COMPLETELY DISJUNCTIVE LANGUAGES

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Abstract

A language over a finite alphabet X is called disjunctive if the principal congruence P_L determined by L is the equality. A dense language is a language which has non-empty intersection with any two-sided ideal of the free monoid X^* generated by the alphabet X. We call an infinite language L completely disjunctive (completely dense) if every infinite subset of L is disjunctive (dense). For a language L, if every dense subset of L is disjunctive, then we call L quasi-completely disjunctive. In this paper, (for the case $|X| \geq 2$) we show that every completely disjunctive language is completely dense and conversely. Characterizations of completely disjunctive languages and quasi-completely disjunctive languages were obtained. We also discuss some operations on the families of languages.

1. Introduction and preliminary

Let X^* be the free monoid generated by the alphabet X. Every element of X^* is called a *word* and every subset of X^* is called a *language*. Let $X^+ = X^* \backslash 1$, where 1 is the empty word. For a given language $L \subseteq X^*$, the relation P_L defined on X^* as

$$x \equiv y(P_L) \Leftrightarrow (uxv \in L \Leftrightarrow uyv \in L, \ \forall u, v \in X^*)$$

is a congruence. We call L *regular* if P_L is of finite index and L is said to be *disjunctive* if P_L is the equality. L regular is equivalent to the fact that L is recognized by an automaton. A *dense* language is a language which has non-empty intersection with any two sided ideal of X^* ([4]). L dense is equivalent to the fact that L contains a disjunctive language (see [5]). We will call an infinite language *completely disjunctive (completely dense)* if every infinite subset of the language is disjunctive (dense). A *quasi-completely disjunctive* language is a dense language L in which every dense subset of L is disjunctive. The purpose of this paper is to characterize completely disjunctive, completely dense and quasi-completely disjunctive languages. We also discuss some operations on those families of languages.

In this paper, some time the free monoid X^* needs to be equipped with a total order $\leq$ on X^*. We call a total order $\leq$ defined on X^* *strict* if for every $u \neq v \in X^*$, $u < v$ if $\lg(u) < \lg(v)$. A *standard total order* defined on X^* is a particular strict total order $\leq$ such that for any $u, v \in X^*$, $u < v$ if $\lg(u) < \lg(v)$ and $\leq$ is the lexicographic order on X^n for all $n \geq 1$.

Now if $\leq$ is a total order on X^*, and if $A = \{x_1 < x_2 < \ldots\}$, $B = \{y_1 < y_2 < \ldots\}$ are two infinite languages over X, then following Shyr we define the *ordered catenation* of A and B to be the set $A \triangle B = \{x_i y_i | i = 1, 2, 3, \ldots\}$. We extend the notion of ordered catenation to finite languages in a natural way. To approach this if a finite language, say $A = \{a_1, a_2, \ldots, a_n\}$, then we consider A as $\{a_1, a_2, \ldots, a_n, 1, 1, \ldots\}$ and $A \triangle B$ means the same as ordered catenation for infinite languages.

We call a word $x \in X^+$ *primitive* if $x = f^n$, $f \in X^+$ implies $n = 1$. Let Q be the collection of all primitive words over X and let $Q^{(i)}$ be the order catenation of i copies of Q. For convenience we let $Q^{(1)} = Q \cup \{1\}$. Let $|X| \geq 2$, where $|X|$ means the cardinality of the alphabet X. Then for $u, v \in X^*$, $uv \in Q^{(i)}$ if and only if $vu \in Q^{(i)}$ for all $i \geq 1$, and it is known that for $i \geq 1$, each $Q^{(i)}$ is disjunctive ([5], [6]). For a given language L, if for every $f \neq g \in X^+$, $\lg(f) = \lg(g)$, there exist $u, v \in X^*$ such that $ufv \in L$ $ugv \notin L$, or vice versa, then L is disjunctive ([6]). Here $\lg(x)$ means the length of the word x.

2. Characterization of completely disjunctive languages

Let us define the completely disjunctive and completely dense languages formally.

Definition. An infinite language L is called *completely disjunctive* (*completely dense*) if every infinite subset of L is disjunctive (dense).

