ON TWO CONJECTURES RELATED TO ADMISSIBLE GROUPS AND QUASIGROUPS

To the memory of Prof. K. Buzási

J. DÉNES and A. D. KEEDWELL*

1122 Budapest, Hungary
*Dept. of Mathematical and Computing Sciences Univ. Surrey, Guildford, England

Received August 11, 1988

A finite group (quasigroup) is said to be admissible if it has a permutation mapping of the form $g \to \alpha(g)$ such that $g \to g\alpha(g)$ is also a permutation.

The study of admissible groups is an important subject. To mention just one of the applications of admissible groups, we call the reader's attention to the fact that if L denotes the latin square which represents the multiplication table of a group of odd order, then L has an orthogonal mate (see Theorems 1.4.3, 5.1.1 in [2]).

More than thirty years ago L. J. PAIGE and M. HALL proposed two conjectures for non-soluble groups:

(1) if the product of all elements, in some order, is equal to the identity element, then the group is admissible;

(2) if the Sylow 2-groups are non-cyclic, then the group is admissible (see [9], [4] and Problems 1.5, 1.6 in [2]). In [3] it has been proved that the two conjectures above can be replaced by the following one: All non-soluble finite groups are admissible.

The aim of the present note is to formulate a new conjecture and to prove that the above-mentioned conjecture is implied by the new one.

A square array of size n by n with n^2 elements in which each row contains each distinct element once is called a row-latin square (Similarly, a column-latin square is a square where each column contains each distinct element once). A square which is both column-latin and row-latin is a latin square. (For a detailed description of properties of latin squares and their generalizations see [2]).

In [7] and [8], a product operation for latin squares has been introduced.

Let the distinct elements of an $n \times n$ row-latin square be designated by the integers $0, 1, \ldots, n-1$. Then the i-th row of the square determines a permutation α_i of these integers from their natural ordering. The square is completely determined by the permutations $\alpha_1 = \varepsilon, \alpha_2, \alpha_3, \ldots, \alpha_n$ which define its rows as permutations of the first row. The product of two row-latin squares $A = (\alpha_1\alpha_2 \ldots \alpha_n)^T$ and $B = (\beta_1\beta_2 \ldots \beta_n)^T$ is defined to be the row-latin square $AB = (\alpha_1\beta_1, \alpha_2\beta_2 \ldots \alpha_n\beta_n)^T$ whose i-th row is given by the product permutation $\alpha_i\beta_i$. Clearly, a consequence of this definition is that $\sqrt{A} = (\sqrt{\alpha_1}\sqrt{\alpha_2} \ldots \sqrt{\alpha_n})$ and that this row-latin square exists if and only if each of the permutations α_i, $i = 1, 2, \ldots, n$, has a square root.

Now, the square of a cycle of odd length is another cycle of the same odd length, the square of a cycle of singly-even length $4t + 2$ is a product of two cycles of odd length $2t + 1$ and the square of a cycle of doubly-even length $4t$ is a product of two cycles of even length $2t$. Thus, the square of any permutation has an even number of cycles of even length. Moreover, any permutation with the latter property has at least one square root since

$$(a_1a_2 \ldots a_{2t+1}) = (a_1a_{t+2}a_2a_{t+3} \ldots a_ta_{2t+1}a_{t+1})^2$$

and

$$(a_1a_2 \ldots a_{2t})(b_1b_2 \ldots b_{2t}) = (a_1b_1a_2b_2 \ldots a_{2t}b_{2t})^2.$$

If the permutation has more than one cycle of equal odd length then it has at least two (and possibly many) square roots since each product of two cycles of equal odd length has two distinct square roots. For example,

$$(0\ 5\ 1\ 6\ 2\ 7\ 3\ 8\ 4\ 9) \quad \text{and} \quad (0\ 3\ 1\ 4\ 2)(5\ 8\ 6\ 9\ 7)$$

are both square roots of the cycle product $(0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)$. For further results on square or higher roots of permutations, see [1], [5], [6].

Next, we remak that, if $(G, \cdot)$ is a finite non-soluble group, then its multiplication table is a latin square L such that $\sqrt{L}$ exists.

Proof. Let $L = (\alpha_1\alpha_2 \ldots \alpha_n)^T$. Then the permutation α_i is the Cayley representation of the i-th element g_i of G and so it is a regular permutation with cycles of length equal to the order of the element g_i in G. Let us suppose that, for at least one value of i, $\sqrt{\alpha_i}$ does not exist. In that case, α_i consists of an odd number k of cycles of even length, where k divides ord G. Let ord $G = n = 2^rhk$, where h and k are odd integers and $r \geq 1$. Then each of the cycles of α_i has length 2^rh and so α_i^h has order 2^r. That is, the element g_i^h of G which is represented by α_i^h generates a cyclic Sylow 2-subgroup of G. However, it is well-known (see, for example, Theorem 2.10 of [10]) that the Sylow 2-subgroups of a non-soluble group are not cyclic. This contradiction shows that $\sqrt{\alpha_i}$ must exist.

Combining this with our previous conjecture in [3] that all finite non-soluble groups are admissible, we are led to make the following further conjecture:

Conjecture 1. If L is the multiplication table of a non-soluble group then, not only does L have at least one, and possibly many, square roots $\sqrt{L}$, but at least one of these square-root squares is a latin square.

If the conjecture were true, it would follow that the latin squares L and $\sqrt{L}$ are orthogonal and consequently that G is an admissible group. To see this, we remark that $(\sqrt{L})^{-1}L = \sqrt{L}$ and so the result follows from a theorem of H. B. MANN [7].

We end this note by combining Mann's result with our conjecture above to give:

Conjecture 2. A necessary and sufficient condition for a latin square A to have an orthogonal mate is that either A^2 is a latin square or A can be represented as the product $A = BC$, of two not-necessarily-distinct latin squares B and C.

The sufficiency is clear: only the necessity remains in doubt.

References

1. ČUPONA, G., SAMARDŽISKI, A., CELAKOSKI, N.: Roots of permutations, Proceedings of the Symposium n-ary structures, Skopje 1982, 85—95.
2. DÉNES, J., KEEDWELL, A. D.: Latin squares and their applications, Academic Press, New York, Akadémiai Kiadó, Budapest, English Universities Press, London, 1974.
3. DÉNES, J., KEEDWELL, A. D.: A new conjecture concerning admissibility of groups, European Journal of Combinatorics. 18 (1989) 171—174.
4. HALL, M., PAIGE, L. J.: Complete mappings of finite groups, Pacific J. Math. *5* (1955), 541—549.
5. HIGGINS, P. M.: A method for constructing square roots in finite full transformation semigroup, Bull. Canadian Math. Soc. *29* (1968), 344—351.
6. HOWIE, I. M., SNOWDEN, M.: Square roots in finite full transformation semigroups, Glasgow Math. J. *23* (1982), 137—149.
7. MANN, H. B.: The construction of orthogonal latin squares, Ann. Math. Statist., *13* (1942) 418—423.
8. NORTONS, D. A.: Groups of orthogonal row-latin squares, Pacific J. Math. *2* (1952), 335—431.
9. PAIGE, L. J.: A note on finite abelian groups, Bull. Amer. Math. Soc. *53* (1947), 590—593.
10. SUZUKI, M.: Group theory II, Springer Verlag, New York, 1968

J. DÉNES 1122 Budapest, Csaba u. 10. Hungary

A. D. KEEDWELL Dept. of Mathematical and Computing Sciences Univ. Surrey, Guildford, England