MODULAR GROUP RINGS AND WREATH PRODUCTS OF GROUPS

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Now we give some definitions and notations. Let A and B be two groups. Define A^B to be the set of all functions f: $B \to A$ such that $f(b) \neq 1$ for all but a finite number of $b \in B$, and with componentwise multiplication

$$fg(b) = f(b)g(b) \text{ for all } b \in B.$$

Then A^B is a restricted direct product of copies of A. The standard restricted wreath product of A by B, $A \wr B$ is defined as the extension of A^B by B given by

$$f^b(b') = f(b'b^{-1}) \text{ for all } f \in A^B;\ b, b' \in B.$$

As we only use the standard restricted wreath product, we will simply call it the wreath product.

In the following let Z_{p^n} be the ring of the integers mod p^n, where p is a prime number, and let $G = A \wr B$ be a wreath product of a cyclic group A by a finite p-group B. We denote by $K = A^B$ the base group of G.

If we consider K as an additive group, then an arbitrary $b \in B$ generates an automorphism φ_b of group K defined in the following way

$$k\varphi_b = b^{-1}kb; \quad b \in B; \quad k \in K.$$

Denoting $k\,\varphi_b = kb$, the additive group K can be identified with a B-module, just as $G = A \wr B$ may be realized as the group ring $Z_{p^n}[B]$ extended by B in its right regular representation. Using automorphism φ_b element $[k, b]$ can be written in the form $k(1 - b)$, where $k \in K$, $b \in B$.

As it is known, the augmentation ideal $I(Z_{p^n}, B)$ of the group ring $Z_{p^n}[B]$ as an additive group can be generated by elements of form $1 - b$, $b \in B$ and $I^k(Z_{p^n}, B)$ can be generated by elements

$$(1 - b_{i,1}) \dots (1 - b_{i,k})$$

where $b_{i,j}$ are arbitrary elements of group B.
It can be seen from these ([2]) that

$$\gamma_i(G) = [KI^{i-1}(Z_{p^n}, B)]\gamma_i(B) \tag{1}$$

where $\gamma_i(G)$ and $\gamma_i(B)$ denote the i-th member of lower central series of corresponding groups.

Denoting by $\nu(n, B)$ the nilpotency class of $I(Z_{p^n}, B)$ and using (1) we obtain

$$\nu(n, B) = cl \quad (C_{p^n} \wr B).$$

Therefore we get the $cl\,(A \wr B)$ by determining $\nu(n, B)$, where A is an abelian group with finite exponent p^n and B is an arbitrary finite p-group.

Now we show some simple connections between nilpotent wreath products and modular group rings.

Let $B = \gamma_1(B) \supset \gamma_2(B) \supset \ldots \supset \gamma_{L+1}(B) = 1$ be the lower central series of B. We denote by $r(j)$ the rank of $\gamma_i(B)/\gamma_{i+1}(B)$ and $B_i = \{b(i, j), 1 \leq j \leq r(j)\} \subseteq \subseteq \gamma_i(B) - \gamma_{i+1}(B)$ denotes an independent set of generators for $\gamma_i(B)$ modulo $\gamma_{i+1}(B)$, such that $|b(i, j)| = p(i, j)$ where $p(i, 1), \ldots, p(i, r(i))$ are the orders, indescending powers of p, of cyclic factors of $\gamma_i(B)/\gamma_{i+1}(B)$.

Theorem 1. [5]. If A is an abelian p-group of exponent p^n and $W = A \wr B$, then

$$cl\,(W) = \sum_{i=1}^{L} \sum_{j=1}^{r(j)} i(p(i, j) - 1) + (n - 1)mp(m, 1)(p - 1)p^{-1} + 1$$

$$mp(m, 1) = \max_{1 \leq i \leq L} \{i\, p(i, 1)\}.$$

We define the Jennings-series of a nilpotent p-group G by the equation

$$K_i(G) = \prod_{n \cdot pj \geq i} \gamma_n(G)^{p^j},$$

where $\gamma_n(G)$ is the n-ts member of the lower central series of $G (i \geq 1)$.
Let B be a finite p-group and d a maximal number such that $K_d(B) \neq 1$.
Let $p^{e(j)} = |K_j(B)/K_{j+1}(B)|$ and $a = 1 + (p - 1) \sum_{j=1}^{d} je(j)$, $b = (p - 1)d$.

Theorem 2. [6]. If A is a nilpotent p-group of class c, B is a finite p-group and the exponent of $\gamma_\omega(A)$ equals $p^{s(\omega)}$ $(1 \leq \omega \leq c)$, then

$$cl\,(A \wr B) = \max \{a\omega + (s(\omega) - 1)b \mid 1 \leq \omega \leq c\}.$$

This shows that to know the lower central series is important for the investigation of nilpotent wreath products.

Corollary 3. If A is an abelian p-group, and exponent of A equals p^n, then

$$\nu(n, B) = \nu(1, B) + (n - 1)(p - 1)d.$$

Proof. Applying theorem 2. with $c = 1$ the statement follows from $a = \nu(1, B)$ ([1]).