By definition, it is clear that every completely disjunctive language is a disjunctive language. Certainly, every completely dense language is dense. And, clearly every infinite subset of a completely disjunctive (dense) is completely disjunctive (dense).

The following are some examples of completely disjunctive and completely dense languages. If $X = \{a\}$, then the disjunctive language $A = \bigcup_{n \geq 1}(a^{2^n})$ is completely disjunctive, and the regular language $B = (a^n)^+$ is completely dense but not disjunctive for $n \geq 1$.

For $|X| \geq 2$, let $\leq$ be any total order defined on X^* and let $X^+ = \{x_1 < x_2 < \ldots\}$. The language $L = \{x_1 x_2 \ldots x_i | i \geq 1\}$ is dense and discrete and hence disjunctive. Clearly every infinite subset of L is disjunctive and by definition L is completely disjunctive.

The following Proposition is immediate.

Proposition 1. Let $|X| = 1$. Then every infinite subset of X^* is completely dense.

We call a language $L \subseteq X^*$ *regular free* if every infinite subset contained in L is not regular.

Proposition 2. Let $|X| = 1$ and let $L \subseteq X^*$ be an infinite language. Then the following are equivalent;

(1) L is completely disjunctive;

(2) L is regular free;

(3) L is quasi-completely disjunctive.

Proof. Since every subset of X^* is either regular or disjunctive ([5]), the equivalences of (1), (2) and (3) are immediate.

We call a language L *semi-discrete* if there exists $k \geq 1$ such that $|L \cap X^n| \leq k$ for all $n \geq 1$. If $k = 1$, then L is a discrete language. Let $|X| \geq 2$. For a semi-discrete language over X we have.

Proposition 3. ([3]) If L is a dense semi-discrete language, then L is disjunctive.

In the rest of this paper, we assume that the cardinality of the alphabet X consists of more than one letter.

Proposition 4. Every infinite regular language over X contains a language, which is neither regular nor disjunctive.

Proof. Let $L \subseteq X^*$ be an infinite regular language. Then L contains a regular language ux^+v, where $x \in X^+$, $u, v \in X^*$. Let $L' = \{ux^pv | p \text{ is a prime number}\}$. Clearly, L′ is not a regular language which is also not disjunctive. Thus L′ is a language in L, which is neither regular nor disjunctive.

A word $u \in X^+$ is said to be *non-overlapping* if $vx = u = yv$ for some $v, x, y \in X^*$ imples $v = 1$.

In order to show the equivalence of completely disjunctivity and completely density we first show the following lemma.

Lemma 5. Let $u, v \in X^*$ with $lg(u) = lg(v)$. Then there exist $x, y \in X^*$ such that xuy and xvy are non-overlapping.

Proof. Let $a, b \in X$ with $a \neq b$ and $n = lg(u) = lg(v)$. Obviously, $b^{n+2}auba^{n+2}$, $b^{n+2}avba^{n+2}$ are non-overlapping.

Proposition 6. Let $L \subseteq X^*$. Then L is completely disjunctive if and only if L is completely dense.

Proof. ($\Rightarrow$) Obvious. ($\Leftarrow$) Let L′ be an infinite subset of L. We prove that L′ is disjunctive. Suppose $u \equiv v(P_{L'})$ and $u \neq v$. We can assume that $lg(u) = lg(v)$ without loss of generality. Moreover, by Lemma 5, we can assume that u, v are non-overlapping, let $K = L' \backslash X^*vX^*$. We now show that K is an infinite set. Let $w \in L' \cap X^*vX^*$. First, we represent w by the following way:

(i) $w = x_1vx_2vx_3 \ldots x_nvx_{n+1}$.

(ii) $x_i \notin X^*vX^*$, $i = 1, 2, \ldots, n+1$.

Let $f(w) = x_1ux_2ux_3 \ldots x_nux_{n+1}$. Since $u \equiv v(P_{L'})$, $f(w) \in L'$. On the other hand, by the fact that u, v are non-overlapping and $x_i \notin X^*vX^*$, we have $f(w) \notin X^*vX^*$. Hence $f(w) \in L' \backslash X^*vX^*$. Obviously, $\{f(w) | w \in L' \cap X^*vX^*\}$ is an infinite set. Therefore, K is an infinite set. However, K is not dense, a contradiction.

