REFLEXIVE UNITARY SUBSEMIGROUPS OF LEFT SIMPLE SEMIGROUPS

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Introduction

Ideal series of semigroups [1] play an important role in the examination of semigroups which have proper two-sided ideals. But the corresponding theorems cannot be used when left simple (or right simple or simple) semigroups are considered. So it is a natural idea that we try to use the group-theoretical methods (instead of the ring-theoretical ones) for the examination of these semigroups.

The purpose of this paper is to find a suitable type of subsemigroups of left simple semigroups which makes possible for us to generalize some notions (the notion of the normal series and the composition series of groups) and some results concerning the groups to the left simple semigroups. We note that the subsemigroups we are looking for are the reflexive unitary subsemigroups of left simple semigroups.

For notations and notions not defined here, we refer to [1] and [2].

1. Reflexive unitary subsemigroups

As it is known [1], a subset H of a semigroup S is said to be *reflexive* in S if $ab \in H$ implies $ba \in H$ for all $a, b \in S$.

We say that a subset U of a semigroup S is *left* [*right*] *unitary* in S (see [1]) if $ab, a \in U$ implies $b \in U$ [$ab, b \in U$ implies $a \in U$] for all $a, b \in S$. A subset of S which is both left and right unitary in S is said to be *unitary* in S.

We note that if $A \subseteq B$ are subsemigroups of a semigroup S and B is unitary in S, then A is unitary in B if and only if A is unitary in S. In this paper we shall use this fact without comment.

It can be easily verified that, in a group, the unitary subsemigroups are exactly the subgroups, and the reflexive unitary subsemigroups are exactly the normal subgroups. So the notion of the reflexive unitary subsemigroup of a semigroup can be considered as a generalization of the notion of the normal subgroup of a group.

Reflexive unitary subsemigroups are important in the description of the group congruences of left simple semigroups. To show this importance, we need the following.

Let S be a semigroup and H a non-empty subset of S.
Let

$$H \dots a = \{(s, t) \in S \times S: \quad sat \in H\}, \ a \in S.$$

We can define a relation $P_H(S)$ on S as follows:

$$P_H(S) = \{(a, b) \in S \times S: \quad H \dots a = H \dots b\}.$$

It can be easily verified that $P_H(S)$ is a congruence on S, such that $W^H = \{c \in S: \quad H \dots c = \emptyset\}$ is a $P_H(S)$-class and an ideal of S. $P_H(S)$ is called the *principal congruence* on S determined by H (see [1]).

If we consider and fix a semigroup S then, for a subset H of a subsemigroup N of S, the principal congruence on N determined by H will be denoted by $P_H(N)$. We shall use P_H instead of $P_H(S)$. For short, we shall use N/P_H instead of $N/P_H(N)$. $P_H|N$ will denote the restriction of P_H to N, that is $P_H|N = P_H(S) \cap (N \times N)$.

We note that if α a is congruence on a semigroup S then we shall not distinguish the congruence classes of S modulo α from the elements of the factor semigroup S/α.

Let S denote a left simple semigroup. Then, by Theorem 10.34, Corollary 10.35 and Exercise 17 for § 10.2 of [1], if H is a reflexive unitary subsemigroup of S then the factor semigroup S/P_H is a group with identity element H and, conversely, if P is a congruence on S such that S/P is a group with identity H then H is a reflexive unitary subsemigroup of S and $P = P_H$.

The following theorem is important for the next.

Theorem 1. *Every right unitary subsemigroup of a left simple semigroup is left simple.*

Proof. Let N be a right unitary subsemigroup of a left simple semigroup S. Consider two arbitrary elements a and b in N. Then there is an element s in S such that $sa = b$. As N is right unitary in S, it follows that $s \in N$ which implies that N is left simple. Thus the theorem is proved.

The following theorem shows that, fixing a reflexive unitary subsemigroup H of a left simple semigroup S, there is a one-to-one correspondence between the set of all unitary subsemigroups of S containing H and the set of all subgroups of S/P_H. In this connection the reflexive unitary subsemigroups correspond to the normal subgroups of S/P_H.