Remark: The class of the radical of $Z_{p^n}[B]$ is equal to $\nu(n, B)$ if $\nu(n-1, B) < \nu(n, B)$ for all n ([2]).

By corollary 3. the last property is true for all finite p-groups. It means that determination of nilpotency classes of nilpotent wreath products has an important application in the theory of modular group rings.

Theorem 4. If B is the group defined in the theorem 1, then the nilpotency class of the radical of $Z_p[B]$ equals

$$\nu(1, B) = \sum_{i=1}^{L} \sum_{j=1}^{r(j)} (i\, p(i, j) - 1) + 1,$$

and the length of the Jennings-series of B equals

$$\alpha = p^{-1} \max_{1 \le i \le L} i\, p(i, 1)\, .$$

Proof: Combining theorem 1. and corollary 3. we obtain the statement (omitting the term depending on the n).

Example. Similarly to the example 4.3 in [5] determine the $\nu(n, C_{p^m} \wr C_p)$ (where C^{pm} denotes a cyclic group of order p^m).

Let $B = C_{p^m} \wr Cp$, $A = C_{p^n}$ and K be the base group of B. Then $|B| = p^{mp+1}$ and by the arguments in [7] B/K abelian, $\gamma_2(B) \subset K$, $\gamma_1(B)/\gamma_2(B) = C_{p^m} \times C_p$, $r(i) = 1$, $p(i, 1) = p$ for $i = 2,3, \ldots, m(p-1)+1$. Thus by the theorem 1. the class of $G = A \wr B$ is also

$$\nu(n, B) = p^m\bigl(1 + n(p-1)\bigr) + (p-1)\frac{m(p-1)+2}{2}\bigl(m(p-1)+1\bigr).$$

Theorem 5. Let

$$G = (\ldots(A_k \wr A_{k-1}) \wr \ldots \wr A_2) \wr A_1$$

where $A_j = A_{j,1} \times \ldots \times A_{j,r(j)}, |A_{j,i}| = p^{\alpha(j,i)}$,

$$p^{\alpha(j,1)} \ge p^{\alpha(j,2)} \ge \ldots \ge p^{\alpha(j,r(i))} \quad (j = 1,2, \ldots, k)$$

then

$$cl\,(G) \ge \prod_{j=1}^{k-1} \Bigl(\sum_{i=1}^{r(j)} (p^{\alpha(j,i)} - 1) + 1\Bigr) + \prod_{j=1}^{k-2} \Bigl(\sum_{i=1}^{r(j)} (p^{\alpha(j,i)} - 1) + 1\Bigr)\bigl(\alpha(k, 1) - 1\bigr) \cdot \cdot \bigl(p^{\alpha(k-1,1)-1}(p-1) + 1\bigr). \qquad (2)$$

If $r(j) = 1$ for all $j = 1,2, \ldots, k$ or $\alpha(k, 1) = \alpha(k, 2) = \ldots = \alpha\bigl(k, \mathrm{r}(k)\bigr)$ then the equality holds in (2).

Proof. We apply the induction by the number of wreath factors. If $j = k-1$, then

$$cl(A_k \wr A_{k-1}) = \sum_{i=1}^{r(k-1)} (p^{\alpha(k-1,i)} - 1) + \bigl(\alpha(k, 1) - 1\bigr) + \bigl(\alpha(k, 1) - 1\bigr) \cdot \cdot p^{\alpha(k-1,1)-1}(p-1) + 1.$$

If $A = (\ldots(A_k \wr A_{k-1}) \wr \ldots) \wr A_{j-1}$ then $cl(A_p \wr A_j) \geq cl(C_p \wr A_j) \cdot cl(A)$ because in the case of $\exp(Z(A)) = p$ by theorem 2. $cl(A \wr A_j) = \nu(1, A_j)\, cl(A)$ (see also [4] theorem 2.5) and $\exp(Z(A)) \geq p$, $s(\omega) \geq 1$ for all $1 \leq \omega \leq c$.

$$\text{As } cl(C_p \wr A_j) = \sum_{i=1}^{r(j)} (p^{\alpha(j,i)} - 1) + 1$$

the induction hypothesis implies the statement of theorem 5.
If $\exp(Aj) = p$ for all $j = 1, 2, \ldots, k$ then

$$cl(G) = \sum_{i=1}^{k-1} \prod_{j=1}^{r(i)} (p^{\alpha(i,j)} - 1) + 1 \qquad \text{holds.}$$

If every $A_j (j = 1, 2, \ldots, k)$ is cyclic, then reducing the determination of nilpotency class of G to the determination of nilpotency class of two cyclic groups ([8]), we obtain

$$cl(G) = \alpha(k, 1)\left(p^{\sum_{i=1}^{k-1} \alpha(i,1)} - p^{\sum_{i=1}^{k-1} \alpha(i,1)-1}\right) + p^{\sum_{i=1}^{k-1} \alpha(i,1)-1}.$$

This completes the proof.

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