Proposition 7. Let $A, B \subseteq X^*$ and let AB be a completely disjunctive language. If A (B) is infinite, then A (B) is completely disjunctive.

Proof. Let A′ be an infinite subset of A. Then for any finite subset $B' \subseteq B$, A′B′ is infinite and thus disjunctive. This implies that A′ is disjunctive and A is completely disjunctive.

Proposition 8. Let A and B be two infinite languages. Then AB is completely disjunctive if and only if both A and B are completely disjunctive.

Proof. ($\Rightarrow$) Proposition 7.

($\Leftarrow$) Suppose AB is not completely disjunctive. Then by Proposition 6, AB is not completely dense. Therefore there exists $L \subseteq AB$, an infinite language which is not dense. Let

$A' = \{x \in A | xy \in L$, for some $y \in B\}$ and let $B' = \{y \in B | xy \in L$ for some $x \in A\}$.

Since L is not dense, we have that both A′ and B′ are not dense. But A′ or B′ is infinite, and this in turn implies that not both A and B are completely dense, a contradiction. This shows that if both A and B are completely disjunctive, then AB is completely disjunctive.

The following can be easily proved:

Proposition 9. Let $A, B \subseteq X^*$, where $(A, \leq_1)$, $(B, \leq_2)$ are strictly ordered sets. If A or B is completely disjunctive then $A \triangle B$ is completely disjunctive.

Proof. Suppose A is a completely disjunctive language. Let L be an infinite subset of $A \triangle B$ and let $A_1 \triangle B_1 = L$, where $A_1 \subseteq A$ is an infinite subset of A and $B_1 \subseteq B$. Since A is completely disjunctive, A_1 is dense. Thus L is a disjunctive language ([7]). Therefore, $A \triangle B$ is completely disjunctive. Similarly, we can show that $A \triangle B$ is completely disjunctive if B is completely disjunctive.

The converse of the above proposition is not true as can be seen from the following example.

Example 1. Let $\leq$ be the standard total order defined on X^* and let $X^+ = \{x_1 < x_2 < \ldots\}$, where $x_1 = a \in X$. Let the languages A and B be defined as the following two sets:

$$A = \{x_1x_2 \ldots x_i | i \geq 1\} \cup \{a^j | j = \lg(x_1x_2 \ldots x_n) + 1, n \text{ is even}\};$$

$$B = \{x_1x_2 \ldots x_i | i \geq 1\} \cup \{a^j | j = \lg(x_1x_2 \ldots x_n) + 1, n \text{ is odd}\}.$$

For the word $x_1x_2 \ldots x_m$, let $j(m) = \lg(x_1x_2 \ldots x_m) + 1$. Then

$$A = \{x_1 < x_1x_2 < a^{j(2)} < x_1x_2x_3 < x_1x_2x_3x_4 < a^{j(4)} < \ldots\} \text{ and}$$

$B = \{x_1 < a^{j(1)} < x_1x_2 < x_1x_2x_3 < a^{j(3)} < x_1x_2x_3x_4 < x_1x_2x_3x_4x_5 < a^{j(5)} < \ldots\}$.

It is clear that both A and B are not completely disjunctive while

$A \triangle B = \{x_1x_1, x_1x_2a^{j(1)}, a^{j(2)}x_1x_2, x_1x_2x_3x_1x_2x_3, x_1x_2x_3x_4a^{j(3)},$

$a^{j(4)}x_1x_2x_3x_4, \ldots\}$ *is completely disjunctive.*

Proposition 10. Let $L \subseteq X^*$, *where* $(L, \leq)$ *is an infinite strictly ordered set. Then the following are equivalent:*

(1) L is a completely disjunctive language;

(2) $L^{(n)}$ *is completely disjunctive for some* $n \geq 2$;

(3) $L^{(n)}$ *is completely disjunctive for all* $n \geq 2$.