Theorem 2. *Let $H \subseteq N$ be subsemigroups of a left simple semigroup S such that H is reflexive and unitary in S. Then N is unitary in S if and only if N is saturated by P_H and N/P_H is a subgroup of S/P_H. In this case $P_H(N) = P_H|N$. If N is unitary in S, then it is also reflexive in S if and only if N/P_H*

is a normal subgroup of S/P_H. If this is the case, then $(S/P_H)/(N/P_H)$ is isomorphic with S/P_N.

Proof. Assume that N is a unitary subsemigroup of S with $H \subseteq N$. Let $a \in N$ be arbitrary. As S is left simple, there are elements u, v in N such that $uav \in H$. As H is reflexive in S, $vua \in H$. Since N is unitary in S, $vu \in N$. So, for every $b \notin N$, it follows that $vub \notin N$ from which we get $ubv \notin N$. Thus $ubv \notin H$ and so $(a, b) \notin P_H$ which means that N is saturated by P_H. Since the inclusion $P_H|N \subseteq P_H(N)$ follows from the definition of $P_H|N$ and $P_H(N)$, we must only show that $P_H(N) \subseteq P_H|N$. To show this last inclusion, let a and b be arbitrary elements in N with $(a, b) \in P_H(N)$. We prove $(a, b) \in P_H|N$. Assume, in an indirect way, that $(a, b) \notin P_H|N$. Then there are elements x, y in S such that either $xay \in H$ and $xby \notin H$ or $xay \notin H$ and $xby \in H$; we consider only the first case. Since H is reflexive in S, it follows that $yxa \in H$ and $yxb \notin H$. As $a \in N$ and N is unitary in S, we get $yx \in N$. Thus, for an arbitrary element h in H, it follows that $yxah \in H$ and $yxbh \notin H$. Since yx and h are in N, we have $(a, b) \notin P_H(N)$ which contradicts the assumption $(a, b) \in P_H(N)$. Consequently, $(a, b) \in P_H|N$, that is $P_H(N) = P_H|N$.

It is evident that N/P_H is a subsemigroup of S/P_H. Let $[b] \in N/P_H$ be arbitrary, where $[b]$ denotes the congruence class of S modulo P_H containing b. From $[b]\,[b]^{-1} \in H$, it follows that $[b]^{-1} \in N$, because N is unitary in S. Thus N/P_H is a subgroup of S/P_H. Conversely, if $N \supseteq H$ is saturated by P_H and N/P_H is a subgroup of S/P_H, then N is a unitary subsemigroup of S. Next, consider two arbitrary elements a and b in S, and let ω denote the canonical homomorphism of S onto S/P_H. Then $ab \in N$ if and only if $a\omega b\omega \in N\omega$. So N is reflexive in S if and only if $N\omega$ is a normal subgroup of S/P_H. To prove the isomorphism between S/P_N and $(S/P_H)/(N/P_H) = G$, let β denote the canonical homomorphism of S/P_N onto G. Then

$$S' = \{s \in S:\ \ s\omega\beta \text{ is the identity of } G\}$$

equals N and $(\omega\beta)^{-1}o(\omega\beta) = P_N$ (see Theorem 10.34 of [1]). Thus S/P_N is isomorphic with G. The theorem is proved.

Next we consider other properties of reflexive unitary subsemigroups of left simple semigroups.

Theorem 3. *If H and N are subsemigroups of an arbitrary semigroup S such that H is reflexive and unitary in S and $H \cap N \neq \emptyset$, then $H \cap N$ is a reflexive unitary subsemigroup of N and $\langle H, N\rangle/P_H$ is isomorphic with $N/P_{H\cap N}$.*

Proof. Let H and N be subsemigroups of a semigroup S such that H is reflexive and unitary in S and $H \cap N \neq \emptyset$. It can be easily proved that $H \cap N$ is a reflexive unitary subsemigroup of N. We may assume $S = \langle H, N\rangle$. Let $(a, b) \in P_H$ for some $a, b \in N$. We note that P_H is the principal congruence on $S = \langle H, N\rangle$ determined by H. Then, for every $x, y \in N$, $xay \in H \cap N$ if and only if $xby \in H \cap N$ as H is reflexive in S and N is a subsemigroup. So

