GROUP AND SEMIGROUP THEORETICAL PROBLEMS IN APPROXIMATION THEORY

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1. Introduction

Let A be a normed algebra. If $f, g \in A$ and ε is a positive number we shall say that f equals to g at ε level: $f = g$ (lev ε), if and only if $\|f - g\| \leq \varepsilon$.

It is easy to see that $f = f$(lev ε), and if $f = g$ (lev ε) and $g = h$ (lev ε), then generally it is not true that $f = h$ (lev ε).

If $x, a_k \in A$, (k = 0, 1, 2, . . ., n) then we shall say that

$$P(x) = \sum_{k=0}^{n} a_k x^k = 0 \ \ (\text{lev } \varepsilon) \tag{1}$$

is an algebraic quasi equation at ε level.

Denote by $E(A)$ the set of all algebraic quasi equations (over A).

In this lecture the special ε-invariants of $P(x)$, and the structure of $E(A)$ are investigated.

The material of this paper is strongly related to the works [1], [2].

2. Basic lemmas

Three simple lemmas will be given first.

Lemma *1.* If $f, g, h \in A$ and $f = g$ (lev ε), $g = h$ (lev $\tilde{\varepsilon}$), then $f = h\ (\text{lev}(\varepsilon + \tilde{\varepsilon}))$.

Proof. Since $\|f - g\| \leq \varepsilon$ and $\|g - h\| \leq \tilde{\varepsilon}$, we have

$$\|f - h\| = \|(f - g) + (g - h)\| \leq \|f - g\| + \|g - h\| \leq \varepsilon + \tilde{\varepsilon}. \tag{2}$$

Lemma *2.* If $f, g, h, k \in A$ and $f = h$ (lev ε) and $g = k$ (lev $\tilde{\varepsilon}$), then $f + g = h + k\ (\text{lev}\ (\varepsilon + \tilde{\varepsilon}))$.

Proof. Since $\|f - h\| \leq \varepsilon$, $\|g - k\| \leq \tilde{\varepsilon}$, we have

$$\|(f + g) - (h + k)\| = \|(f - h) + (g - k)\| \leq \|f - h\| + \|g - k\| \leq$$

$$\leq \varepsilon + \tilde{\varepsilon}. \tag{3}$$

Lemma *3.* If $f, g, h, k \in A$ and $f = h$ (lev ε) and $g = k$ (lev $\tilde{\varepsilon}$), then $fg = hk$ (lev $(\varepsilon||g|| + \tilde{\varepsilon}||h||)$).

Proof. Since $||f - h|| \leqq \varepsilon$, $||g - k|| \leqq \tilde{\varepsilon}$, we get

$$||fg - hk|| = ||(f-h)g + h(g-k)|| \leqq ||f-h|| \cdot ||g|| + ||g-k|| \, ||h|| \leqq \leqq \varepsilon||g|| + \tilde{\varepsilon}||h||. \tag{4}$$

Lemma *4.* If $f, g \in A$ and $f = g$ (lev ε), then $mf = mg$ (lev $m\varepsilon$), where m is a non-negative integer number.

Proof. From $||f - g|| \leqq \varepsilon$, we have

$$||mf - mg|| = ||m(f-g)|| \leqq m||f-g|| \leqq m\varepsilon. \tag{5}$$

Lemma 5. If $f, g \in A$, $fg = gf$ and $f = g$ (lev ε), then for arbitrary non-negative integer n,

$$f^n = g^n(\text{lev } (\varepsilon \cdot n \cdot [\text{Max}\,(||f||, ||g||)]^{n-1})). \tag{6}$$

Proof. Since $||f - g|| \leqq \varepsilon$, and $fg = gf$, we have

$$||f^n - g^n|| \leqq ||f-g||(||f||^{n-1} + ||f||^{n-2}\,||g|| + \ldots + ||f||^{n-j-1} \cdot ||g||^j + + \ldots + ||g||^{n-1}) \leqq \varepsilon \cdot n\,[\text{Max}(||f||, ||g||)]^{n-1}, \tag{7}$$

from which (6) follows.

3. Invariants

Now we give an important basic definition.

Definition. Let α be a one to one map of A onto itself. We shall say that α is an ε-invariant transformation (shortly ε-invariant) of $P(x)$ if the following condition holds: $P(\alpha x) = P(x)$(lev ε), for all $x \in A$.

Denote by $G(A)$ the set of all one to one transformations of A onto itself.

It is easy to see that $G(A)$ is a group.

Denote by $I_\varepsilon(P(x))$, the set of all ε-invariants of $P(x)$.