Proof. (1) $\Rightarrow$ (3) Proposition 9.

(3) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (1) Let $L^{(n)} = \{w^n | w \in L\}$ be a completely disjunctive language for some $n \geq 2$ and let A be an infinite subset of L. Then $A^{(n)}$ is an infinite subset of $L^{(n)}$ and thus a dense language. It follows that A is a dense subset of L and L is completely dense. By Proposition 6, L is completely disjunctive.

We are now able to prove the main characterization of completely disjunctive languages.

Proposition 11. Let $\{a, b\} \subseteq X$ *and let* $L \subseteq X^*$, *where* $(L, \leq)$ *is an infinite strictly ordered set. Then the following are equivalent:*

(1) L is completely disjunctive;

(2) L is completely dense;

(3) Every subset of L is either regular or disjunctive;

(4) $L \backslash X^*wX^*$ *is finite for all* $w \in X^+$;

(5) L^n *is completely disjunctive for every* $n \geq 2$;

(6) $L^{(n)}$ *is completely disjunctive for every* $n \geq 2$;

(7) For every infinite language S, $L \cap S$ *is finite or disjunctive.*

Proof. (1) $\Leftrightarrow$ (2). Proposition 6.

(1) $\Rightarrow$ (3). Immediate.

(3) $\Rightarrow$ (1). Let D be an infinite subset of L. Then by (3) D is either regular or disjunctive. If D is disjunctive, then we are done. On the other hand if D is regular, then by Proposition 4, D contains a language which is neither regular nor disjunctive. This contradicts the condition (3).

(2) $\Rightarrow$ (4). Let $L \backslash X^*wX^*$ be an infinite language for some $w \in X^+$. Then $L \backslash X^*wX^*$ is an infinite language contained in L and by (2) $L \backslash X^*wX^*$ is dense, a contradiction.

(4) $\Rightarrow$ (2). Suppose D is an infinite subset of L which is not dense. Then there exists $w \in X^+$ such that $D \cap X^*wX^* = \emptyset$. Since $D \subseteq L \backslash X^*wX^*$ and by (4) D is finite, a contradiction.

(1) $\Leftrightarrow$ (5). Proposition 8.

(1) $\Leftrightarrow$ (6). Proposition 10.

(1) $\Rightarrow$ (7). Trivial.

(7) $\Rightarrow$ (2). Suppose L is not completely dense. Then there exists an infinite subset A of L such that A is not dense. Thus $A = A \cap L$ is neither finite nor dense, a contradiction.

3. Characterization of quasi-completely disjunctive languages

We give the definition of quasi-completely disjunctive language formally. Recall that the alphabet X consists of more than one letter.

Definition. A dense language L is called *quasi-completely disjunctive* if every dense subset of L is disjunctive.

It is clear that quasi-completely disjunctive languages are disjunctive languages and every dense subset of a quasi-completely disjunctive language is quasi-completely disjunctive.

For any $L \subseteq X^*$ and $x \in X^*$, let $L \ldots x = \{(u, v) | uxv \in L\}$. The following is a characterization of the quasi-completely disjunctive language.

Proposition 12. Let $L \subseteq X^*$ be a dense language. Then L is quasi-completely disjunctive if and only if for every $x \neq y \in X^+$, the language $L_{xy} = \{uv | uxv \in L$ and $uyv \in L\}$ is not dense.

Proof. ($\Rightarrow$) Let $x \neq y \in X^+$ and suppose $L_{xy} = \{uv | uxv \in L \text{ and } uyv \in L\}$ is dense. Then the language

$$L_1 = \{uxv | (u, v) \in L \ldots x \cap L \ldots y\} \cup \{uyv | (u, v) \in L \ldots x \cap L \ldots y\}$$

is dense. Indeed, by the assumption that L_{xy} is dense for every $w \in X^*$, there exist $u', v' \in X^*$ such that $u'wwv' \in L_{xy}$. Thus $u'wwv' = uv \in L_{xy}$ for some $u'v' \in X^*$ and $uxv, uyv \in L$. This then implies that $uxv, uyv \in L_1$. Since either u or v contains w as a subword, we see that $L_1 \cap X^*wX^* \neq \emptyset$ and L_1 is dense. Now, by the definition of the set L_1, we see that $x \equiv y(P_{L_1})$. Then L_1 is a dense subset of L which is not disjunctive, a contradiction. This shows that L_{xy} is not dense.