$(a, b) \in P_{H\cap N}(N)$, that is $P_H|N \subseteq P_{H\cap N}(N)$. Next we show that $P_{H\cap N}(N) \subseteq \subseteq P_H|N$. Let $a, b \in N$ with $(a, b) \in P_{H\cap N}(N)$. Assume, in an indirect way, that $(a, b) \notin P_H|N$. Then there are elements x, y in S such that, for example, $xay \in H$ and $xby \notin H$. Since H is the identity element of S/P_H and $S = \langle H, N\rangle$, there are elements u and v in N with $(x, u) \in P_H$ and $(y, v) \in P_H$. Using $(xay, uav) \in P_H$ and $(xby, ubv) \in P_H$ several times, we have $uav \in H$ and $ubv \notin H$. Since $u, v \in N$, the last result contradicts $(a, b) \in \in P_{H\cap N}(N)$. Noting that, for any $x \in S$, there exists $u \in N$ such that $(x, u) \in \in P_H$, we can see that $\langle H, N\rangle/P_H$ is isomorphic with $N/P_{H\cap N}$.

Theorem 4. *If H and N are unitary subsemigroups of a left simple semigroup S such that H is reflexive in S, then $\langle H, N\rangle$ is a unitary subsemigroup of S and $\langle H, N\rangle = HN$. If N is also reflexive in S, then HN is reflexive in S. In this case $HN = NH$.*

Proof. Let H and N be unitary subsemigroups of a left simple semigroup S. Assume that H is reflexive in S. Consider two elements b_1 and b_2 in N with $(b_1, b_2) \in P_H$. By Theorem 1, N is left simple. Thus there is an element x in N such that $xb_1 = b_2$. So $(xb_1, ub_2) \in P_H$, for every $u \in H$. Since P_H is a group congruence on S, we have $(x, u) \in P_H$ which means that $H \cap N \neq \emptyset$. Evidently $H \cap N$ is a unitary and reflexive subsemigroup of N. Let

$$K = \{s \in S: \ (s, n) \in P_H \text{ for some } n \in N\}.$$

Then K is a subsemigroup of S. We show that K is unitary in S. Let a, b be arbitrary elements in S. Assume $a, ab \in K$. Then there is an element a_1 in N such that $(a, a_1) \in P_H$. Since $H \cap N$ is a reflexive unitary subsemigroup of N and N is left simple, $P_{H\cap N}(N)$ is a group congruence on N. Thus there is an element r in N such that $ra_1 \in H \cap N$. Then $ra \in H$ which implies $(rab, b) \in \in P_H$. Since $rab = r(ab) \in NK \subseteq K$, it follows that $b \in K$. Thus K is left unitary in S. To show that K is right unitary in S, assume $a, ba \in K$. Let k denote the product ba. Then there are elements $a_2, k_1 \in N$ such that $(a_2, a) \in \in P_H$ and $(k_1, k) \in P_H$. Since N is left simple, there is an element q in N such that $qa_2 = k_1$. Then $(ba, qa) \in P_H$. Since P_H is a group congruence on S, it follows that $(b, q) \in P_H$, that is $b \in K$. Thus K is right unitary in S. So K is unitary in S. The proof of the first assertion will be complete if we show that $HN = K$. Since $H, N \subseteq K$, it follows that $HN \subseteq K$. Let k be an arbitrary element of K. Fix an element v of N such that $(v, k) \in P_H$. Since S is left simple, there is an element s in S such that $sv = k$ which means that $s \in H$ and so $k \in HN$. Thus $K \subseteq HN$, that is $K = HN$. Assume that N is also reflexive in S. Then $H \cap N$ is a reflexive unitary subsemigroup of S and both H and N are saturated by $P_{H\cap N}$. Let ω denote the canonical homomorphism of S onto $S/P_{H\cap N}$. Then $H\omega$ and $N\omega$ are normal subgroups of $S/P_{H\cap N}$ and so $N\omega H\omega = NH\omega$ is a normal subgroup of $S/P_{H\cap N}$. Let $L = \{s \in S:$

$s\omega \in NH\omega\}$. Then, by Theorem 2, L is a reflexive unitary subsemigroup of S and $L = HN = NH$.

