simple Mealy-automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ be characteristically output-free and let G be a characteristically output-free system of $\mathbf{A}$. Take the sets

$$A_b = \{bp \mid p \in X^*\} \ \ (b \in B).$$

Assume that $b_1, b_2 \in G$ and $A_{b1} \cap A_{b_2} \neq \Phi$. If $b_1 p = b_2 q (p, q \in X^*)$, then for $x \in X$:

$$\overline{\lambda(b_1, px)} = \lambda(b_1 p, x) = \lambda(b_2 q, x) = \overline{\lambda(b_2, px)}.$$

Thus $b_1 = b_2$. That is, $A_{b_1} = A_{b_2}$.

For every $b \in G$, let

$$Y_b = \{\overline{\lambda(b, q)} \mid q \in X^+\}.$$

$\mathbf{A}_b = (A_b, X, Y_b, \delta_b, \lambda_b)$ is a characteristically output-free cyclic Mealy-automaton. It is evident that for every $b_1 \neq b_2 \in G$, $Y_{b_1} \cap Y_{b_2} = \Phi$ and $Y = \bigcup_{b \in G} Y_b$. Let $A_1 = \bigcup_{b \in G} A_b$. $\mathbf{A}_1 = (A_1, X, Y_1, \delta_1, \lambda_1)$ is an A-subautomaton of $\mathbf{A}$.

Let $b_1, b_2 \in G$. We define the following mappings φ and ψ:

$$\varphi\colon\ b_1 p \to b_2 p \ (p \in X^*),$$

$$\psi\colon\ \overline{\lambda(b_1, q)} \to \overline{\lambda(b_2, q)} \ (q \in X^+).$$

It is obvious that ψ is one-to-one. If $b_1 p = b_1 p' (p, p' \in X^*)$, then for every $r \in X^+$:

$$\overline{\lambda(b_1, pr)} = \overline{\lambda(b_1 p, r)} = \overline{\lambda(b_1 p', r)} = \overline{\lambda(b_1, p'r)}.$$

Since $\mathbf{A}$ is characteristically output-free, thus $(pr, p'r) \in \varrho_A$. That is,

$$\overline{\lambda(b_2 p, r)} = \overline{\lambda(b_2, pr)} = \overline{\lambda(b_2, p'r)} = \overline{\lambda(b_2 p', r)}$$

for every $r \in X^+$. But $\mathbf{A}$ is simple, thus $b_2 p = b_2 p'$. This means that φ is one-to-one.

$$\varphi(\delta_{b_1}(b_1 p, x)) = \varphi(b_1 px) = b_2 px = \delta_{b_2}(b_2 p, x) = \delta_{b_2}(\varphi(b_1 p), x))$$

and

$$\psi(\lambda_{b_1}(b_1p, x)) = \psi(\overline{\lambda(b_1, px)}) = \overline{\lambda(b_2, px)} = \lambda_{b_2}(b_2p, x) = \lambda_{b_2}(\varphi(b_1p), x)$$

$(p \in X^*, x \in X)$. This means that (φ, ι, ψ) is an isomorphism of $\mathbf{A}_{b_1}$ onto $\mathbf{A}_{b_2}$, where ι is the identity mapping of X. We get that $\mathbf{A}_1$ is the direct sum of isomorphic characteristically output-free cyclic Mealy-automata $\mathbf{A}_b (b \in G)$.

Conversely, let $\mathbf{A}_1$ be an A-subautomaton of $\mathbf{A}$ and let $\mathbf{A}_1$ be a direct sum of isomorphic characteristically output-free cyclic Mealy-automata $\mathbf{A}_{b_i} (i \in I)$. ($b_i$ is a generating element of $\mathbf{A}_{b_i}$. We note that b_i is an output-generating element of $\mathbf{A}_{b_i}$, too.) Let $(\varphi_{i,j}, \iota, \psi_{i,j})$ be isomorphic mappings of $\mathbf{A}_{b_i}$ onto $\mathbf{A}_{b_j} (i \neq j \in I)$. Then every $p \in X^*$:

$$\varphi_{i,j}(b_i p) = \varphi_{i,j}(b_i) p = b_j p.$$

Thus $\varphi_{i,j}(b_i) = b_j$. Let $G = \{b_i \mid i \in I\}$. G is an output-generating system of $\mathbf{A}$. Let

$$\overline{\lambda(b_i, p)} = \overline{\lambda(b_j, q)} \quad (p, q \in X^+).$$

Then $b_i = b_j$ and thus $i = j$. Let $k \in I$. Then

$$\overline{\lambda(b_k, p)} = \overline{\lambda_k(b_k, p)} = \overline{\lambda_k(\varphi_{i,k}(b_i), p)} = \overline{\psi_{i,k}(\lambda_i(b_i, p))} =$$
$$= \overline{\psi_{i,k}(\lambda_i(b_i, q))} = \overline{\lambda_k(\varphi_{i,k}(b_i), q)} = \overline{\lambda(b_k, q)}.$$

Thus $(p, q) \in \varrho_{\mathbf{A}}$. This means that $\mathbf{A}$ is characteristically output-free.

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