($\Leftarrow$) Let L_1 be a dense subset in L. Since $L_{xy} = \{uv | uxv \in L \text{ and } uyv \in L\}$ is not dense, there exists w such that $X^*wX^* \cap L_{xy} = \emptyset$. Now for every $u, v \in X^*$, if $uw_1xw_2v \in L$ then $uw_1yw_2v \notin L$ where $w = w_1w_2$, $w_1, w_2 \in X^*$. Since L_1 is dense, there exist $u', v' \in X^*$ such that $u'w_1xw_2v' \in L_1$. Thus $u'w_1x_2v' \in L_1$ and $u'w_1yw_2v' \notin L_1$. Therefore L_1 is disjunctive and L is qusi-completely disjunctive.

Proposition 13. Every semi-discrete disjunctive language is a quasi-completely disjunctive language.

Proof. Follows from ([3]).

Proposition 14. Let A, $B \subseteq X^*$, where $(A, \leq_1)$, $(B, \leq_2)$ are two strictly ordered sets. Then the following are equivalent:

(1) A or B is dense;

(2) AB is dense;

(3) $A \triangle B$ is dense;

(4) $A \triangle B$ is disjunctive;

(5) $A \triangle B$ is completely disjunctive.

Proof. The equivalences of (1), (2) and (3) are immediate.

(1) $\Leftrightarrow$ (4). Theorem 3 of ([7]).

(4) $\Rightarrow$ (5). Assume that $A \triangle B$ is disjunctive. Let $L_1 \triangle L_2$ be a dense subset of $A \triangle B$. Then by the equivalence of (3) and (4), we have that $L_1 \triangle L_2$ is disjunctive and we are done.

(5) $\Rightarrow$ (3). Trivial.

It has been shown that the language $\bigcup_{i\geq 2} Q^{(i)}$ is quasi-completely disjunctive ([1]). But the language Q is not quasi-completely disjunctive.

For example, let the language $L = \{q \in Q | \lg(q)$ is a prime number$\}$ is a dense subset of Q, which is not disjunctive.

Proposition 15. Let A, B be two languages. If AB is quasi-completely disjunctive, then one of A and B is quasi-completely disjunctive.

Proof. Certainly AB is dense. Then clearly one of A or B is dense. Let us assume that A is dense. Let $A' \subseteq A$ be dense and let $B' \subseteq B$ be finite. Then $A'B'$ is a dense subset of AB and therefore disjunctive. That A' disjunctive follows from the fact that $A'B'$ is disjunctive and B' is finite (see [10]). Thus A is quasi-completely disjunctive.

Similarly, we can show that if B is dense then B is quasi-completely disjunctive.

From the above we can conclude that for two languages A and B, if AB is quasi-completely disjunctive, then both A and B are quasi-completely disjunctive.

In general, the catenation of two quasi-completely disjunctive languages may not be quasi-completely disjunctive. This can be seen from the following example.

Example 2. The language $\bar{Q} = \bigcup_{i\geq 2} Q^{(i)}$ is quasi-completely disjunctive but $\bar{Q}\bar{Q}$ is not quasi-completely disjunctive. Indeed, $\bar{Q}\bar{Q} = \{f^i | \in Q, i \geq 4\} \cup \cup \{p^i q^j | p \neq q \in Q, i, j \geq 2\}$ and there exist $x \neq y \in X^+$ such that $(\bar{Q}\bar{Q})_{xy} = \{uv | uxv \in \bar{Q}\bar{Q}$ and $uyv \in \bar{Q}\bar{Q}\}$ is dense. Let $A = \{uaav | (u, v) \in \bar{Q}\bar{Q} \ldots aa \cap \cap \bar{Q}\bar{Q} \ldots bb\} \cup \{ubbv | (u, v) \in \bar{Q}\bar{Q} \ldots aa \cap \bar{Q}\bar{Q} \ldots bb\}$. It is clear that for every $x \in X^+$, $aaxx$, $bbxx \in A$ and hence A is not a quasi-completely disjunctive language.