We can see that the identical map of $G(A)$ belongs to $I_\varepsilon(P(x))$, and if $\alpha \in I_\varepsilon(P(x))$ then $\alpha^{-1} \in J_\varepsilon(P(x))$.

Since

$||P(\alpha x) - P(x)|| \leqq \varepsilon$ holds for all $x \in A$, we have

$$||P(\alpha\alpha^{-1}x) - P(\alpha^{-1}x)|| = ||P(\alpha^{-1}x) - P(x)|| \leqq \varepsilon. \tag{8}$$

Lemma 6. If $\alpha \in I_\varepsilon(P(x))$ and $\beta \in I_{\tilde{\varepsilon}}(P(x))$, $(\varepsilon \geqq 0, \tilde{\varepsilon} \geqq 0)$, then $\alpha\beta \in \in I_{\varepsilon+\tilde{\varepsilon}}(P(x))$.

Proof. Since

$$||P((\alpha\beta)x) - P(x)|| = ||P((\alpha\beta)x) - P(\beta x) + P(\beta x) - P(x)|| \leqq ||P(\alpha(\beta x) - P(\beta x)|| + ||P(\beta x) - P(x)|| \leqq \varepsilon + \tilde{\varepsilon}, \tag{9}$$

we obtain $\alpha\beta \in I_{\varepsilon+\tilde{\varepsilon}}(P(x))$, from which

$$I_\varepsilon(P(x)) \cdot I_{\tilde{\varepsilon}}(P(x)) \subseteq I_{\varepsilon+\tilde{\varepsilon}}(P(x)), \tag{10}$$

follows.

The structure of the set of all $I_\varepsilon(P(x))$ where $\varepsilon \geqq 0$, is very simple, because if $\varepsilon \leqq \tilde{\varepsilon}$, then

$$I_\varepsilon(P(x)) \subseteq I_{\tilde{\varepsilon}}(P(x)). \tag{11}$$

Example 1. Now we suppose that $P(x) = a_0x + a_1$.

Consider the folllowing transformation

$$\alpha x = x + b,\ (x, b \in A).$$

In this case

$$P(\alpha x) = a_0(x + b) + a_1, \tag{12}$$

and

$$||P(\alpha x) - P(x)|| = ||a_0 b||. \tag{13}$$

Therefore if $||a_0b|| \leqq \varepsilon$, then

$$P(\alpha x) = P(x),\ \text{lev}\ (\ \varepsilon), \tag{14}$$

and α is an ε-invariant.

Example 2.

Consider the following transformation:

$$\alpha x = \begin{cases} x & \text{if}\ \ x \neq u, v, \\ \mathrm{u} & \text{if}\ \ x = v, \\ v & \text{if}\ \ x = u, \end{cases} \tag{15}$$

where $x,\ u,\ v \in A$ and $u \neq v$.

If $P(x) = x^2 + b$, $(b \in A)$, and $||u^2 - v^2|| \leqq \varepsilon$, then

$$P(\alpha x) = P(x)\ \ (\text{lev}\ \varepsilon). \tag{16}$$

It is easy to see that

$$P(\alpha x) - P(x) = \begin{cases} 0 & \text{if}\ \ x \neq u, v, \\ v^2 - u^2 & \text{if}\ \ x = u, \\ u^2 - v^2 & \text{if}\ \ x = v. \end{cases} \tag{17}$$

Since $||u^2 - v^2|| \leqq \varepsilon$ and (17), we have

$$P(\alpha x) = P(x)\ (\text{lev}\ \varepsilon). \tag{18}$$

Therefore α is an ε-invariant of $P(x)$. Next we give three theorems.

Theorem 1. If $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, where $x, a_0, a_1, \ldots, a_n \in A$ and $u, v \in A$, $u \neq v$, further $||P(u) - P(v)|| \leq \varepsilon$, then

$$\alpha x = \begin{cases} x & \text{if } x \neq u, v, \\ u & \text{if } x = v, \\ v & \text{if } x = u, \end{cases}$$

is an ε-invariant of $P(x)$.

