LATTICES WITH 0-DISTRIBUTIVE OR 0-MODULAR REES FACTORS

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Rees factors of a lattice have been defined and discussed in [4]. In this paper distributive lattices will be characterized by their Rees factors and the analogous problem will be examined for modular lattices.

We use the notations and definitions of the books [1] and [3].

Definition 1. Let L be a lattice with respect to the meet operation $\frown$ and join operation $\smile$. Let, further, I be an ideal of L and θ_I the equivalence of L defined as follows: For any elements a and b of L, let $a \equiv b(\theta_I)$ mean that either $a = b$ or else both a and b belong to I. For convenience, let any one-element θ_I-class $\{a\}$ $(a \in L)$ be identified with the element a of L. Then the set of all θ_I-classes forms a lattice with respect to the operations $\wedge$ and $\vee$ defined by

$$a \wedge b = \begin{cases} a \frown b & \text{if} \quad a \neq I, b \neq I \text{ and } a \frown b \notin I, \\ I & \text{otherwise} \end{cases}$$

and

$$a \vee b = \begin{cases} a \smile b & \text{if} \quad a \neq I \quad \text{and} \quad b \neq I, \\ a & \text{if} \quad b = I, \\ b & \text{if} \quad a = I \end{cases}$$

(see [4]); this lattice is called the **Rees factor** (**lattice**) of L modulo I and is denoted by L/I.

Definition 2. Let L be a lattice with the least element o. If, for any elements x, y, z of L,

$$x \frown z = o \text{ and } y \frown z = o \text{ imply } (x \smile y) \frown z = o,$$

then L is called **0-distributive** [5, p. 41].

Definition 3. Let L be a lattice with the least element o. If, for any elements x, y, z of L,

$$x \leq z \text{ and } y \frown z = o \text{ imply } (x \smile y) \frown z = x,$$

then L is called **0-modular** [5, p. 28].

In other words, the 0-modularity of L means that $x \cap z = x$ and $y \cap z = o$ imply $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

Theorem 1. *A lattice is distributive if and only if every Rees factor of it is 0-distributive.*

Proof. Assume that L is distributive and consider an ideal I of L. Let $x, y, z \in L/I$ such that

$$x \wedge z = y \wedge z = I,$$

that is,

$$x \cap z,\ y \cap z \in I. \tag{1}$$

If at least one of the elements x, y, z is equal to I (in L/I), then $(x \vee y) \wedge z = I$ trivially. In the opposite case we have, by the distributivity of L and by the relation (1),

$$(x \cup y) \cap z = (x \cap z) \cup (y \cap z) \in I.$$

Thus $(x \vee y) \wedge z = I$ also in this case. This means that L/I is 0-distributive, indeed.

Assume that L/I is 0-distributive for any ideal I of L. Let a, x, y, z be any elements of L. Then, by the 0-distributivity of the Rees factor $L/(a]$, the inequalities

$$x \cap z \leqq a \qquad \text{and} \qquad y \cap z \leqq a$$

imply

$$(x \cup y) \cap z \leqq a.$$

It follows, by a result of [2] (see also [5, p. 42]) that L is distributive.

Thus Theorem 1 is proved.

For modular lattices only the following weaker assertion is true:

Theorem 2. *If any Rees factor of a lattice is 0-modular, then the lattice is modular.*

Proof. Let x, y, z be any elements of the lattice L such that $x \leqq z$. Let us denote by $\cap$ and $\cup$ the operations in L, by $\wedge$ and $\vee$ the operations in the Rees factor $R = L/(y \cap z]$. Then $y \wedge z$ is equal to the least element of R and $x \leqq z$ also in R. Assume R to be 0-modular. Then we have

$$(x \vee y) \wedge z = x$$

which means that either

$$(x \cup y) \cap z \leqq y \cap z \qquad \text{and} \qquad x \leqq y \cap z \tag{2}$$

or

$$(x \cup y) \cap z = x. \tag{3}$$

In case (2) we have

$$y \cap z \leqq (x \cup y) \cap z \leqq y \cap z \quad \text{and} \quad y \cap z \leqq x \cup (y \cap z) = y \cap z$$

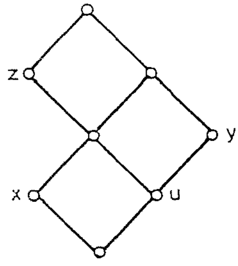
whence

$$(x \cup y) \cap z = y \cap z \quad \text{resp.} \quad x \cup (y \cap z) = y \cap z.$$

In case (3) we have (by $x \leqq z$)

$$x \leqq x \cup (y \cap z) \leqq (x \cup y) \cap z = x.$$

In both cases we have the result that $x \leqq z$ implies $(x \cup y) \cap z = x \cup \cup (y \cap z)$ in L. This means that L is modular, indeed.



Remark. Unfortunately, the converse statement is false. Consider, for example, the lattice L given by the diagram: L is modular (even distributive) but the Rees factor $S = L/(u]$ is not 0-modular (see the elements x, y, z). Since S is 0-distributive (in accordance with our Theorem 1), this example shows also that 0-distributivity does not imply 0-modularity.

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