4. Operations on the quasi-completely disjunctive languages

We now study some operations on the family of quasi-completely disjunctive languages. Let $CD(X)$ be the family of all completely disjunctive languages over X (which is equivalent to the family of all completely dense languages over X), and let $QCD(X)$ be the family of all quasi-completely disjunctive languages over X.

Proposition 16. Let $L \in QCD(X)$. *If* $L = A \cup B$ *with* $A \cap B = \emptyset$, *then* A *or* B *is disjunctive.*

Proof. Immediate.

The converse of the above proposition is not true in general as can be seen from the following proposition. Let us first present a lemma, which is due to ITO, KATSURA and SHYR ([2]).

Lemma 17. ([5]) *Let* $x, y, u, v \in X^+$ $(x \neq y)$ *and let* $a, b \in X$ $(a \neq b)$. *If* $m \geq max\{\lg(x), \lg(y)\}$ *then* $uxab^mv \in Q$ *or* $uyab^mv \in Q$.

Proposition 18. Let $Q = A \cup B$ *with* $A \cap B = \emptyset$. *If* A *is not disjunctive, then* B *is disjunctive.*

Proof. Let $x \neq y \in X^n$, $n \geq 1$, $x \equiv y(P_A)$. Let $w \neq z$, $\lg(w) = \lg(z)$. Suppose $a \neq b \in X$ and $m \geq \lg(xw)$. Because Q is disjunctive, we can find $u, v \in X$ such that $uxwab^mv \notin Q$. Then $uywab^mv$ and $uxzab^mv$ are primitive. Since $x \equiv y(P_A)$ and $uxwab^mv \notin A$ we have $uywab^mv \notin A$ and hence $uywab^mv \in B$.

Now if $uxzab^mv \in B$, then since $uxwab^mv \notin Q$, we have that $w \not\equiv z(P_B)$ and we are done. If on the other hand $uxzab^mv \notin B$ then $uxzab^mv \in A$ and $uyzab^mv \in A$ $(\notin B)$. Since $uywab^mv \in B$, we have $w \not\equiv z(P_B)$. This shows that $w \not\equiv z(P_B)$ and B is disjunctive.

Proposition 19. Let $A, B \in QCD(X)$. *Then* $L = A \cup B$ *is disjunctive.*

Proof. Let $A, B \in QCD(X)$. Suppose L is not disjunctive and there exist $x \neq y \in X^*$, $x \equiv y(P_L)$. Since $A, B \in QCD(X)$, by Proposition 12, both A_{xy} and B_{xy} are not dense. Thus there exist w and w' such that $X^*wX^* \cap A_{xy} = \emptyset$ and $X^*w'X^* \cap B_{xy} = \emptyset$. Now for every $u, v \in X^*$, if $uxwv \notin A$ then $uywv \in A$ or vice versa, and if $uw'xv \in B$ then $uw'yv \notin B$ or vice versa. Since A is dense, there exist $u, v \in X^*$ such that $uxww'yv \in A$ and $uyww'yv \notin A$. By the assumption that $x \equiv y(P_L)$, $uyww'yv \in B$ and $uyww'xv \notin B$ hold. We then have $uyww'xv \in A$.

Similarly, if $uyww'xv \in A$ then $uxww'xv \in B$ and $uxww'yv \in A$. We thus have $xww'y \equiv yww'x(P_L)$, a contradiction. Therefore, $A \cup B$ is disjunctive.

The following is immediate.

Corollary 20. *Let* A *be a regular language and let* $L \subseteq A$. *Then* $L \in QCD(X)$ *implies that* $A \setminus L \notin QCD(X)$.

Certainly, if L is a quasi-completely disjunctive language then $\bar{L} = X^* \setminus L$ is not quasi-completely disjunctive.

Dense languages have been characterized by Shyr ([8]). We give another characterization for the dense languages.