Theorem 5. (Zassenhaus lemma). *Let S be a left simple semigroup and A, B, M, N subsemigroups of S. Assume that A and B are unitary in S and $A \cap B \neq \emptyset$. If N and M are reflexive unitary subsemigroups in A and B, respectively, then $N(A \cap M)$ and $M(B \cap N)$ are reflexive unitary subsemigroups in $N(A \cap B)$ and $M(B \cap A)$, respectively, such that $N(A \cap B)/P_{N(A \cap M)}$ is isomorphic with $M(A \cap B)/P_{M(B \cap N)}$.*

Proof. First we prove that $A \cap M \neq \emptyset$. By the assumption for A and B, $A \cap B$ is a unitary subsemigroup in B. Since M is a reflexive unitary subsemigroup in B, we get that $M(A \cap B)$ is a unitary subsemigroup in B and so in S such that $\langle M, A \cap B \rangle = M(A \cap B)$. So, for every element x in M, there are elements $y \in M$ and $c \in A \cap B$ such that $x = yc$. Since M is unitary in B, it follows that $c \in M$ and so $A \cap M = A \cap B \cap M \neq \emptyset$. We can prove $B \cap N \neq \emptyset$, in a similar way. By the assumption for M and N, $A \cap M$ and $B \cap N$ are reflexive unitary subsemigroups in $A \cap B$. Then, by Theorem 4, $\langle A \cap M, B \cap N \rangle = (A \cap M)(B \cap N)$ is a reflexive unitary subsemigroup in $A \cap B$.

Let C, D and H denote the semigroups $A \cap B$, $(A \cap M)(B \cap N)$ and C/P_D, respectively. Since C is left simple (Theorem 1), H is a group. We give a homomorphism p of NC onto H. Let x be an arbitrary element in NC. Then there exist elements $a \in N$ and $c \in C$ such that $x = ac$. Let xp be equal to the congruence class of C modulo $P_D(C)$ containing the element c. We prove that p is uniquely determined. Let $a_1 \in N$ and $c_1 \in C$ be arbitrary elements with $a_1c_1 = x$. We prove that $(c, c_1) \in P_D(C)$. Since H is a group, there is an element c' in C such that $c_1c' \in D$. Thus $acc' = a_1c_1c' \in ND$. Since D is unitary in $A \cap B$, it is unitary in A. Consequently $\langle N, D \rangle = ND$ is a unitary subsemigroup of A and so of S. Since $acc' \in ND$ and $a \in \langle N, D \rangle = ND$, it follows that $cc' \in ND$. This implies that $cc' \in D$, because N and C are unitary subsemigroups of S and

$$ND \cap C = (((N \cap D) \cup (N - D))D) \cap C = (((N \cap D)D) \cap C) \cup$$

$$(((N - D)D) \cap C) = ((N \cap D)D) \cap C = D \cap C = D.$$

Consequently $(c, c_1) \in P_D$. Thus p is uniquely determined. It can be easily verified that p maps NC onto H. To prove that p is a homomorphism, let x and y be arbitrary elements of NC. Then $xy \in NC$. Thus there are elements c_1, c_2, c_3 in C and a_1, a_2, a_3 in N such that $x = a_1c_1, y = a_2c_2$ and $xy = a_3c_3$. We show that $(c_1c_2, c_3) \in P_D(C)$. Since $a_1c_1a_2c_2 = a_3c_3$, we have $(a_1c_1a_2c_2, a_3c_3) \in P_N(A)$. Since N is the identity element of A/P_N, it follows that $(c_1c_2, c_3) \in P_N(A)$. To prove $(c_1c_2, c_3) \in P_D(C)$, we must show that, for every $u, v \in A \cap B$, both uc_1c_2v and uc_3v are either in D or in $C - D$. First we show that, for every $t, s \in C$, the condition $(t, s) \in P_N(A)$ implies that both t and s are either in D or in $C - D$. Since $N \subseteq \langle N, D\rangle = ND$ and ND is a unitary subsemigroup of S, we get that ND is saturated by $P_N(A)$. Since $ND \cap C = D$ (as we have shown above), it follows that $(t, s) \in P_N(A)$ implies either $t, s \in D$ or $t, s \notin D$, for every $t, s \in C$. Let u and v be arbitrary elements in C. As u and v are in A, taking into consideration that $(c_1c_2, c_3) \in P_N(A)$, we get $(uc_1c_2v, uc_3v) \in P_N(A)$ and uc_1c_2v, uc_3v are in C. Thus both uc_1c_2v and uc_3v are either in D or in $C - D$. Consequently $(c_1c_2, c_3) \in P_D(C)$ which means that p is a homomorphism of NC onto H. Let $Y = \{y \in NC\colon\ yp = D\}$. We prove that $Y = N(A \cap M)$. Since $A \cap M \subseteq B$, we have $N(A \cap M) \subseteq Y$. To prove $Y \subseteq N(A \cap M)$, let y be an arbitrary element in Y. Then there are elements $a \in A$, $d \in D$ with $y = ad$. Then $y \in ND = N(A \cap M)\ (B \cap N) \subseteq$ $\subseteq N(A \cap M)N \subseteq N^2(A \cap M) = N(A \cap M)$ as $\langle N, A \cap M\rangle = N(A \cap M)$ and N is left simple. Thus $Y \subseteq N(A \cap M)$. Consequently $Y = N(A \cap M)$.