Proof. It is easy to see that in our case

$$||P(\alpha x) - P(x)|| = \begin{cases} 0 & \text{if } \quad x = u, v, \\ ||P(u) - P(v)|| & \text{if } \quad x = u, v. \end{cases} \tag{20}$$

Since $||P(u) - P(v)|| \leq \varepsilon$, we have

$$P(\alpha x) = P(x) \text{ (lev } \varepsilon). \tag{21}$$

Theorem 2. If $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, where $x, a_0, a_1, \ldots, a_n \in A$ and u, v, p, q are different elements of A further on

$$\alpha x = \begin{cases} x & \text{if } \quad x \neq u, v \\ u & \text{if } \quad x = v, \\ v & \text{if } \quad x = u, \end{cases} \tag{22}$$

$$\beta x = \begin{cases} x & \text{if } \quad x \neq p, q \\ p & \text{if } \quad x = q, \\ q & \text{if } \quad x = p, \end{cases} \tag{23}$$

$$||P(u) - P(v)|| \leq \varepsilon, \tag{24}$$

$$||P(p) - P(q)|| \leq \varepsilon, \tag{25}$$

then $\{e, \alpha, \beta, \alpha\beta\}$ is a group and its elements are ε-invariants.

Proof. If $x \neq u, v$ then $\alpha x = x$, $\alpha^2 x = x$. If $x = u$, then $\alpha u = v$ and $\alpha(\alpha u) = \alpha v = u$. If $x = v$, then $\alpha v = u$ and $\alpha(\alpha v) = \alpha u = v$. Therefore $\alpha^2 = e$, $\big(e$ denotes the unit element of $G(A)\big)$. We can see that $\beta^2 = e$.

Since

$$(\alpha\beta)\, x = \begin{cases} x & \text{if } \quad x \neq u, v, p, g, \\ v & \text{if } \quad x = u, \\ u & \text{if } \quad x = v, \\ q & \text{if } \quad x = p, \\ p & \text{if } \quad x = g, \end{cases} \tag{26}$$

and

$$||P(u) - P(v)|| \leq \varepsilon, \tag{27}$$

$$||P(p) - P(q)|| \leq \varepsilon, \tag{28}$$

we have

$$||P((\alpha\beta)x) - P(x)|| = \begin{cases} 0 & \text{if } \quad x \neq u, v, p, q, \\ ||P(u) - P(v)|| & \text{if } \quad x = u, v, \\ ||P(p) - P(q)|| & \text{if } \quad x = p, q, \end{cases} \tag{29}$$

from which

$$||P((\alpha\beta)x - P(x)|| \leq \varepsilon, \tag{30}$$

follows.

From (22) and (23) we have

$$\alpha\beta = \beta\alpha . \tag{31}$$

Therefore $\{e, \alpha, \beta, \alpha\beta\}$ is a group and its elements are ε-invariants.

Problem 1. Find all ε-invariants of $P(x)$.

Problem 2. Find all groups whose elements are ε-invariants of $P(x)$.

4. The structure of E(A)

Now we suppose that A is a commutative normed algebra.

Let $P(x) = 0$ (lev ε), $Q(x) = 0$ (lev $\bar{\varepsilon}$), and $R(x) =$ (lev $\hat{\varepsilon}$) be elements of $E(A)$, on which two operations will be defined as follows:

$$\big(P(x) = 0 (\text{lev } \varepsilon)\big) \oplus \big(Q(x) = 0 \ (\text{lev } \bar{\varepsilon})\big) = \big(P(x) + Q(x) = 0 \ (\text{lev } (\varepsilon + \bar{\varepsilon}))\big), \tag{32}$$

$$\big(P(x) = 0 \ (\text{lev } \varepsilon)\big) \circledcirc \big(Q(x) = 0 \ (\text{lev } \bar{\varepsilon})\big) = \big(P(x)\, Q(x) = 0 \ (\text{lev } \varepsilon\bar{\varepsilon})\big). \tag{33}$$

It is easy to see that the operations $\oplus$, $\circledcirc$ are commutative and associative.

Therefore $(E(A), \oplus)$, $(E(A), \circledcirc)$ are commutative semigroups.

We get easily the following distribution law:

$$[\big(P(x) = 0 \ (\text{lev } \varepsilon)\big) \oplus \big(Q(x) = 0 \ (\text{lev } \bar{\varepsilon})\big)] \circledcirc \big(R(x) = 0 \ (\text{lev } \hat{\varepsilon})\big) =$$
$$= \big(P(x)R(x) = 0 \ (\text{lev } \varepsilon\hat{\varepsilon})\big) \oplus \big(Q(x)R(x) = 0 \ (\text{lev } \bar{\varepsilon}\hat{\varepsilon})\big). \tag{34}$$

Denote by $S(E(A), \varepsilon)$ the set of all $P(x) = 0$ (lev ε) elements of $E(A)$.

Lemma 7. $F_1 = \big(S(E(A), 1), \circledcirc\big)$ is a semigroup and F_ε, where $0 \leqq \varepsilon \leqq 1$ is an ideal of F_1.