Proposition 21. Let $L \subseteq X^*$. Then the following are equivalent:

(1) L is dense;

(2) L contains a completely disjunctive language;

(3) L contains a quasi-completely disjunctive language;

(4) L contains a disjunctive language.

Proof. (1) ⇒ (2). Let $\leq$ be a total order defined on X^* and let $X^+ = \{x_1, x_2, x_3, \ldots\}$. Let

$L' = \{u_i x_1 x_2 \ldots x_i v_i | u_i x_1 x_2 \ldots x_i v_i \in L, i \geq 1\} \subseteq L.$

Since L is dense, L′ is dense. It is clear that every infinite subset of L′ is dense. Therefore L′ is completely dense and hence L′ is a completely disjunctive language.

(2) ⇒ (3) and (3) ⇒ (4) are immediate.

(4) ⇒ (1). Proposition 4.20 of ([6]).

It is obvious that $CD(X) \subseteq QCD(X)$. Since $\bar{Q} \in QCD(X)$ and $\bar{Q} \notin CD(X)$, we have $CD(X)$ is a proper subfamily of $QCD(X)$.

5. Lattice properties

In this section we consider the family of languages

$$M(X) = \{\emptyset\} \cup \{F \subseteq X^* | F \text{ is a finite set}\} \cup CD(X).$$

Then by the previous result we see that $M(X)$ is a semigroup under catenation operation. The relation $\subseteq$ on $M(X)$ is clearly a partial order, and the semigroup $M(X)$ has a lattice property. Indeed,

Proposition 22. If $A, B \in M(X)$, then $A \cup B \in M(X)$ and $A \cap B \in M(X)$.

Proof. If A or B is finite or empty, then we are done. Assume that $A, B \in CD(X)$. For every infinite subset $S \subseteq A \cup B$. S contains an infinite subset of A or B. Thus S is dense. By Proposition 11, $A \cup B \in CD(X)$. If $A \cap B$ is finite, then $A \cap B \in M(X)$. If $A \cap B$ is infinite, then $A \cap B$ is an infinite subset of A. Thus $A \cap B \in CD(X)$.

We have the following proposition.

Proposition 23. $(M(X), \subseteq, \cap, \cup)$ forms a distributive lattice for every finite alphabet X.

Proof. For every $A, B \in M(X)$, $A \cup B$ is the minimum set such that $A, B \subseteq A \cup B$ and $A \cap B$ is the maximal set such that $A \cap B \subseteq A, B$. It is easy to see that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Therefore, $(M(X), \subseteq, \cap, \cup)$ forms a distributive lattice.

References

1. Ito, M., Jurgensen, H., Shyr, H. J. and Thierrin, G.: Anti-commutative Languages and n-Codes (to be submitted).
2. Ito, M., Katsure, M. and Shyr, H. J.: Relation Between Disjunctive Languages and Regular Languages (under preparation).
3. Kunze, M., Shyr, H. J. and Thierrin, G., H-bounded and Semi-discrete Languages, Information and Control, Vol. *51*, No. 2 (1981) 174—187.
4. Lallement, G.: Semigroups and Combinatorial Applications, John Wiley and Sons, New York (1978).
5. Shyr, H. J.: Disjunctive Languages on a Free Monoid, Information and Control, Vol. *34* (1977) 123—129.
6. Shyr, H. J.: Free Monoids and Languages, Lecture Notes, Department of Mathematics, Soochow University, Taipei, Taiwan (1979).
7. Shyr, H. J.: Ordered Catenation and Regular Free Disjunctive Languages, Information and Control, vol. *46*, No. 3 (1980) 257—269.
8. Shyr, H. J.: A characterization of Dense Languages, Semigroup Forum, vol. 30 (1984) 237—240.
9. Shyr, H. J. and Yu, S. S.: Solid m-codes and Disjunctive Domains, Semigroup Forum (submitted for publication).
10. Shyr, H. J. and Yu, S. S.: Some Properties of Left Cancellative Languages, Proc. 10 Symposium on Semigroups, held at Josai University, Japan (1986) 15—25.

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