Let $q = p^{-1}op$. Then q is a group congruence on NC and $NC/q = H$. Since $N(A \cap M) = Y$ is the identity element of NC/q, we get that $N(A \cap M)$ is a reflexive unitary subsemigroup of $NC = N(A \cap B)$ and the factor group $N(A \cap B)/P_{N(A \cap M)}$ is isomorphic with H. We can prove, in a similar way, that $M(N \cap B)$ is a reflexive unitary subsemigroup of $M(A \cap B)$ and that $M(A \cap$ $\cap B)P_{M(B \cap N)}$ is isomorphic with H. By the transitivity of the isomorphism, the theorem is completely proved.

2. Normal series and composition series

Definition 6. *Let S be a left simple semigroup. By a normal series of S we mean a finite sequence*

$$S = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_k \tag{1}$$

of subsemigroups S_i of S with the property that S_i is a reflexive unitary subsemigroup of S_{i-1}, for $i = 1, \ldots, k$. The integer k is called the length of the normal series (1)*. The factors of* (1) *are S_{i-1}/P_{S_i}, $i = 1, \ldots, k$.*
We say that the normal series (1) *and*

$$S = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \tag{2}$$

are isomorphic if $n = k$ and there exists a permutation $i \to i^*$ of the integers $1, 2, \ldots, k$ such that S_{i-1}/P_{S_i} and $H_{i^*-1}/P_{H_{i^*}}$ are isomorphic, for $i = 1, \ldots, k$. Moreover, we say that (2) *is a refinement of* (1) *if $n \geqq k$ and every H_i equals some S_j, $i = 1, \ldots, n$; $j = 1, \ldots, k$.*

Theorem 7. *In a left simple semigroup S, every two normal series have refinements ending with the same subsemigroup of S.*

Proof. Let

$$S = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_k \tag{3}$$

and

$$S = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \tag{4}$$

be two normal series of a left simple semigroup S. It is evident that $S_i \cap H_j$ is a reflexive unitary subsemigroup of $S_i \cap H_{j-1}$ and $S_i \cap H_j$ is a reflexive unitary subsemigroup of $S_{i-1} \cap H_j$, $i = 1, \ldots, k$ and $j = 1, \ldots, n$. So

$$S = S_0 \supseteq S_1 \supseteq \ldots S_k \supseteq S_k \cap H_1 \supseteq S_k \cap H_2 \supseteq \ldots \supseteq S_k \cap H_n \tag{5}$$

and

$$S = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \supseteq H_n \cap S_1 \supseteq H_n \cap S_2 \supseteq \ldots \supseteq H_n \cap S_k \tag{6}$$

are two normal series of S such that (5) and (6) are refinements of (3) and (4), respectively.

Theorem 8. (Schreier refinement theorem) *In a left simple semigroup, every two normal series have isomorphic refinements.*

Proof. Let

$$S = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_k \tag{7}$$

and

$$S = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \tag{8}$$

be two normal series of a left simple semigroup S. By Theorem 7, we may assume $S_k = H_n$.