Proof. If $P(x) = 0$ (lev 1) and $Q(x) = 0$ (lev 1), then $[P(x) = 0 \text{ (lev 1)}] \circledcirc [Q(x) = 0 \text{ (lev 1)}] = [P(x)\, Q(x) = 0 \text{(lev 1)}]$.

Since the operation $\circledcirc$ is associative, F_1 is a semigroup.

If $P(x) = 0$ (lev 1) and $Q(x) = 0$ (lev ε), $(0 \leqq \varepsilon \leqq 1)$, then

$$[P(x) = 0 \ (\text{lev } 1)] \circledcirc [Q(x) = 0 \ (\text{lev } \varepsilon)] = [P(x)\, Q(x) = 0 \ (\text{lev } \ \varepsilon)]. \tag{35}$$

Therefore F_ε is an ideal of F_1.

Theorem 3. The semigroup $\big(E(A), \circledcirc\big)$ can be represented as a union of two disjoint subsemigroups.

Proof. Since Lemma 7. F_1, $(0 \leqq \varepsilon \leqq 1)$ is a subsemigroup of $\big(E(A), \circledcirc\big)$. Denote by $\widetilde{F}_1$ the set of all elements $Q(x) = 0$ (lev δ) of $E(A)$ where $\delta > 1$.

If $R(x) = 0$ (lev η), where $\eta > 1$ then $\big(R(x) = 0 \ (\text{lev } \eta)\big) \in \widetilde{F}_1$,

and

$$[Q(x) = 0 \ (\text{lev } \delta)] \circledcirc [R(x) = 0 \ (\text{lev } \eta)] = [Q(x)R(x) = 0 \ (\text{lev } \eta\delta)] \in \widetilde{F}_1, \tag{36}$$

because in our case $\eta\delta > 1$. Since the operation $\circledcirc$ is associative $\hat{F}_1 = (\tilde{F}_1, \circledcirc)$ is a subsemigroup of $(E(A), \circledcirc$.

Therefore

$$(E(A), \circledcirc) = F_1 \cup \hat{F}_1. \tag{37}$$

Denote by $E_k(A)$, the set of all equations $ax^k = 0$ (lev δ) where $a, x \in A$, (k $= 0, 1, 2, \ldots$) and δ is an arbitrary nonnegative number.

It can be proved that $S_k(A) = \big(E_k(A), \oplus\big)$ is a commutative semigroup.

If $ax^k = 0$ (lev δ), bx^k(lev $\tilde{\delta}$) $\in E_k(A)$, then

$$[ax^k = 0\,(\text{lev}\ \delta)] \oplus [bx^k = 0\ (\text{lev}\ \tilde{\delta})] = [(a+b)x^k = 0\ \big(\text{lev}\ (\delta + \tilde{\delta})\big)] \in E_k(A)\ , \tag{38}$$

and the operation $\oplus$ is commutative and associative.

It is easy to see that

$$E(A) = \bigcup_{k=0}^{\infty} E_k(A), \tag{39}$$

and

$$E_k(A) \cap E_j(A) = \emptyset \quad \text{if} \quad k \neq j\,. \tag{40}$$

If $P(x) = 0$ (lev ε) $\in E(A)$ and

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n\,,$$

then

$$[P(x) = 0(\text{lev}\ \varepsilon)] = \left[a_0x^n = 0\left(\text{lev}\frac{\varepsilon}{n+1}\right)\right] \oplus \left[a_1x^{n-1} = 0\left(\text{lev}\left(\frac{\varepsilon}{n+1}\right)\right)\right] \oplus \ldots \oplus$$

$$\oplus \left[a_n = 0\left(\text{lev}\ \frac{\varepsilon}{n+1}\right)\right] \tag{41}$$

where

$$\left[a_kx^k = 0\left(\text{lev}\ \frac{\varepsilon}{n+1}\right)\right] \in E_k(A)\ . \tag{42}$$

Thus we have proved the following theorem:

Theorem 4. The semigroup $\big(E(A), \oplus\big)$ can be represented in the form

$$\big(E(A), \oplus\big) = \bigcup_{k=0}^{\infty} (E_k(A), \oplus). \tag{43}$$

where

$$E_k(A) \cap E_j(A) = \emptyset \quad \text{if} \quad k \neq j.$$

and $\big(E_k(A), \oplus\big)$, $(k = 0, 1, 2, \ldots)$ are subsemigroups of $\big(E(A), \oplus\big)$.

Finally we set up two problems:

Problem 3. Find all the substructures of $\big(E(A), \oplus, \odot\big)$.

Problem 4. Find the automorphism group of $\big(E(A), \oplus, \odot\big)$.

References

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