For every $i = 1, \ldots, k$ and $j = 0, 1, \ldots, n$, let

$$S_{ji} = S_i(S_{i-1} \cap H_j)$$

and, for every $i = 0, 1, \ldots, k$ and $j = 1, \ldots, n$, let

$$H_{ji} = H_j(H_{j-1} \cap S_i).$$

By Theorem 5, S_{ji} and H_{ji} are reflexive unitary subsemigroups of $S_{j-1,i}$ and $H_{j,i-1}$, respectively, such that $S_{j-1,i}/P_{S_{ji}}$ is isomorphic with $S_{j,i-1}/P_{H_{ji}}$. Since

$$H_{j-1} = H_{j0} \supseteq H_{j,i-1} \supseteq H_{ji} \supseteq H_{jk} = H_j$$

and

$$S_{i-1} = S_{0i} \supseteq S_{j-1,i} \supseteq S_{ji} \supseteq S_{ni} = S_i,$$

$i = 1, \ldots, k$ and $j = 1, \ldots, n$, it follows that

$$S = S_{01} \supseteq S_{11} \supseteq S_{21} \supseteq \ldots \supseteq S_1 = S_{02} \supseteq S_{12} \supseteq \ldots \supseteq S_k \tag{9}$$

and

$$S = H_{10} \supseteq H_{11} \supseteq H_{12} \supseteq \ldots \supseteq H_1 = H_{20} \supseteq H_{21} \supseteq \ldots \supseteq H_n \qquad (10)$$

are isomorphic normal series of S such that (9) and (10) are refinements of (7) and (8), respectively.

Definition. 9. *A normal series*

$$S = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_k$$

of a left simple semigroup S will be called a composition series of S if $S_{i-1} \neq S_i$ and S_{i-1} has no reflexive unitary subsemigroup T with $S_{i-1} \supset T \supset S_i$, $i = 1, \ldots, k+1$. Here S_{k+1} denotes the empty set.

Remark. It can be easily verified that a normal series of a left simple semigroup is a composition series if and only if $S_{i-1} \neq S_i$, every factor $S_{i-1}|P_{S_i}$ is a simple group, $i = 1, \ldots, k$, and S_k has no proper reflexive unitary subsemigroups.

Theorem 10. (Jordan—Hölder theorem). *If a left simple semigroup S has a composition series, then every two composition series of S are isomorphic with each other.*

Proof. By Theorem 8, it is trivial.

Example. Let $(S_1;o)$ and $(S_2;+)$ be left simple semigroups such that $S_1 \cap S_2 = \emptyset$ and there is an isomorphism α of S_1 onto S_2. On the set $F = S_1 \cup S_2$, define an operation as follows

$$ef = \begin{cases} eof & \text{if } e, f \in S_1 \\ e + f\alpha & \text{if } e \in S_2 \text{ and } f \in S_1 \\ e\alpha + f & \text{if } e \in S_1 \text{ and } f \in S_2 \\ e\alpha^{-1} o f\alpha^{-1} & \text{if } e, f \in S_2 \end{cases}$$

for every $e, f \in F$. It can be shown that F is a left simple semigroup such that S_j is a reflexive unitary subsemigroup of F.

Remarks. If

$$S = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_k \qquad (*)$$

is a normal series of a left simple semigroup S, then every S_i, $i = 0, 1, \ldots, k$, is a unitary subsemigroup of S and so they are left simple semigroups (see Theorem 1). If (*) is a composition series, then S_k has no proper reflexive unitary subsemigroups. So it is a natural problem to describe the structure of left simple semigroups which have no proper reflexive unitary subsemigroups.

By Theorem 1.27 of [1], a semigroup is left simple and contains an idempotent if and only if it is a direct product of a left zero semigroup and a group. So we can prove easily that *a left simple semigroup with idempotents has no proper reflexive unitary subsemigroups if and only if it is a left zero semigroup.*

Problem. Find all left simple semigroups without idempotents which have no proper reflexive unitary subsemigroups.

References

1. CLIFFORD, A. H. and PRESTON, G. B.: The Algebraic Theory of Semigroups, Amer. Math. Soc. Providence R.I. 1961 (vol. *1*) and 1967 (vol. *2*)
2. KUROS, A. G.: Theory of Groups, Moscow, 1953 (Russian).

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